Cooperative Games

Lecture 1: Introduction

Stéphane Airiau

ILLC - University of Amsterdam

Why study coalitional games?

Cooperative games are a branch of game theory that models **cooperation** or collaboration between agents.

Coalitional games can also be studied from a computational point of view (e.g., the problem of succint representation).

A coalition may represent a set of:

- persons or group of persons (labor unions, towns)
- objectives of an economic project
- artificial agents

We have a population N of n agents.

Definition (Coalition)

A **coalition** $\mathcal{C}$ is a set of agents: $\mathcal{C} \in 2^N$.

The classic problem

- N is the set of all agents (or players)
- $v : 2^N \to \mathbb{R}$ is the **valuation function**. For $\mathcal{C} \subseteq N$, $v(\mathcal{C})$ is the value obtained by the coalition $\mathcal{C}$

problem: a game (N,v), and we assume all agents in N want to cooperate.

solution: a **payoff distribution** $x \in \mathbb{R}^n$ that provides a value to individual agents.

What are the interesting **properties** that x should satisfy?

How to **determine** the payoff vector x?

An example

$$N = \{1,2,3\}$$
$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$$
$$v(\{1,2\}) = 90$$
$$v(\{1,3\}) = 80$$
$$v(\{2,3\}) = 70$$
$$v(\{1,2,3\}) = 105$$

What should we do?

- form $\{1,2,3\}$ and share equally $\langle 35,35,35 \rangle$?
- 3 can say to 1 "let's form $\{1,3\}$ and share $\langle 40,0,40 \rangle$".
- 2 can say to 1 "let's form $\{1,2\}$ and share $\langle 45,45,0 \rangle$".
- 3 can say to 2 "OK, let's form $\{2,3\}$ and share $\langle 0,46,24 \rangle$".
- 1 can say to 2 and 3, "fine! $\{1,2,3\}$ and $\langle 33,47,25 \rangle$
- ... is there a "good" solution?

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1- Games with Transferable Utility (TU games)

- Any two agents can **compare** their utility
- Utility can be **transferred** between agents

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(\mathcal{C})$ to any subset $\mathcal{C} \subseteq N$, i.e., $v : 2^N \to \mathbb{R}$

Without loss of generality, we will assume that $\begin{cases} v(\emptyset) = 0 \\ v(\mathcal{C}) \geqslant 0 \end{cases}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

2- Games with Non Transferable Utility (NTU games)

It is **not** always possible to compare the utility of two agents or to transfer utility (e.g., no price tags). Agents have preference over coalitions.

Today

- We provide some examples of TU games.
- We discuss some desirable solution properties.
- We end with a quick overview of the course and practicalities

Informal example: a task allocation problem

- A set of tasks needs to be performed,
  - they require different expertises
  - they may be decomposed.
- Agents do not have enough resource on their own to perform a task.
- Find complementary agents to perform the tasks
  - robots have the ability to move objects in a plant, but multiple robots are required to move a heavy box.
  - transportation domain: agents are trucks, trains, airplanes, ships... a task is a good to be transported.
- **Issues:**
  - What coalition to form?
  - How to reward each each member when a task is completed?

Market games (classic application in Economics)

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$, $a_i \in \mathbb{R}^m_+$ is the initial endowment vector
- $F = (f_i)_{i \in N}$, $f_i : \mathbb{R}^m_+ \to \mathbb{R}$ is the valuation function vector

Assumptions of the model:

- The **utility** of agent i for possessing $x \in \mathbb{R}^m_+$ and an amount of money $p \in \mathbb{R}$ is $u_i(x, p) = f_i(x) + p$. The money models side payments.
- Initially, agents have **no money**.
- p_i can be **positive** or **negative** (like a bank account).
- Agents can increase their utility by **trading**: after a trade among the members of S, they have an endowment $(b_i)_{i \in S}$ and money $(p_i)_{i \in S}$ such that $\sum_{i \in S} a_i = \sum_{i \in b} b_i$ and $\sum_{i \in S} p_i = 0$.

Market games (cont.)

Definition (Market game)

A game (N,v) is a market game if there exists a market (N,M,A,F) such that, for every $S \subseteq N$,

$$v(S) = \max\left\{ \sum_{i\in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i\in S} x_i = \sum_{i\in S} a_i \right\}$$

Shapley. The solutions of a symmetric market game, in Contributions to the Theory of Games, Luce and Tuckers editors, 1959

Shapley and Shubik. On market games, *Journal of Economic Theory, 1, 9-25, 1969*

Cost allocation games

Definition (Cost allocation game)

