Lecture 7

The Shapley value

The Shapley value is designed to provide a fair payoff distribution in a coalition [18] whereas the solution concepts we have introduced thus far focus on stability. Two ways can be used to define the Shapley value: either by an axiomatic approach or by using a formulation based on marginal surplus. We are presenting both, then we present computational issues and a variant of the Shapley value that reduces the computational choice.

7.1 A Definition Based on the Ordinal Marginal Contribution

A first interpretation of the Shapley value is based on the notion of ordered marginal contribution. We recall that the marginal contribution of an agent i to a coalition $\mathcal{C} \subseteq N$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$. Let us consider that a coalition $\mathcal{C}$ is built incrementally with one agent at a time entering the coalition. Also consider that the payoff of each agent i is its marginal contribution[1]. For example, $\langle mc_1(\emptyset), mc_2(\{1\}), mc_3(\{1,2\})\rangle$ is an efficient payoff distribution for a game $\{1,2,3\}, v)$. In this case, the value of each agent depends on the order in which the agents enter the coalition. This may not be fair! For example, consider agents that form a coalition to take advantages of price reduction when buying large quantities of a product. Agents that start the coalition may have to spend large setup cost, and agents that come later benefits from the already large number of agents in the coalition. To alleviate this issue, the Shapley value averages each agents' payoff over all possible orderings: the value of agent i in coalition $\mathcal{C}$ is the average marginal value over all possible orders in which the agents may join the coalition.

Let π represent a joining order of the grand coalition N: π can also be viewed as a permutation of $\langle 1, \ldots, n\rangle$. We write $mc(\pi)$ the payoff vector where agent i obtains

[1] We used this payoff distribution in the proof showing that a convex game has a non-empty core (Theorem 2.2.3)

$mc_i(\{\pi(j) \mid j < i\})$. The payoff vector $mc(\pi)$ is called the *marginal vector*. Let us denote the set of all permutations of the sequence $\langle 1, \ldots, n \rangle$ as $\Pi(N)$. The Shapley values can then be defined as

$$Sh(N, v) = \frac{\sum_{\pi \in \Pi(N)} mc(\pi)}{n!}.$$

	$v(\{1\}) = 0$	$v(\{2\}) = 0$	$v(\{3\}) = 0$
$N = \{1, 2, 3\}$	$v(\{1, 2\}) = 90$	$v(\{1, 3\}) = 80$	$v(\{2, 3\}) = 70$
		$v(\{1, 2, 3\}) = 120$	

	1	2	3
$1 \leftarrow 2 \leftarrow 3$	0	90	30
$1 \leftarrow 3 \leftarrow 2$	0	40	80
$2 \leftarrow 1 \leftarrow 3$	90	0	30
$2 \leftarrow 3 \leftarrow 1$	50	0	70
$3 \leftarrow 1 \leftarrow 2$	80	40	0
$3 \leftarrow 2 \leftarrow 1$	50	70	0
total	270	240	210
Shapley value ϕ	45	40	35

Let $y = \langle 50, 40, 30 \rangle$

$\mathcal{C}$	$e(\mathcal{C}, \phi)$	$e(\mathcal{C}, y)$
$\{1\}$	-45	0
$\{2\}$	-40	0
$\{3\}$	-35	0
$\{1, 2\}$	5	0
$\{1, 3\}$	0	0
$\{2, 3\}$	-5	0
$\{1, 2, 3\}$	0	0

This example shows that the Shapley value may not be in the core, and may not be the nucleolus.

Table 7.1: Example of a computation of Shapley value

We provide an example in Table 7.1 in which we list all the orders in which the agents can enter the grand coalition. The sum is over all joining orders, which may contains a very large number of terms. However, when computing the Shapley value for one agent, one can avoid some redundancy by summing over all coalitions and noticing that:

- There are $|\mathcal{C}|!$ permutations in which all members of $\mathcal{C}$ precede i.
- There are $|N \setminus (\mathcal{C} \cup \{i\})|!$ permutations in which the remaining members succede i, i.e. $(n - |\mathcal{C}| - 1)!$.

These observations allow us to rewrite the Shapley value as:

$$Sh_i(N, v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(n - |\mathcal{C}| - 1)!}{n!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right).$$

Note that the example from Table 7.1 also demonstrates that in general the Shapley value is not in the core or in the nucleolus.

7.2 An Axiomatic Characterisation

In the first lecture, we provided some concepts some desirable concepts that can be satisfied by a payoff distribution. The core is defined by using two of these concepts: efficiency and group rationality. In the following, we are going to use some properties, called axioms in this context, to define a payoff distribution that guarantees some elements of fairness.

