Lecture 4

The nucleolus

The nucleolus is based on the notion of excess and has been introduced by Schmeidler [3]. The excess measures the amount of "complaints" of a coalition for a payoff distribution. We already mentionned the excess and gave a definition of the core using the excess. We now recall the definition.

4.0.2. DEFINITION. [Excess] Let (N, v) be a TU game, $\mathcal{C} \subseteq N$ be a coalition, and x be a payoff distribution over N. The *excess* $e(\mathcal{C}, x)$ of coalition $\mathcal{C}$ at x is the quantity $e(\mathcal{C}, x) = v(\mathcal{C}) - x(\mathcal{C})$.

When a coalition has a positive excess, some utility is not provided to the coalition's members, and the members complain about this. for a payoff distribution in the core, there cannot be any complaint. The goal of the nucleolus is to reduce the amount of complaint, and we are now going to see in what sense it is reduced.

4.1 Motivations and Definitions

Let us consider the game in Table 4.1 and we want to compare two payoff distributions x and y. A priori, it is not clear which payoff should be preferred. To compare two vectors of complaints, we can use the lexicographical order[1].

4.1.1. DEFINITION. [Lexicographical ordering] Let $(x, y) \in \mathbb{R}^m$, $x \geq_{lex} y$. We say that x is greater or equal to y in the *lexicographical ordering*, and we note $x \geq_{lex} y$, when $\begin{cases} x = y \text{ or} \\ \exists t,\ 1 \leq t \leq m \text{ such that } \forall i\ 1 \leq i < t\ x_i = y_i \text{ and } x_t > y_t \end{cases}$

For example, we have $\langle 1, 1, 0, -1, -2, -3, -3 \rangle \geq_{lex} \langle 1, 0, 0, 0, -2, -3, -3 \rangle$. Let l be a sequence of m reals. We denote by $l^{\blacktriangleright}$ the *reordering* of l in *decreasing* order. In the example, $e(x) = \langle -3, -3, -2, -1, 1, 1, 0 \rangle$ and then $e(x)^{\blacktriangleright} = \langle 1, 1, 0, -1, -2, -3, -3 \rangle$.

[1] the order used for the names in a phonebook or words in a dictionary

Using the lexicographical ordering, we are now ready to compare the payoff distributions x and y and we note that y is better than x since $e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright}$: there is a smaller amount of complaints in y than in x given the lexicographical ordering.

$$\begin{gathered} N = \{1,2,3\}, \\ v(\{i\}) = 0 \text{ for } i \in \{1,2,3\} \\ v(\{1,2\}) = 5,\, v(\{1,3\}) = 6,\, v(\{2,3\}) = 6 \\ v(N) = 8 \end{gathered}$$

Let us consider two payoff vectors $x = \langle 3,3,2\rangle$ and $y = \langle 2,3,3\rangle$.

$x = \langle 3,3,2\rangle$

coalition $\mathcal{C}$	$e(\mathcal{C}, x)$
$\{1\}$	-3
$\{2\}$	-3
$\{3\}$	-2
$\{1,2\}$	-1
$\{1,3\}$	1
$\{2,3\}$	1
$\{1,2,3\}$	0

$y = \langle 2,3,3\rangle$

coalition $\mathcal{C}$	$e(\mathcal{C}, y)$
$\{1\}$	-2
$\{2\}$	-3
$\{3\}$	-3
$\{1,2\}$	0
$\{1,3\}$	1
$\{2,3\}$	0
$\{1,2,3\}$	0

Table 4.1: A motivating example for the nucleolus

The first entry of $e(x)^{\blacktriangleright}$ is the maximum excess: the agents involved in the corresponding coalition have the largest incentive to leave their current coalition and form a new one. Put another way, the agents involved in that coalition have the most valid complaint. If one selects the payoff distribution minimizing the most valid complaint, there can be a large number of candidates. To refine the selection, among those payoff distribution with the smallest largest complaint, one can look at minimizing the second largest complaint. A payoff distribution is in the nucleolus when it yields the "least problematic" sequence of complaints according to the lexicographical ordering. The nucleolus tries to minimise the possible complaints (or minimise the incentives to create a new coalition) over all possible payoff distributions.

