Cooperative Games

Lecture 3: The core

Stéphane Airiau

ILLC - University of Amsterdam

Today

- Characterize the set of games with non-empty core (Bondareva Shapley theorem), and we will informally introduce linear programming.
- Application of the Bondareva Shapley theorem to market games.

The Bondareva Shapley theorem:

a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Notations:

- Let $\mathcal{V}(N) = \mathcal{V}$ the set of all coalition functions on 2^N.
- Let $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid Core(N,v) \neq \emptyset\}$.

Can we characterize $\mathcal{V}_{Core}$?

$Core(N,v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$

The core is defined by a set of linear constraints.

➩ The idea is to use results from linear optimization.

Linear programming

A linear program has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$$

- x is a vector of n variables
- c is the objective function
- A is a $m \times n$ matrix
- b is a vector of size n
- A and b represent the linear constraints

Linear programming

A linear program has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to } \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$$

- x is a vector of n variables
- c is the objective function
- A is a $m \times n$ matrix
- b is a vector of size n
- A and b represent the linear constraints

example: maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to } \begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$$

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}.$$

A **feasible solution** is a solution that satisfies the constraints.
Example: maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to} \begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$$

A **feasible solution** is a solution that satisfies the constraints.

Example: maximize $8x_1 + 10x_2 + 5x_3$

subject to $\begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$

- $\langle 0,1,1 \rangle$ is feasible, with objective function value 15.
- $\langle 1,1,0 \rangle$ is feasible, with objective function value 18.

A **feasible solution** is a solution that satisfies the constraints.

Example: maximize $8x_1 + 10x_2 + 5x_3$

subject to $\begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$

- $\langle 0,1,1 \rangle$ is feasible, with objective function value 15.
- $\langle 1,1,0 \rangle$ is feasible, with objective function value 18.

The **dual** of a LP: finding an upper bound to the objective function of the LP.

$(1) \times 1 + (2) \times 6$ ➩ $9x_1 + 10x_2 + 8x_3 \leqslant 19$

$(1) \times 2 + (2) \times 2$ ➩ $8x_1 + 10x_2 + 6x_3 \leqslant 18$

The coefficients are as large as in the obective function,

➩ the bound is an upper bound for the objective function.

A **feasible solution** is a solution that satisfies the constraints.

Example: maximize $8x_1 + 10x_2 + 5x_3$

subject to $\begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & \leqslant & 7 & (1) \\ x_1 & + & x_2 & + & x_3 & \leqslant & 2 & (2) \end{cases}$

- $\langle 0,1,1 \rangle$ is feasible, with objective function value 15.
- $\langle 1,1,0 \rangle$ is feasible, with objective function value 18.

The **dual** of a LP: finding an upper bound to the objective function of the LP.

$(1) \times 1 + (2) \times 6$ ➯ $9x_1 + 10x_2 + 8x_3 \leqslant 19$

$(1) \times 2 + (2) \times 2$ ➯ $8x_1 + 10x_2 + 6x_3 \leqslant 18$

The coefficients are as large as in the obective function,

➯ the bound is an upper bound for the objective function.

Hence, the solution cannot be better than 18, and we found one: problem solved! ✔

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to} \begin{cases} y^T A \geqslant c^T, \\ y \geqslant 0 \end{cases} \end{cases}$

Theorem (Duality theorem)

When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.

Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\ S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Example: $n = 3$, $\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

	1	2	3
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\, S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\, S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

The dual of (LP):

$$(DLP) \begin{cases} \max \sum_{\mathcal{C} \subseteq N} y_{\mathcal{C}} v(\mathcal{C}) \\ \text{subject to } \begin{cases} \sum_{\mathcal{C} \subseteq N} y_{\mathcal{C}} \chi_{\mathcal{C}} = \chi_N \text{ and,} \\ y_{\mathcal{C}} \geqslant 0 \text{ for all } \mathcal{C} \subseteq N,\, \mathcal{C} \neq \emptyset. \end{cases} \end{cases}$$

Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N, S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

The dual of (LP):

$$(DLP) \begin{cases} \max \sum_{\mathcal{C} \subseteq N} y_{\mathcal{C}} v(\mathcal{C}) \\ \text{subject to } \begin{cases} \sum_{\mathcal{C} \subseteq N} y_{\mathcal{C}} \chi_{\mathcal{C}} = \chi_N \text{ and,} \\ y_{\mathcal{C}} \geqslant 0 \text{ for all } \mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset. \end{cases} \end{cases}$$

It follows from the duality theorem of linear programming: (N,v) has a non empty core iff $v(N) \geqslant \sum_{\mathcal{C} \subseteq N} y_{\mathcal{C}} v(\mathcal{C})$ for all feasible vector $(y_{\mathcal{C}})_{\mathcal{C} \subseteq N}$ of (DLP).

