Cooperative Games

Lecture 10: Challenging the transferable utility assumption

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Today

- Hedonic games: a class of games with non-transferable utility
- NTU games: the general framework for games with non-transferable utility.

Hedonic games

Agents have preferences over coalitions, i.e. agent only cares about the other members of the coalition: "enjoying the pleasure of each other's company".

Let N be a set of agents and $\mathcal{N}_i$ be the set of coalitions that contain agent i, i.e., $\mathcal{N}_i = \{\mathcal{C} \cup \{i\} \mid \mathcal{C} \subseteq N \setminus \{i\}\}$.

Definition (Hedonic games)

An **Hedonic game** is a tuple $(N, (\succeq_i)_{i \in N})$ where

- N is the set of agents
- $\succeq_i \subseteq 2^{\mathcal{N}_i} \times 2^{\mathcal{N}_i}$ is a complete, reflexive and transitive preference relation for agent i, with the interpretation that if $S \succeq_i T$, agent i prefers coalition T at most as much as coalition S.

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Stability concepts of Hedonic Games

Let $\Pi \in \mathcal{S}_N$ be a coalition structure, and Π_i denotes the coalition in Π that contains i.

- A coalition structure Π is **core stable** iff $\nexists C \subseteq N \mid \forall i \in C, C \succ_i \Pi_i$.
- A coalition structure s is **Nash stable** $(\forall i \in N)\ (\forall C \in \Pi \cup \{\emptyset\})\ \Pi_i \succsim_i C \cup \{i\}$. No player would like to join any other coalition in Π assuming the other coalitions did not change.
- A coalition structure Π is **individually stable** iff $\nexists i \in N\ \nexists C \in \Pi \cup \emptyset \mid \Big((C \cup \{i\} \succ_i \Pi_i) \wedge (\forall j \in C, C \cup \{i\} \succsim_j C) \Big)$. No player can move to another coalition that it prefers without making some members of that coalition unhappy.
- A coalition structure Π is **contractually individually stable** iff $\nexists i \in N\ \nexists C \subseteq N \mid$ $(C \cup \{i\} \succ_i \Pi_i) \wedge (\forall j \in C, C \cup \{i\} \succsim_j C) \wedge (\forall j \in \Pi_i \setminus \{i\}, \Pi_i \setminus \{i\} \succsim_j \Pi_i)$ No player can move to a coalition it prefers so that the members of the coalition it leaves and it joins are better off

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Example 1

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\}$$

$$\{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \succ_2 \{2,3\}$$

$$\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}$$

{{1,2},{3}} is in the core and is individually stable.
There is no Nash stable partitions.

{{1},{2},{3}}	{1,2} is preferred by both agent 1 and 2, hence not NS, not IS.
{{1,2},{3}}	{1,2,3} is preferred by agent 3, so it is not NS, as agents 1 and 3 are worse off, it is not a possible move for IS. no other move is possible for IS.
{{1,3},{2}}	agent 1 prefers to be on its own (not NS, then, not IS).
{{2,3},{1}}	agent 2 prefers to join agent 1, and agent 1 is better off, hence not NS, not IS.
{{1,2,3}}	agents 1 and 2 have an incentive to form a singleton, hence not NS, not IS.

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The core is empty.

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{{1,3},{2}}	{1,2} is blocking
{{2,3},{1}}	{1,3} is blocking
{{1,2,3}}	{1,2}, {1,3}, {2,3} are blocking

{{1,2,3}} is the unique Nash stable partition, unique individually stable partition (no agent has any incentive to leave the grand coalition).

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The core is empty (similar argument as for example 2).

There is no Nash stable partition or individually stable partition. But there are three contractually individually stable CSs: {{1,2},{3}}, {{1,3},{2}},{{2,3},{1}}.

For {{1,2},{3}}:

- {{1},{2,3}}: agents 2 and 3 benefit, hence {{1,2},{3}} is not Nash or individually stable. however, agent 1 is worse off, hence not a possible move for CIS.
- {{2},{1,3}}: agent 1 has no incentive to join agent 3.
- {{1},{2},{3}}: neither agent 1 or 2 has any incentive to form a singleton coalition.

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Nash stability $\Rightarrow$ Individual stability $\Rightarrow$ contractual individual stability

Core stability $\not\Rightarrow$ Nash stability $\not\Rightarrow$ Core stability

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Some classes of games have a non-empty core,
other classes have Nash stable coalition structures.