4.1.2. DEFINITION. Let $\mathcal{I}mp$ be the set of all imputations. The *nucleolus* $Nu(N, v)$ is the set

$$Nu(N, v) = \{x \in \mathcal{I}mp \mid \forall y \in \mathcal{I}mp \;\; e(y)^{\blacktriangleright} \geq_{lex} e(x)^{\blacktriangleright}\}.$$

Intuitively, this definition makes sense. It is another solution concept that focuses on stability, and it relaxes the stability requirement of the core: the core requires no complaint at all. The nucleolus may allow for some complaints, but tries to minimize them.

4.2 Some properties of the nucleolus

We now provide some properties of the nucleolus. First, we consider the relationship with the core of a game. The following theorem guarantees that the nucleolus of a game is always included in the core. A payoff distribution in the core does not have any complaint. If there are more than one payoff in the core, it is possible to use the excess and the lexicographical ordering to rank the payoff according to the satisfaction of the agents.

4.2.1. THEOREM. *Let* (N, v) *be a TU game with a non-empty core. Then* $Nu(N, v) \subseteq core(N, v)$

Proof. This will be an assignment of Homework 2. □

if the core of a game in non-empty, one can use the nucleolus to discriminate between different core members. Now, we turn to the important issue of the existence of payoff distributions in the nucleolus. The following theorem guarantees that the nucleolus is non-empty in most games:

4.2.2. THEOREM. *Let* (N, v) *be a TU game and* $\mathcal{I}mp$ *is the set of imputations. If* $\mathcal{I}mp \neq \emptyset$*, then the nucleolus* $Nu(N, v)$ *is* non-empty.

This property ensures that the agents will always find an agreement if they use this method, which is a great property. The assumption that the set of imputation is a very mild assumption: if the game does not have any efficient and individually rational payoff distributions, it is not such an interesting game. The following theorem shows that in addition to always exist, the nucleolus is in fact unique.

4.2.3. THEOREM. *The nucleolus has* at most one *element.*

The *proofs* of both theorems are a bit involved, and are included in the next section.

The nucleolus is guaranteed to be non-empty and it is unique. These are two important property in favour of the nucleolus. Moreover, when the core is non-empty, the nucleolus is in the core.

One drawback, however, is that the nucleolus is difficult to compute. It can be computed using a sequence of linear programs of decreasing dimensions. The size of each of these groups is, however, exponential. In some special cases, the nucleolus can be computed in polynomial time [2, 1], but in the general case, computing the nucleolus is not guaranteed to be polynomial. Only a few papers in the multiagent systems community have used the nucleolus, e.g., [4].

4.3 Proofs of the main theorem

The results that the nucleolus is a unique payoff distribution is quite an important result. We will also use this result to show that some other solution concepts are non-empty. For this reason, it is worth stating one proof of this theorem, although it is quite a technical result. To prove the theorem, one needs to use results from analysis. In the following, we informally recall some definitions and theorems that will be used in the proofs.

4.3.1 Elements of Analysis

Let $E = \mathbb{R}^m$ and $X \subseteq E$. $||.||$ denote a distance in E, e.g., the euclidean distance.

We consider functions of the form $u : \mathbb{N} \to \mathbb{R}^m$. Another viewpoint on u is an infinite *sequence* of elements indexed by natural numbers $(u_0, u_1, \ldots, u_k, \ldots)$ where $u_i \in X$. We recall some definitions:

- *convergent sequence:* A sequence (u_t) converges to $l \in \mathbb{R}^m$ iff for all $\epsilon > 0$, $\exists T \in \mathbb{N}$ s.t. $\forall t \geq T$, $||u_t - l|| \leq \epsilon$.
- *extracted sequence:* Let (u_t) be an infinite sequence and $f : \mathbb{N} \to \mathbb{N}$ be a monotonically increasing function. The sequence v is extracted from u iff
  $v = u \circ f$, i.e., $v_t = u_{f(t)}$.
- *closed set:* a set X is closed if and only if it contains all of its limit points. In other words, for all converging sequences $(x_0, x_1 \ldots)$ of elements in X, the limit of the sequence has to be in X as well.

  For example, if $X = (0, 1]$, $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots)$ is a converging sequence. However, 0 is not in X, and hence, X is not closed.

