# Lecture 5

# The Kernel

The Kernel is another stability concept that weakens the stability requirements of the core. It was first introduced by Davis and Maschler [3]. The definition of the kernel is based on the excess of a coalition. For the nucleolus, a positive excess was interpreted as an amount of complaint as by forming a coalition with positive excess, some payoff was lost. In the kernel, a positive excess is interpreted as a measure of threat: in the current payoff distribution, if some agents deviate by forming coalition with positive excess, they are able to increase their payoff by redistributing the excess between them. When any two agents in a coalition have similar threatening powers, the kernel considers that the payoff is stable. In the following, we will see two definitions of the kernel and we will see that it is guaranteed to be non-empty.

## 5.1 Definition of the Kernel

We recall that the *excess* related to coalition $\mathcal{C}$ for a payoff distribution x is defined as $e(\mathcal{C}, x) = v(\mathcal{C}) - x(\mathcal{C})$. We saw that a positive excess can be interpreted as an amount of complaint for a coalition. We can also interpret the excess as a potential to generate more utility. Let us consider that the agents are forming a CS $\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$, and let consider that the excess of a coalition $\mathcal{C} \notin \mathcal{S}$ is positive. Agent $i \in \mathcal{C}$ can view the positive excess as a measure of his strength: if she leaves its current coalition in $\mathcal{S}$ and forms coalition $\mathcal{C} \subseteq N$, she has the power to generate some surplus $e(\mathcal{C}, x)$. When two agents want to compare their strength, they can compare the maximum excess of a coalition that contains them and excludes the other agent, and the kernel is based on this idea.

**5.1.1.** DEFINITION. [Maximum surplus] For a TU game $(N, v)$, the *maximum surplus* $s_{k,l}(x)$ of *agent* $k$ *over agent* $l$ with respect to a payoff distribution $x$ is

$$s_{k,l}(x) = \max_{\mathcal{C} \subseteq N \mid k \in \mathcal{C},\, l \notin \mathcal{C}} e(\mathcal{C}, x).$$

For two agents $k$ and $l$, the *maximum surplus* $s_{k,l}$ of agent $k$ over agent $l$ with respect to $x$ is the *maximum excess* from a coalition that *includes* $k$ but does *exclude* $l$. This maximum surplus can be used by agent $k$ to show its strength over agent $l$: assuming it is positive and that agent $k$ can claim all of it, she can argue that she will be better off without agent $l$; hence she should be compensated with more utility for staying in the current coalition. When any two agents in a coalition have the same maximum surplus (except for a special case), the agents are said to be in equilibrium. A payoff distribution is in the Kernel when all agents are in equilibrium. The formal definitions follow:

**5.1.2.** DEFINITION. [kernel] Let $(N, v, \mathcal{S})$ be a TU game with coalition structure. The *kernel* is the set of imputations $x \in \mathcal{I}mp(\mathcal{S})$ such that for every coalition $\mathcal{C} \in \mathcal{S}$, if $(k, l) \in \mathcal{C}^2$, $k \neq l$, then we have either $s_{kl}(x) \geq s_{lk}(x)$ or $x_k = v(\{k\})$.

First, note that the definition of the kernel is for a particular coalition structure. The agents do not try to change the structure, but they argue about the payoff distribution. Another observation is that the kernel is a subset of the set of imputations(we could define a pre-kernel and show this solution does not satisfy individual rationality). Finally, the stability condition must be satisfied by any two agents that are in the same coalition. This condition may appear to surprising at first as one would expect an equality between the maximum surpluses of the two agents. The condition $s_{kl}(x) < s_{lk}(x)$ calls for a transfer of utility from $k$ to $l$ unless it is prevented by individual rationality, i.e., by the fact that $x_k = v(\{k\})$. Hence, it is possible that agent $l$ has a strictly greater maximum excess than $k$ when the payoff of $k$ is her minimum payoff $v(\{k\})$.

## 5.2 Another definition

As for the bargaining set and the nucleolus, the kernel can be defined using objections and counter objections. For the kernel, objections and counter-objections are exchanged between two *members of the same coalition* in $\mathcal{S}$. Objections and counter-objections take the form of *coalitions* only (unlike for the the bargaining set and the nucleolus for which a payoff distribution was part of an objection and counter-objection).

Let us consider a game with CS $\langle N, v, \mathcal{S} \rangle$ and let us consider a coalition $\mathcal{C} \in \mathcal{S}$ such that both agents $k$ and $l$ are in $\mathcal{C}$.