A cost allocation game is a game (N,c) where

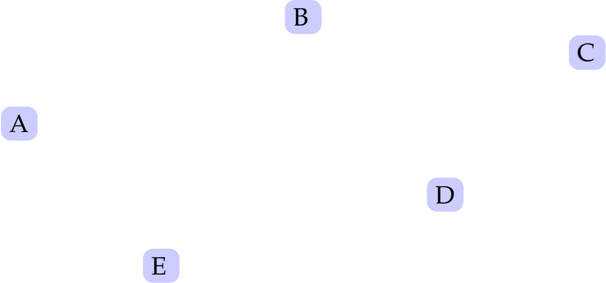
- N represents the potential customers of a public service or a public facility.
- $c(S)$ is the cost of serving the members of S

Mathematically speaking, a cost game is a game. The special status comes because of the different intuition (worth of a coalition vs. cost of a coalition).
We can associate a cost game with a "traditional game" using the corresponding saving game (N,v) given by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S).$$

Examples of cost allocation games

- **Airport game:** n types of planes can land on a runway. The cost to accommodate a plane of type k is c_k. The cost is defined as $c(S) = \max_{k \in S}\{c_k\}$
- **Sharing a water supply system:** n towns considers building a common water treatment facility. The cost of a coalition is the minimum cost of supplying the coalition members by the most efficient means.



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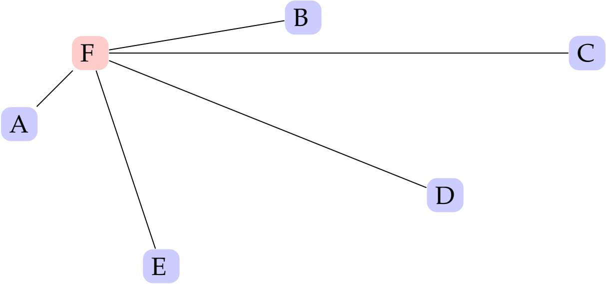

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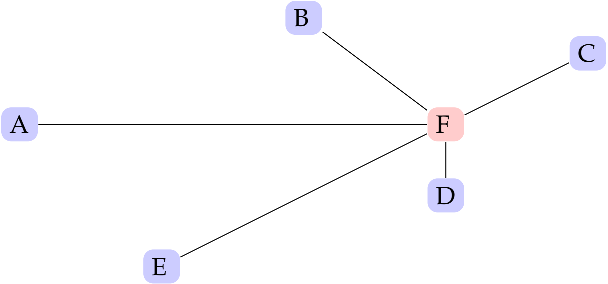

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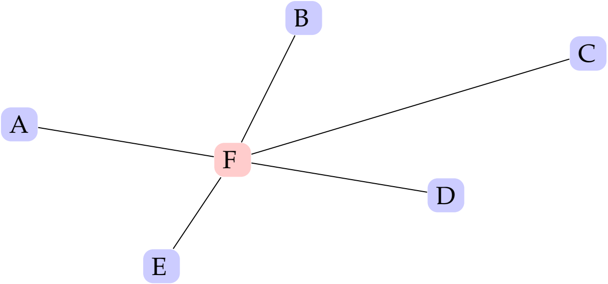
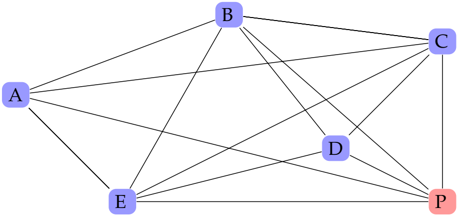


Examples of cost allocation games

- **Minimum cost spanning tree games:** a set H of houses have to be connected to a power plant P. The houses can be linked directly to P or to another house. The cost of connecting two locations $(i,j) \in H \cup \{P\}$ is c_{ij}. Let $S \subseteq H$. $\Gamma(S)$ is the minimum cost spanning tree spanning over the set of edges $S \cup \{P\}$. The cost function is $c(S) = \sum_{\text{all edges of } \Gamma(S)} c_{ij}$.



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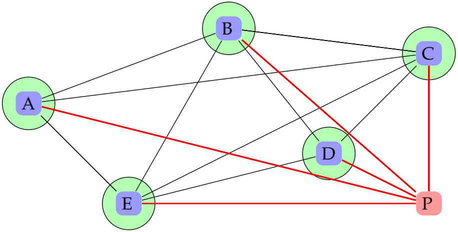

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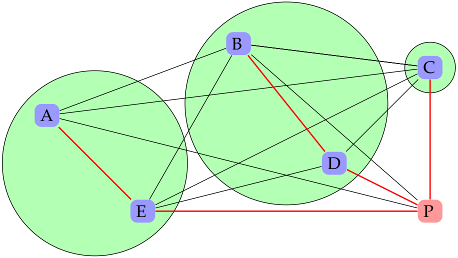