  One way to think about a closed set is by saying “A closed set contains its borders”.
- *bounded set:* A subset $X \subseteq \mathbb{R}^m$ is *bounded* if it is contained in a ball of finite radius, i.e. $\exists c \in \mathbb{R}^m$ and $\exists r \in \mathbb{R}^+$ s.t. $\forall x \in X$ $||x - c|| \leq r$.
- *compact set:* A subset $X \subseteq \mathbb{R}^m$ is a *compact* set iff from all sequences in X, we can extract a convergent sequence in X.

➩ A set is *compact* set of $\mathbb{R}^m$ iff it is *closed* and *bounded*.

- *convex set:* A set X is convex iff $\forall (x, y) \in X^2$, $\forall \alpha \in [0, 1]$, $\alpha x + (1 - \alpha) y \in X$ (i.e. all points in a line from x to y is contained in X).
- *continuous function:* Let $X \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$.

  f is *continuous at* $x_0 \in X$ iff

$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in X \quad ||x - x_0|| < \delta \Rightarrow ||f(x) - f(x_0)|| < \epsilon$. In other words, the function f does not contain any jump.

We now state some theorems. Let $X \subseteq \mathbb{R}^n$.

Thm A_1 If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $X \subseteq E$ is a non-empty compact subset of $\mathbb{R}^n$, then $f(X)$ is a non-empty compact subset of $\mathbb{R}^m$.

Thm A_2 Extreme value theorem: Let X be a non-empty compact subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$ a *continuous* function. Then f is bounded and it reaches its supremum.

Thm A_3 Let X be a non-empty compact subset of $\mathbb{R}^n$. $f : X \to \mathbb{R}$ is continuous iff for every closed subset $B \subseteq \mathbb{R}$, the set $f^{-1}(B)$ is compact.

4.3.2 Proofs

Let us assume that the following two theorems are valid. We will prove them later.

4.3.1. THEOREM. *Assume we have a TU game (N, v), and consider its set $\mathcal{I}mp$. If $\mathcal{I}mp \neq \emptyset$, then set $B = \{e(x)^{\blacktriangleright} \mid x \in \mathcal{I}mp\}$ is a non-empty compact subset of* $\mathbb{R}^{2^{|N|}}$

4.3.2. THEOREM. *Let A be a non-empty compact subset of* $\mathbb{R}^m$.
$\{x \in A \mid \forall y \in A \; x \leq_{lex} y\}$ *is non-empty.*

We can use these theorems to prove that the nucleolus is non-empty.
Proof.
Let us take a TU game (N, v) and let us assume $\mathcal{I}mp \neq \emptyset$. From theorem 4.3.1, we know that $B = \{e(x)^{\blacktriangleright} \mid x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$.
Now let us apply the result of theorem 4.3.2 to B. We then have that $\{e(x)^{\blacktriangleright} \mid (x \in \mathcal{I}mp) \;\wedge\; (\forall y \in \mathcal{I}mp \; e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright})\}$ is non-empty. From this, it follows that: $Nu(N, v) = \{x \in \mathcal{I}mp \mid \forall y \in \mathcal{I}mp \; e(y)^{\blacktriangleright} \geq_{lex} e(x)^{\blacktriangleright}\} \neq \emptyset$. ✔ □

In the following, we need to prove both theorems 4.3.1 and 4.3.2. We start by the first one.
Proof. Let (N, v) be a TU game and consider its set $\mathcal{I}mp$. Let us assume that $\mathcal{I}mp \neq \emptyset$. We want to prove that $B = \{e(x)^{\blacktriangleright} \mid x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$.
First, let us prove that $\mathcal{I}mp$ is a non-empty compact subset of $\mathbb{R}^{|N|}$.

- $\mathcal{I}mp$ non-empty by assumption.
- To see that $\mathcal{I}mp$ is bounded, we need to show that for all i, x_i is bounded by some constant (independent of x). We have $v(\{i\}) \leq x_i$ by individual rationality and $x(N) = v(N)$ by efficiency. Then $x_i + \sum_{j=1, j\neq i}^{n} v(\{j\}) \leq v(N)$, hence $x_i \leq v(N) - \sum_{j=1, j\neq i}^{n} v(\{j\})$.