Instead of considering properties of a payoff distribution, we will state the properties of a function that maps a game to a payoff distribution.

7.2.1. DEFINITION. [value function] Let $\mathcal{G}_N$ the set of all valuation functions $2^N \to \mathbb{R}$ that maps a coalition of N to a real number. A *value function* $\phi : N \times \mathcal{G}_N \to \mathbb{R}^n$ assigns an efficient allocation x to a TU game (N, v) (i.e., an allocation that satisfies $\sum_{i \in N} \phi(N, v)_i = v(N)$).

We have already seen one value function for games that have at least one imputation: the nucleolus provides a single payoff distribution for such game.

We now consider some axioms that may be desirable for such a value function. We consider a TU game (N, v) and we simply note $\phi(N, v)$ as ϕ. The first axiom uses the definition of a dummy agent. Intuitively, there is no synergy between a dummy agent and any other agent, in other words the marginal contribution of a dummy agent i to any coalition is $v(\{i\})$.

DUM ("Dummy actions") : if agent i is a dummy then $\phi_i = v(\{i\})$. In other words, if the presence of agent i improves the value of any coalition by exactly $v(\{i\})$, this agent should obtain precisely $v(\{i\})$.

SYM ("Symmetry") : When two agents generate the same marginal contributions, they should be rewarded equally: for $i \neq j$ and $\forall \mathcal{C} \subseteq N$ such that $i \notin \mathcal{C}$ and $j \notin S$, if $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$, then $\phi_i = \phi_j$.

ADD ("Additivity") : For any two TU games (N, v) and (N, w) and their corresponding value functions $\phi(N, v)$ and $\phi(N, w)$, the value function for the TU game $(N, v + w)$ is $\phi(N, v + w) = \phi(N, v) + \phi(N, w)$.

These three axioms appear as good properties that should be satisfied by a payoff distribution. Dummy states that an agent that does not have any synergy with any other agent should get the value of its singleton coalition. SYM provides an element of fairness: two agents that are substitutes should get the same payoff. The last axiom demands a value function to be additive. Shapley [18] showed that actually, there is a unique value that satisfies this three axioms.

7.2.2. THEOREM. *The Shapley value is the unique value that is budget-balanced and that satisfies axioms 1, 2, and 3.*

These axioms are independent. One can prove that if one of the three axioms is dropped, it is possible to find multiple value functions satisfying the other two axioms. To prove this results, one needs to show the existence of a value function that satisfies the three axioms, and then prove the unicity of the value function. We will follow the proof from [14]. We will need the following type of games called in the proof.

7.2.3. DEFINITION. [Unanimity Game] Let N be a set of agents and $T \subseteq N \setminus \emptyset$. The *unanimity game* (N, v_T) is the game such that $\forall \mathcal{C} \subseteq N$, $v_T(\mathcal{C}) = \begin{cases} 1, \text{ if } T \subseteq \mathcal{C}, \\ 0 \text{ otherwise.} \end{cases}$

One can note that if $i \in N \setminus T$, i is a null player. In addition, if $(i, j) \in T^2$, i and j are substitutes. The following lemma will be useful for the proof.

7.2.4. LEMMA. *The set* $\{v_T \in \mathcal{G}_N \mid T \subseteq N \setminus \emptyset\}$ *is a linear basis of* $\mathcal{G}_N$.

The lemma means that a TU game (N, v) can be represented by a unique set of values $(\alpha_T)_{T \subseteq N \setminus \emptyset}$ such that $\forall \mathcal{C} \subseteq N$, $v(\mathcal{C}) = \left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T\right)(\mathcal{C})$.
Proof. There are $2^n - 1$ unanimity games and the dimension of $\mathcal{G}_N$ is also $2^n - 1$. We only need to prove that the unanimity games are linearly independent. Towards a contradiction, let us assume that $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T = 0$ where $(\alpha_T)_{T \subseteq N \setminus \emptyset} \neq 0_{\mathbb{R}^{2^n - 1}}$. Let T_0 be a minimal set in $\{T \subseteq N \mid \alpha_T \neq 0\}$. Then, $\left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T\right)(T_0) = \alpha_{T_0} \neq 0$, which is a contradiction. □

We can now turn to proving that the Shapley value is the unique value that satisfies SYM, DUM and ADD.
Proof. We start by proving the uniqueness. Let ϕ a feasible solution on $\mathcal{G}_N$ that is non-empty and satisfies the axioms SYM, DUM and ADD. Let us prove that ϕ is a value function. Let $(N, v) \in \mathcal{G}_N$.