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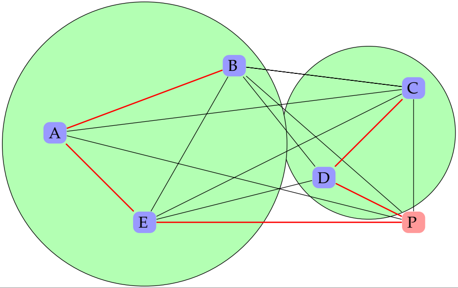

Simple or Voting games

Definition (voting games)

A game (N,v) is a **voting game** when

the valuation function takes two values

- 1 for a winning coalitions
- 0 for the losing coalitions

v satisfies *unanimity*: $v(N) = 1$

v satisfies *monotonicity*: $S \subset T \Rightarrow v(S) \leqslant v(T)$

Weighted Voting games

Definition (weighted voting games)

A game $(N, w_{i\in N}, q, v)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i\in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

Example: 1958 European Economic Community: Belgium, Italy, France, Germany, Luxembourg and the Netherlands. Each country gets the following number of votes:

- Italy, France, Germany: 4
- Belgium, the Netherlands: 2
- Luxembourg: 1

The threshold of the game is $q = 12$.

Some types of TU games

$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq N \mid \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, $i \in N$, $i \notin \mathcal{C}_1$

- **additive (or inessential):** $v(\mathcal{C}_1 \cup \mathcal{C}_2) = v(\mathcal{C}_1) + v(\mathcal{C}_2)$
  trivial from the game theoretic point of view
- **superadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \geqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$ satisfied in many applications: it is better to form larger coalitions.
- **weakly superadditive:** $v(\mathcal{C}_1 \cup \{i\}) \geqslant v(\mathcal{C}_1) + v(\{i\})$
- **subadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \leqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$
- **convex:** $\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$,
  $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.
  Convex game appears in some applications in game theory and have nice properties.
- **monotonic:** $\forall \mathcal{C} \subseteq \mathcal{T} \subseteq N$ $v(\mathcal{C}) \leqslant v(\mathcal{T})$.

Some properties

Let $x \in \mathbb{R}^n$ be a solution of the TU game (N, v)

Feasible solution: $\sum_{i \in N} x(i) \leqslant v(N)$

Anonymity: a solution is independent of the names of the player.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

Let mc_i^{min} and mc_i^{max} denote the minimal and maximal marginal contribution.

x is **reasonable from above** if $\forall i \in N \; x^i \leqslant mc_i^{max}$

- ➪ mc_i^{max} is the strongest **threat** that an agent can use against a coalition.

x is **reasonable from below** if $\forall i \in N \; x^i \geqslant mc_i^{min}$

- ➪ mc_i^{min} is a minimum acceptable reward.

Some properties

Let x, y be two solutions of a TU-game (N,v).

Efficiency: $x(N) = v(N)$

- ➪ the payoff distribution is an allocation of the entire worth of the grand coalition to all agents.

Individual rationality: $\forall i \in N, x(i) \geqslant v(\{i\})$

- ➪ agent obtains at least its self-value as payoff.

Group rationality: $\forall \mathcal{C} \subseteq N$, $\sum_{i\in\mathcal{C}} x(i) = v(\mathcal{C})$

- ➪ if $\sum_{i\in\mathcal{C}} x(i) < v(\mathcal{C})$ some utility is lost
- ➪ if $\sum_{i\in\mathcal{C}} x(i) > v(\mathcal{C})$ is not possible

Pareto Optimal: $\sum_{i\in N} x(i) = v(N)$

- ➪ no agent can improve its payoff without lowering the payoff of another agent.

An **imputation** is a payoff distribution x that is efficient and individually rational.

Summary

- Two main classes of games: TU games and NTU games
- Examples of TU games: market games, cost allocation games, voting games
- Some classes of TU games: superadditive, convex, etc.
- Some desirable properties of a solution

Coming next

A first solution concept to ensure stable coalitions: the **core**.

Definition (Core for superadditive games)

The **core** of a game (N, v) is the set:
$\{x \in \mathbb{R}^n \mid x(S) \geqslant v(S) \text{ for all } S \subseteq N\}$

Course overview

- Game theory stability concepts: the core, the nucleolus, the kernel
- A fair solution concept: the Shapley value
- Special types of games: Voting games
- Representation and complexity
- Other model of cooperation: NTU games and hedonic games.
- Issues raised by practical approaches (seach for optimal CS, uncertainty, overlapping coalition, etc).

Practicalities

- Webpage: `http://staff.science.uva.nl/~stephane/Teaching/CoopGames/2012/`
  It will contain the lecture notes and the slides, posted shortly before class.
- Evaluation: $6\text{ECTS} = 6 \cdot 28\text{h} = 168\text{h}$.
  - some homeworks (every two or three weeks) 40% of the grade. LaTeXis preferred, but you can hand-write your solution.
  - final paper 50% of the grade (more details at the end of the first block)
  - final presentation 10% of the grade
  - **no** exam.
- Attendance: not part of the grade.