Characterization of games with non-empty core

Definition (Balanced game)

A game is **balanced** iff for each balanced map λ we have $\sum_{\mathcal{C}\subseteq N, \mathcal{C}\neq\emptyset} \lambda(\mathcal{C})v(\mathcal{C}) \leqslant v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Application to Market Games

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}_+^m, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Theorem

Every Market Game is balanced

Coalition Structure

Definition (Coalition Structure)

A **coalition structure (CS)** is a partition of the grand coalition into coalitions.
$\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ where $\cup_{i \in \{1..k\}} \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.
We note $\mathscr{S}_N$ the set of all coalition structures over the set N.

ex: $\{\{1,3,4\}\{2,7\}\{5\}\{6,8\}\}$ is a coalition structure for $n = 8$ agents.

We will study three solution concepts: the **bargaining set**, the **nucleolus** and the **kernel**. They form the **"bargaining set family"** and we will see later why. In addition, the definition of each of these solution concepts uses a CS.

We start by defining a game with coalition structure, and see how we can define the core of such a game. Then, we'll start studying the bargaining set family.

Game with Coalition Structure

Definition (TU game)

A TU game is a pair (N,v) where N is a set of agents and where v is a valuation function.

Definition (Game with Coalition Structures)

A **TU-game with coalition structure** $(N,v,\mathcal{S})$ consists of a TU game (N,v) and a CS $\mathcal{S} \in \mathscr{S}_N$.

- We assume that the players agreed upon the formation of $\mathcal{S}$ and only the payoff distribution choice is left open.
- The CS may model affinities among agents, may be due to external causes (e.g. affinities based on locations).
- The agents may refer to the value of coalitions with agents outside their coalition (i.e. opportunities they would have outside of their coalition).
- (N,v) and $(N,v,\{N\})$ represent the same game.

Definition (*core* of a game (N,v))

The core of a TU game (N,v) is defined as
$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}$

The set of **feasible** payoff vectors for $(N,v,\mathcal{S})$ is
$X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \text{ for every } \mathcal{C} \in \mathcal{S}\ \ x(\mathcal{C}) \leqslant v(\mathcal{C})\}$.

Definition (Core of a game with CS)

The **core** $Core(N,v,\mathcal{S})$ of $(N,v,\mathcal{S})$ is defined by
$\{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leqslant v(\mathcal{C})) \text{ and } (\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geqslant v(\mathcal{C}))\}$

We have $Core(N,v,\{N\}) = Core(N,v)$.

The next theorems are due to Aumann and Drèze.

R.J. Aumann and J.H. Drèze. **Cooperative games with coalition structures**, *International Journal of Game Theory*, 1974

Definition (Superadditive cover)

The **superadditive cover** of (N, v) is the game $(N, \hat{v})$ defined by

$$\begin{cases} \hat{v}(\mathcal{C}) = \max_{\mathcal{P} \in \mathscr{S}_{\mathcal{C}}} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\} \forall \mathcal{C} \subseteq N \setminus \emptyset \\ \hat{v}(\emptyset) = 0 \end{cases}$$

- We have $\hat{v}(N) = \max_{\mathcal{P} \in \mathscr{S}_N} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\}$, i.e., $\hat{v}(N)$ is the maximum value that can be produced by N. We call it the **value of the optimal coalition structure**.
- The superadditive cover is a superadditive game (**why?**).

Theorem

Let $(N, v, \mathcal{S})$ be a game with coalition structure. Then

a) $Core(N, v, \mathcal{S}) \neq \emptyset$ iff $Core(N, \hat{v}) \neq \emptyset \wedge \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$

b) if $Core(N, v, \mathcal{S}) \neq \emptyset$, then $Core(N, v, \mathcal{S}) = Core(N, \hat{v})$

Definition (Substitutes)

Let (N,v) be a game and $(i,j) \in N^2$. Agents i and j are **substitutes** iff $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

A nice property of the core related to fairness:

Theorem

Let $(N,v,\mathcal{S})$ be a game with coalition structure,
let i and j be substitutes, and let $x \in Core(N,v,\mathcal{S})$.
If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.

Summary

- We introduced a stability solution concept: the core.
- we looked at examples:
  - individual games: some games have an empty core.
  - classes of games have a non-empty core: e.g. convex games and minimum cost spanning tree games.
- We look at a characterization of games with non-empty core: the Shapley Bondareva theorem, which relies on a result from linear programming.
- We Apply the Bondareva-Shapley to market games.
- We considered the core of games with coalition structures.

Coming next

- Bargaining sets.