- if $v = 0_{\mathcal{G}_N}$, all players are dummy. Since the solution is non-empty, $0^{\mathbb{R}^{|N|}}$ is the unique member of $\phi(N, v)$.
- otherwise, $(N, -v) \in \mathcal{G}_N$. Let $x = \phi(N, v)$ and $y = \phi(N, -v)$. By ADD, $x + y = \phi(v - v)$, and then since $v - v = 0_{\mathcal{G}}$, we have that $x = -y$ is unique. Moreover, $x(N) \leq v(N)$ as ϕ is a feasible solution. Also $y(N) \leq -v(N)$. Since $x = -y$, we have $v(N) \leq x(N) \leq v(N)$, i.e. x is efficient.

Hence, ϕ is a value function. We now show that the value function of a unanimity game is unique. Let $T \subseteq N \setminus \emptyset$ and $\alpha \in \mathbb{R}$. Let us prove that $\phi(N, \alpha \cdot v_T)$ is uniquely defined.

- Let $i \notin T$. We have trivially $T \subseteq \mathcal{C}$ iff $T \subseteq \mathcal{C} \cup \{i\}$. Then $\forall \mathcal{C} \subseteq N \setminus \{i\}$, $\alpha v_T(\mathcal{C}) = \alpha v_T(\mathcal{C} \cup \{i\})$. Hence, all agent $i \notin T$ are dummies. By DUM, $\forall i \notin T$, $\phi_i(N, \alpha \cdot v_T) = 0$.

- Let $(i, j) \in T^2$. Then for all $\mathcal{C} \subseteq N \setminus \{i, j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$. By SYM, $\phi_i(N, \alpha \cdot v_T) = \phi_j(N, \alpha \cdot v_T)$.
- Since ϕ is a value function, it is efficient. Then, $\sum_{i \in N} \phi_i(N, \alpha \cdot v_T) = \alpha v_T(N) = \alpha$.
  Hence, $\forall i \in T$, $\phi_i(N, \alpha \cdot v_T) = \frac{\alpha}{|T|}$.

This proves that $\phi(N, \alpha \cdot v_T)$ is uniquely defined. Since any TU game (N, v) can be written as $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T$ and because of ADD, there is a unique value function that satisfies the three axioms.

We now turn to show the existence of the Shapley value, i.e. we need to show that there exist a value function that satisfies the three axioms. Let (N, v) a TU game and we consider the Shapley value as defined in the previous section. We are going to check whether this function satisfies the three axioms.

SYM Let i and j be substitutes, i.e., $\forall \mathcal{C} \subseteq N \setminus \{i, j\}$, we have $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$. Then $\forall \mathcal{C} \subseteq N \setminus \{i, j\}$, we have

- $mc_i(\mathcal{C}) = mc_j(\mathcal{C})$
- $v(\mathcal{C} \cup \{i, j\}) - v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{i, j\}) - v(\mathcal{C} \cup \{j\})$, hence, we have $mc_j(\mathcal{C} \cup \{j\}) = mc_i(\mathcal{C} \cup \{i\})$.
- ⇒ $Sh_i(N, v) = Sh_j(N, v)$, Sh satisfies SYM.

DUM Let i be a dummy agent, i.e., for all coalition $\mathcal{C} \subseteq N \setminus \{i\}$, we have $v(\mathcal{C}) + v(\{i\}) = v(\mathcal{C} \cup \{i\})$. Then, each marginal contribution of player i is $v(\{i\})$, and it follows that $Sh_i(N, v) = v(\{i\})$. Sh satisfies DUM.

ADD Sh is clearly additive.

We proved that a unique function satisfies all three axioms at the same time and we found one function that satisfies them all, hence the Shapley value is well defined.✔ □

The axioms SYM and DUM are clearly desirable. The last axiom, ADD, is harder to motivate in some cases. If the valuation function of a TU game is interpreted as an expected payoff, then ADD is desirable (as you want to be able to add the value of different states of the world). Also, if we consider cost-sharing games and that a TU game corresponds to sharing the cost of one service, then ADD is desirable as the cost for a joint-service should be the sum of the cost of the separate services. However, if we do not make any assumptions about the games (N, v) and (N, w), the axiom implies that there is no interaction between the two games. In addition, the game $(N, v + w)$ may induce a behavior that may be unrelated to the behavior induced by either (N, v) or (N, w), and in this case ADD can be questioned.

Other axiomatisations that do not use the ADD axiom have been proposed by Young in [23] and by Myerson in [13].

7.2.5. DEFINITION. [Marginal contribution axiom] Let (N, v) and (N, v) be two TU games. A value function ϕ satisfies the *marginal contribution axiom* iff for all $i \in N$, *if* for all $\mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C} \cup \{i\}) - v(C) = u(\mathcal{C} \cup \{i\}) - u(C)$, *then* $\phi(u) = \phi(v)$.