A. Bogomolnaia and M.O. Jackson, *The stability of hedonic coalition structure*. **Games and Economic Behavior**, 2002.

$$\text{Nash stability} \Rightarrow \text{Individual stability} \Rightarrow \text{contractual individual stability}$$

$$\text{Core stability} \not\Rightarrow \text{Nash stability} \not\Rightarrow \text{Core stability}$$

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Representations

- **Anonymous preferences** player's preferences depend only on the size of the coalition.
- **Individually rational coalition lists (IRCLs)** list only those coalitions that are preferred to singletons.
- **Additively separable game (ASG)**. A game G is AS if there exists an $|N| \times |N|$ matrix M such that
  $$\mathcal{C}_1 \succeq_i \mathcal{C}_2 \text{ iff } \sum_{j \in \mathcal{C}_1} M(i,j) \geqslant \sum_{k \in \mathcal{C}_2} M(i,k).$$
- **Friends and Enemies**: each player has a set of friends F and a set of enemies E and use the number of members that are friends or enemies to evaluate their preferences (friends appreciation: which coalition has most friends, break ties with number of enemies, and enemies aversion: do the opposite)
- $\mathcal{B}$ and $\mathcal{W}$-preferences: use a ranking over individuals and base the preferences on the best and worst member of the coalition.
- **Hedonic Coalition Nets** based on MC-nets.

A general model for NTU games (Non-transferable utility games)

It is **not** always possible to compare the utility of two agents or to transfer utility.

Definition (NTU game)

A NTU game is a tuple $(N, X, V, (\succeq_i)_{i \in N})$ where

- X set of outcomes
- $\succeq_i$ a preference relation (transitive and complete) for agent i over the set of outcomes.
- $V(\mathcal{C})$ a set of outcomes that a coalition $\mathcal{C}$ can bring about (set of choices available to $\mathcal{C}$)

Interpretation of $V(\mathcal{C})$

- not all outcomes may be realized, but the coalition can choose on outcome in $V(\mathcal{C})$.
- **Effectivity function**: set of choices that can be enforced by coalition $\mathcal{C}$.

- **Example 1:** hedonic games as a special class of NTU games.
  Let $(N, (\succeq_i^H)_{i \in N})$ be a hedonic game.
  • For each coalition $\mathcal{C} \subseteq N$, create a unique outcome $x_{\mathcal{C}}$.
  • For any two outcomes x_S and x_T corresponding to coalitions S and T that contains agent i, We define $\succeq_i$ as follows: $x_S \succeq_i x_T$ iff $S \succeq_i^H T$.
  • For each coalition $\mathcal{C} \subseteq N$, we define $V(\mathcal{C})$ as $V(\mathcal{C}) = \{x_C\}$
- **Example 2:** a TU game can be viewed as an NTU game.
  Let (N, v) be a TU game.
  • We define X to be the set of all allocations, i.e., $X = \mathbb{R}^n$.
  • For any two allocations $(x, y) \in X^2$, we define $\succeq_i$ as follows: $x \succeq_i y$ iff $x_i \geqslant y_i$.
  • For each coalition $\mathcal{C} \subseteq N$, we define $V(\mathcal{C})$ as $V(\mathcal{C}) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leqslant v(\mathcal{C})\}$. $V(\mathcal{C})$ lists all the feasible allocation for the coalition $\mathcal{C}$.

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Core

An outcome $x \in X$ is **blocked** by a coalition $\mathcal{C}$ if there is some outcome $y \in V(\mathcal{C})$ such that all members i of $\mathcal{C}$ strictly prefer y to x, i.e., $\exists \mathcal{C} \subseteq N$, $\exists y \in V(\mathcal{C})$ s.t. $\forall i \in \mathcal{C}$, $y \succ_i x$.

The **core** of an NTU game $(N, X, V, (\succeq_i)_{i\in N})$ is defined as:
$Core(N, X, V, (\succeq)) = \{x \in V(N) \mid \nexists \mathcal{C} \subseteq N, \nexists y \in V(\mathcal{C}), \forall i \in \mathcal{C} : y \succ_i x\}$

A game is **balanced** iff for every balanced collection B, we have $\bigcap_{\mathcal{C}\subseteq B} V(\mathcal{C}) \subset V(N)$

Theorem

The core of a balanced game is non-empty

H. Scarf **The Core of an N Person Game**, in *Econometrica*, 1967.

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An application of NTU games: Exchange Economy

For TU games, we studied *market games* and proved such games have a non-empty core. We now consider the case in which agents do not have a utility function, but have a preference relation over the bundle of goods.

An **exchange economy** is a tuple $(N, M, A, (\succeq_i)_{i \in N})$ where

- N is the set of n agents
- M is the set of k continuous goods
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $(\succeq_i)_{i \in N}$ is the preference profile, in which $\succeq_i$ is a preference relation over bundles of goods.

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Given an exchange economy $(N, M, A, (\succeq_i)_{i\in N})$, we define the associated **exchange economy game** as the following NTU game $(N, X, V, (\succeq_i)_{i\in N})$ where:

- The set of outcomes X is defined as $X = \left\{(x_1, \ldots, x_n) \,|\, x_i \in \mathbb{R}^k_+ \text{ for } i \in N\right\}$.
  Note that $x_i = \langle x_{i1}, \ldots, x_{ik}\rangle$ represents the quantity of each good that agent i possesses in a outcome x.
- The preference relations are defined as follows: for $(x, y) \in X^2$ $x \succeq_i y \Leftrightarrow x_i \succeq_i y_i$.
  Each player is concerned by its own bundle only.
- The value sets are defined as $\forall \mathcal{C} \subseteq N$,
  $$V(\mathcal{C}) = \left\{ x \in X \,\middle|\, \sum_{i\in\mathcal{C}} x_i = \sum_{i\in\mathcal{C}} a_i \wedge x_j = a_j \text{ for } j \in N \setminus \mathcal{C} \right\}.$$
  The players outside $\mathcal{C}$ do not participate in any trading and hold on their initial endowments. When all agents participate in the trading, we have $V(N) = \{x \in X \,|\, \sum_{i\in N} x_i = \sum_{i\in N} a_i\}$.