**Objection:** A coalition $P \subseteq N$ is an objection of $k$ against $l$ to $x$ iff $k \in P$, $l \notin P$ and $x_l > v(\{l\})$.

> "$P$ is a coalition that contains $k$, excludes $l$ and which sacrifices too much (or gains too little)."

An objection of agent $k$ against agent $l$ is simply a coalition $P$ that has some issues according to agent $k$. The only constraint is that agent $l$ is not a member of that coalition. A counter-objection of agent $l$ is then a coalition $Q$ that excludes agent $k$ which has a greater (or equal) excess than $e(P,x)$.

**Counter-objection:** A coalition $Q \subseteq N$ is a counter-objection to the objection $P$ of $k$ against $l$ at $x$ iff $l \in Q$, $k \notin Q$ and $e(Q,x) \geq e(P,x)$.

> "$k$'s demand is not justified: $Q$ is a coalition that contains $l$ and excludes $k$ and that sacrifices even more (or gains even less)."

Note that if the inequality is strict, we can view $Q$ as a new objection against agent $l$. Remember that the set of feasible payoff vectors for $(N, v, \mathcal{S})$ is $X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \text{for every } \mathcal{C} \in \mathcal{S} \ : \ x(\mathcal{C}) \leq v(\mathcal{C})\}$. We are now ready to define the kernel.

**5.2.1.** DEFINITION. [Kernel] Let $(N, v, \mathcal{S})$ be a TU game in coalition structure. The *kernel* is the set of imputations $x \in X_{(N,v,\mathcal{S})}$ s.t. for any coalition $\mathcal{C} \in \mathcal{S}$, for each objection $P$ of an agent $k \in \mathcal{C}$ over any other member $l \in \mathcal{C}$ to $x$, there is a counter-objection of $l$ to $P$.

This definition using objections and counter-objections is very intuitive compared to the first definition that was presented. It should be clear that both definitions coincide.

## 5.3 Properties of the kernel

The Kernel and the nucleolus are linked: the following result shows that the nucleolus is included in the kernel. As a consequence, this guarantees that the Kernel is non-empty. The second part of the result is that the kernel is included in the bargaining set. As a consequence, the nucleolus is also included in the bargaining set and the bargaining set is non-empty, a result we did not prove during the lecture on the bargaining set. A final observation is that the kernel can be seen as a refinement of the bargaining set.

**5.3.1.** THEOREM. *Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. Then we have:*

- *(i) $Nu(N, v, \mathcal{S}) \subseteq K(N, v, \mathcal{S})$*
- *(ii) $K(N, v, \mathcal{S}) \subseteq BS(N, v, \mathcal{S})$*

*Proof.*

Let us start by proving (i).

Let $x \notin K(N, v, \mathcal{S})$, we want to show that $x \notin Nu(N, v, \mathcal{S})$.

Since $x \notin K(N, v, \mathcal{S})$, there exists a coalition $\mathcal{C} \in CS$ and two members $k$ and $l$ of coalition $\mathcal{C}$ such that $s_{lk}(x) > s_{kl}(x)$ and $x_k > v(\{k\})$. Let $y$ be a payoff distribution corresponding to a transfer of utility $\epsilon > 0$ from $k$ to $l$:

$$y_i = \begin{cases} x_i \text{ if } i \neq k \text{ and } i \neq l \\ x_k - \epsilon \text{ if } i = k \\ x_l + \epsilon \text{ if } i = l \end{cases}$$

Since $x_k > v(\{k\})$ and $s_{lk}(x) > s_{kl}(x)$, we can choose $\epsilon > 0$ small enough such that

- $x_k - \epsilon > v(\{k\})$
- $s_{lk}(y) > s_{kl}(y)$

We want to show that $e(y)^{\blacktriangleright} \leq_{lex} e(x)^{\blacktriangleright}$. Note that for any coalition $S \subseteq N$ such that $e(S,x) \neq e(S,y)$ we have either:

- $k \in S$ and $l \notin S$ ($e(S,x) > e(S,y)$ since $e(S,y) = e(S,x) + \epsilon > e(S,x)$)
- $k \notin S$ and $l \in S$ ($e(S,x) < e(S,y)$ since $e(S,y) = e(S,x) - \epsilon < e(S,x)$)

Let $\{B_1(x), \dots, B_M(x)\}$ a partition of the set of all coalitions such that

- $(S,T) \in B_i(x)$ iff $e(S,x) = e(T,x)$. We denote by $e_i(x)$ the common value of the excess in $B_i(x)$, i.e. $e_i(x) = e(S,x)$ for all $S \in B_i(x)$.
- $e_1(x) > e_2(x) > \dots > e_M(x)$

In other words, $e(x)^{\blacktriangleright} = \langle \underbrace{e_1(x), \dots, e_1(x)}_{|B_1(x)| times}, \dots, \underbrace{e_M(x), \dots, e_M(x)}_{|B_M(x)| times} \rangle$.