In other words, the value of a player depends only on its marginal contribution. The following axiomatisation is due to Young [23].

7.2.6. THEOREM. *The Shapley value is the unique value function that satisfies symmetry and marginal contribution axioms.*

We refer by $v \setminus i$ the TU game $(N \setminus \{i\}, v_{\setminus i})$ where $v_{\setminus i}$ is the restriction of v to $N \setminus \{i\}$.

7.2.7. DEFINITION. [Balanced contribution axiom] A value function ϕ satisfies *the balanced contribution axiom* iff for all $(i, j) \in N^2$ $\phi_i(v) - \phi_i(v \setminus j) = \phi_j(v) - \phi_j(v \setminus i)$.

For any two agents, the amount that each agent would win or lose if the other "leaves the game" should be the same. The following axiomatisation is due to Myerson [13].

7.2.8. THEOREM. *The Shapley value is the unique value function that satisfies the balanced contribution axiom.*

7.3 Other properties

At noted before, the Shapley value always exists and is unique.

7.3.1. THEOREM. *For superadditive games, the Shapley value is an imputation.*

Proof. Let (N, v) be a superadditive TU game.

By superadditivity, $\forall i \in N, \forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) > v(\{i\})$. Hence, for each marginal vector, an agent i gets at least $v(\{i\})$. The same is true for the Shapley value as it is the average over all marginal vectors.✔ □

7.3.2. THEOREM. *For convex game, the Shapley value is in the core.*

Proof. Let (N, v) be a convex game. We know that all marginal vectors are in the core (to show that convex games have non-empty core, we used one marginal vector and showed it was in the core). The core is a convex set. The average of a finite set of points in a convex set is also in the set. Finally, the Shapley value is in the core.✔ □

When the valuation function is superadditive, the Shapley value is individually rational, i.e., it is an imputation. When the Core is non-empty, the Shapley value may not be in the Core. However, when the valuation function is convex, the Shapley value is also group rational, hence, it is in the Core.

7.4 Computational Issues

The nature of the Shapley value is combinatorial, as all possible orderings to form a coalition needs to be considered. This computational complexity can sometimes be an advantage as agents cannot benefit from manipulation. For example, it is $\mathcal{NP}$-complete to determine whether an agent can benefit from false names [21]. Nevertheless, some representations allow to compute the Shapley value efficiently. We will surveying few representations in a coming lecture. For now, we just concentrate on a simple proposal.

In order to reduce the combinatorial complexity of the computation of the Shapley value, Ketchpel introduces the Bilateral Shapley Value (BSV) [9]. The idea is to consider the formation of a coalition as a succession of merging between two coalitions. Two disjoint coalitions $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, may merge when $v(\mathcal{C}_1 \cup \mathcal{C}_2) \geq v(\mathcal{C}_1) + v(\mathcal{C}_2)$. When they merge, the two coalitions, called founders of the new coalition $\mathcal{C}_1 \cup \mathcal{C}_2$, share the marginal utility as follows: $BSV(\mathcal{C}_1) = \frac{1}{2}v(\mathcal{C}_1) + \frac{1}{2}\left(v(\mathcal{C}_1 \cup \mathcal{C}_2) - v(\mathcal{C}_2)\right)$ and $BSV(\mathcal{C}_2) = \frac{1}{2}v(\mathcal{C}_2) + \frac{1}{2}\left(v(\mathcal{C}_1 \cup \mathcal{C}_2) - v(\mathcal{C}_1)\right)$. This is the expression of the Shapley value in the case of an environment with two agents. In $\mathcal{C}_1 \cup \mathcal{C}_2$, each of the founders gets half of its 'local' contribution, and half of the marginal utility of the other founder. Given this distribution of the marginal utility, it is rational for $\mathcal{C}_1$ and $\mathcal{C}_2$ to merge if $\forall i \in \{1, 2\}$, $v(\mathcal{C}_i) \leq BSV(C_i)$. Note that symmetric founders get equal payoff, i.e., for $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C} = \mathcal{C}_2 \cap \mathcal{C} = \emptyset$, $v(\mathcal{C} \cup \mathcal{C}_1) = v(\mathcal{C} \cup \mathcal{C}_2) \Rightarrow BSV(\mathcal{C} \cup \mathcal{C}_1) = BSV(\mathcal{C} \cup \mathcal{C}_2)$. Given a sequence of successive merges from the states where each agent is in a singleton coalition, we can use a backward induction to compute a stable payoff distribution [10]. Though the computation of the Shapley value requires looking at all of the permutations, the value obtained by using backtracking and the BSV only focuses on a particular set of permutations, but the computation is significantly cheaper.