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  The players outside $\mathcal{C}$ do not participate in any trading and hold on their initial endowments. When all agents participate in the trading, we have $V(N) = \{x \in X \,|\, \sum_{i\in N} x_i = \sum_{i\in N} a_i\}$.

Given an exchange economy $(N, M, A, (\succeq_i)_{i \in N})$, we define the associated **exchange economy game** as the following NTU game $(N, X, V, (\succeq_i)_{i \in N})$ where:

- The set of outcomes X is defined as $X = \left\{ (x_1, \ldots, x_n) \,|\, x_i \in \mathbb{R}^k_+ \text{ for } i \in N \right\}$.
  Note that $x_i = \langle x_{i1}, \ldots, x_{ik} \rangle$ represents the quantity of each good that agent i possesses in a outcome x.
- The preference relations are defined as follows: for $(x, y) \in X^2$ $x \succeq_i y \Leftrightarrow x_i \succeq_i y_i$.
  Each player is concerned by its own bundle only.
- The value sets are defined as $\forall \mathcal{C} \subseteq N$,

$$V(\mathcal{C}) = \left\{ x \in X \,\middle|\, \sum_{i \in \mathcal{C}} x_i = \sum_{i \in \mathcal{C}} a_i \wedge x_j = a_j \text{ for } j \in N \setminus \mathcal{C} \right\}.$$

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Let us assume we can define a price p_r for a unit of good r.
The idea would be to exchange the goods at a **constant price** during the negotiation.

Let us define a price vector $p \in \mathbb{R}^k_+$.
The amount of each good that agent i possesses is $x_i \in R^k_+$.

The total cost of agent i's bundle is $p \cdot x_i = \sum_{r=1}^{k} p_r x_{i,r}$.

Since the initial endowment of agent i is a_i, the agent has at his disposal an amount $p \cdot a_i$, and i can afford to obtain a bundle y_i such that $p \cdot y_i \leqslant p \cdot a_i$.

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What would be an ideal situation?

A **competitive equilibrium** of an exchange economy is a **pair** (p,x) where $p \in \mathbb{R}_+^k$ is a price vector and $x \in \left\{(x_1,\ldots,x_n) \mid x_i \in \mathbb{R}_+^k \text{ for } i \in N\right\}$ such that

- $\sum_{i\in N} x_i = \sum_{i\in N} a_i$ (the allocation results from trading)
- $\forall i \in N,\ p \cdot x_i \leqslant p \cdot a_i$ (each agent can afford its allocation)
- $\forall i \in N\ \forall y_i \in \mathbb{R}_+^k\ (p \cdot y_i \leqslant p \cdot a_i) \Rightarrow x_i \succeq_i y_i$
  Among all the allocations that an agent can afford, it obtains one of its most favorites outcomes.

Using the price vector and the allocation, each agent believes it possesses the best outcome.

Theorem

Let $(N,M,A,(\succeq_i)_{i\in N})$ be an exchange economy. If each preference relation $\succeq_i$ is continuous and strictly convex, then a competitive equilibrium exists.

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Let $(N, M, A, (\succeq_i)_{i \in N})$ be an exchange economy. If each preference relation $\succeq_i$ is continuous and strictly convex, then a competitive equilibrium exists.

The theorem guarantees the existence, but not how to obtain the price vector or the allocation. The following theorem links the allocation with the core:

Theorem

If (p, x) is a competitive equilibrium of an exchange economy, then x belongs to the core of the corresponding exchange economy game.

Proof

Let us assume x is not in the core of the associated exchange economy game. Then, there is at least one coalition $\mathcal{C}$ and an allocation y such that $\forall i \in \mathcal{C}\ y \succ_i x$. By definition of the competitive equilibrium, we must have $p \cdot y_i > p \cdot a_i$. Summing over all the agents in $\mathcal{C}$, we have $p \cdot \sum_{i \in \mathcal{C}} y_i > p \cdot \sum_{i \in \mathcal{C}} a_i$. Since the prices are positive, we deduce that $\sum_{i \in \mathcal{C}} y_i > \sum_{i \in \mathcal{C}} a_i$, which is a contradiction. □

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It then follows that if each preference relation is continuous and strictly convex, then the core of an exchange economy game is non-empty.
In an economy, the outcomes that are immune to manipulations by groups of agent are competitive equilibrium allocation.

Summary

- We considered Hedonic games, an example of games in which utility cannot be transferred between agents.
- We defined general NTU games
- We studied an important application of NTU games: the exchange economy.

Coming next

- Deriving cooperative games from non-cooperative ones.