Let $i^*$ be the minimal value of $i \in \{1, \dots, M\}$ such that there is a coalition $\mathcal{C} \in B_{i^*}(x)$ with $e(\mathcal{C}, x) \neq e(\mathcal{C}, y)$. For all $i < i^*$, we have $B_i(x) = B_i(y)$ and $e_i(x) = e_i(y)$. Since $s_{lk}(x) > s_{kl}(x)$, $B_{i^*}$ contains

- at least one coalition $S$ that contains $l$ but not $k$, for such coalition, we must have $e(S,x) > e(S,y)$
- no coalition that contains $k$ but not $l$.

*If* $B_{i^*}(x)$ contains either

- coalitions that contain both $k$ and $l$
- or coalitions that do not contain both $k$ and $l$

*Then*, for any such coalitions $S$, we have $e(S,x) = e(S,y)$, and it follows that $B_{i^*}(y) \subset B_{i^*}(x)$.
*Otherwise*, we have $e_{i^*}(y) < e_{i^*}(x)$.

In both cases, we have $e(y)$ is lexicographically less than $e(x)$, and hence $y$ is not in the nucleolus of the game $(N, v, \mathcal{S})$. ✔

We now turn to proving (ii).

Let $(N, v, \mathcal{S})$ a TU game with coalition structure. Let $x \in K(N, v, \mathcal{S})$. We want to prove that $x \in BS(N, v, \mathcal{S})$. To do so, we need to show that for any objection $(P, y)$ from any player $i$ against any player $j$ at $x$, there is a counter objection $(Q, z)$ to $(P, y)$. For the bargaining set, An *objection of* $i$ *against* $j$ is a pair $(P, y)$ where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where $p$ is the size of $P$
- $y(P) \leq v(P)$ (y is a feasible payoff for members of $P$)
- $\forall k \in P$, $y_k \geq x_k$ and $y_i > x_i$

An *counter-objection to* $(P, y)$ is a pair $(Q, z)$ where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.
- $z \in \mathbb{R}^q$ where $q$ is the size of $Q$
- $z(Q) \leq v(Q)$ (z is a feasible payoff for members of $Q$)
- $\forall k \in Q$, $z_k \geq x_k$
- $\forall k \in Q \cap P$ $z_k \geq y_k$

Let $(P, y)$ be an objection of player $i$ against player $j$ to $x$. $i \in P$, $j \notin P$, $y(P) \leq v(P)$ and $y(P) > x(P)$. We choose $y(P) = v(P)$.

- $x_j = v(\{j\})$*:* Then $(\{j\}, v(\{j\}))$ is a counter objection to $(P, y)$. ✔
- $x_j > v(\{j\})$*:* Since $x \in K(N, v, \mathcal{S})$ we have $s_{ji}(x) \geq s_{ij}(x) \geq v(P) - x(P) \geq y(P) - x(P)$ since $i \in P$, $j \notin P$.

  Let $Q \subseteq N$ such that $j \in Q$, $i \notin Q$ and $s_{ji}(x) = v(Q) - x(Q)$.

  We have $v(Q) - x(Q) \geq y(P) - x(P)$. Then, we have

$$\begin{array}{rcl} v(Q) & \geq & y(P) + x(Q) - x(P) \\ & \geq & y(P \cap Q) + y(P \setminus Q) + x(Q \setminus P) - x(P \setminus Q) \\ & > & y(P \cap Q) + x(Q \setminus P) \text{ since } i \in P \setminus Q,\, y(P \setminus Q) > x(P \setminus Q) \end{array}$$

  Let us define z as follows $\begin{cases} x_k \text{ if } k \in Q \setminus P \\ y_k \text{ if } k \in Q \cap P \end{cases}$

  $(Q, z)$ is a counter-objection to $(P, y)$. ✔

Finally $x \in BS(N, v, \mathcal{S})$.

□

These properties allow us to conclude that when the set of imputation is non-empty, both the kernel and the bargaining set are non-empty.

**5.3.2.** THEOREM. *Let* $\langle N, v, \mathcal{S} \rangle$ *a game with coalition structure such that the set of imputations* $\mathcal{I}mp \neq \emptyset$*. The kernel* $K(N, v, \mathcal{S})$ *and the bargaining set* $BS(N, v, \mathcal{S})$ *are non-empty.*

## 5.4 Computational Issues

One method for computing the Kernel is the Stearns method [7]. The idea is to build a sequence of side-payments between agents to decrease the difference of surpluses between the agents. At each step of the sequence, the agents with the largest maximum surplus difference exchange utility so as to decrease their surplus: the agent with smaller surplus makes a payment to an agent with higher surplus so as to decrease their surplus difference. After each side-payment, the maximum surplus over all agents decreases. In the limit, the process converges to an element in the Kernel. Computing an element in the Kernel may require an infinite number of steps as the side payments can become arbitrarily small, and the use of the $\epsilon$-Kernel can alleviate this issue. A criteria to terminate Stearns method is proposed in [6], and we present the corresponding algorithm in Algorithm 1.

**Algorithm 1:** Transfer scheme to converge to a $\epsilon$-Kernel-stable payoff distribution for the CS $S$

```
compute-ϵ-Kernel(ϵ, S)
repeat
    for each coalition 𝒞 ∈ S do
        for each member i ∈ 𝒞 do
            for each member j ∈ 𝒞, j ≠ i, do    // compute the surplus
            for two members of a coalition in S
                s_ij ← max_{R∈𝒞|(i∈R, j∉R)} v(R) − x(R)
    δ ← max_{(i,j)∈N²} |s_ij − s_ji|;
    (i⋆, j⋆) ← argmax_{(i,j)∈N²} s_ij − s_ji;
    if (x_j⋆ − v({j}) < δ/2) then    // payment should be individually
    rational
        d ← x_j⋆ − v({j⋆});
    else
        d ← δ/2;
    x_i⋆ ← x_i⋆ + d;
    x_j⋆ ← x_j⋆ − d;
until δ/v(S) ≤ ϵ;
```

Computing a Kernel distribution is of exponential complexity. In Algorithm 1, computing the surpluses is expensive, as we need to search through all coalitions that contains a particular agent and does not contain another agent. Note that when a side-payment is performed, it is necessary to recompute the maximum surpluses. The derivation of the complexity of the Stearns method to compute a payoff in the $\epsilon$-Kernel can be found in [4, 6], and the complexity for one side-payment is $O(n \cdot 2^n)$. Of course, the number of side-payments depends on the precision $\epsilon$ and on the initial payoff distribution. They derive an upper bound for the number of iterations: converging to an element of the $\epsilon$-Kernel requires $n \log_2(\frac{\delta_0}{\epsilon \cdot v(S)})$, where $\delta_0$ is the maximum surplus difference in the initial payoff distribution. To derive a polynomial algorithm, the number of coalitions must be bounded. The solution used in [4, 6] is to only consider coalitions whose size is bounded in the interval $K_1, K_2$. The complexity of the truncated algorithm is $O(n^2 \cdot n_{coalitions})$ where $n_{coalitions}$ is the number of coalitions with a size between $K_1$ and $K_2$, which is a polynomial of order $K_2$.

## 5.5 Fuzzy Kernel

In order to take into account the uncertainty in the knowledge of the utility function, a fuzzy version of stability concept can be used. Blankenburg et al. consider a coalition to be Kernel-stable with a degree of certainty [2]. This work also presents a side-payment scheme and shows that the complexity is similar to the crisp Kernel, and the idea from [4] can be used for ensuring a polynomial coalition formation algorithm. This approach assumes a linear relationship of the membership and coalition values.

Fuzzy coalitions can also allow agents to be members of multiple coalitions at the same time, with possibly different degrees of involvement [1]. It can be mutually beneficial for an agent to be in two different coalitions. It may be beneficial for the agent to be in both coalition. In addition, the two coalitions may need the competence of the same agent, though the coalitions do not have any incentive to merge as they may not have anything to do with each other. This solution may allow to form coalitions that involve only agents that need to work together. In the previous example, without the possibility of being member of multiple coalitions, the two coalitions should merge to benefit from the agent participation, and agents that would not need to be in the same coalition are forced to be in the same coalition. In [1], the degree of involvement of an agent in different coalition is a function of the risk involved in being in that coalition. The risk is quantified using financial risk measures. This work presents a definition of the Kernel based on partial membership in coalitions and introduces a coalition formation protocol that runs in polynomial time.

# Bibliography


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