- $\mathcal{I}mp$ is closed (this is trivial as the boundaries of $\mathcal{I}mp$ are members of $\mathcal{I}mp$).

- $\mathcal{I}mp$ non-empty, closed and bounded. By definition, it is a non-empty compact subset of $\mathbb{R}^{|N|}$. ✔

$e()^{\blacktriangleright}$ is a continuous function and $\mathcal{I}mp$ is a non-empty and compact subset of $\mathbb{R}^{2^{|N|}}$. Using thm A$_1$, we can conclude that $e(\mathcal{I}mp)^{\blacktriangleright} = \{e(x)^{\blacktriangleright} | x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$, which concludes the proof of theorem 4.3.1. ✔ □

We now turn to the proof of theorem 4.3.2.

Proof. For a non-empty compact subset A of $\mathbb{R}^m$, we need to prove that the set $\{x \in A \mid \forall y \in A \;\; x \leq_{lex} y\}$ is non-empty.

First, let $\pi_i : \mathbb{R}^m \to \mathbb{R}$ be the projection function such that $\pi_i(x_1, \ldots, x_m) = x_i$. Then, let us define the following sets:

$$\begin{cases} A_0 = A \\ A_{i+1} = \underset{x \in A_i}{\operatorname{argmin}}\, \pi_{i+1}(x),\, i \in \{0, 1, \ldots, m-1\} \end{cases}$$

Let us assume that we want to find the minimum of A according to the lexicographic order (say you want to find the last words in a text according to the order in a dictionary). You first take the entire set, then you select only the vectors that have the smallest first entries (set A_1). Then, from this set, you select the vectors that have the smallest second entry (forming the set A_2) and you repeat the process until you reach m steps. At the end, you have $A_m = \{x \in A \mid \forall y \in A \;\; x \leq_{lex} y\}$.

We want to prove by induction that each A_i is non-empty compact subset of $\mathbb{R}^m$ for $i \in \{1, \ldots, m\}$. First, we need to show non-emptiness:

- $A_0 = A$ is non-empty compact of $\mathbb{R}^m$ by hypothesis ✔.

- Let us assume that A_i is a non-empty compact subset of $\mathbb{R}^m$ and let us prove that A_{i+1} is a non-empty compact subset of $\mathbb{R}^m$.

  π_{i+1} is a continuous function and A_i is a non-empty compact subset of $\mathbb{R}^m$. Using the extreme value theorem A_2, $\min_{x \in A_i} \pi_{i+1}(x)$ exists and it is reached in A_i, hence $\operatorname{argmin}_{x \in A_i} \pi_{i+1}(x)$ is non-empty. ✔

Now, we need to show each A_i is compact. We note by $\pi_i^{-1} : \mathbb{R} \to \mathbb{R}^m$ the inverse of π_i. Let $\alpha \in \mathbb{R}$, $\pi_i^{-1}(\alpha)$ is the set of all vectors $\langle x_1, \ldots, x_{i-1}, \alpha, x_{i+1}, \ldots, x_m \rangle$ such that $x_j \in \mathbb{R}$, $j \in \{1, \ldots, m\}$, $j \neq i$. We can rewrite A_{i+1} as:

$$A_{i+1} = \pi_{i+1}^{-1}\left(\min_{x \in A_i} \pi_{i+1}(x)\right) \bigcap A_i$$

$$A_{i+1} = \underbrace{\underbrace{\pi_{i+1}^{-1}\left(\left\{\underbrace{\min_{x\in A_i} \pi_{i+1}(x)}_{\text{closed}}\right\}\right)}_{\text{According to Thm A}_3\text{, it is a compact subset of } \mathbb{R}^m} \bigcap A_i}_{\substack{\text{is a compact subset of } \mathbb{R}^m \text{ since} \\ \text{the intersection of two closed sets is closed and in } \mathbb{R}^m, \\ \text{and a closed subset of a compact subset of } \mathbb{R}^m \\ \text{is a compact subset of } \mathbb{R}^m\ \checkmark}}$$

Hence A_{i+1} is a non-empty compact subset of $\mathbb{R}^m$ and the proof is complete. □

For a TU game (N, v) the nucleolus $Nu(N, v)$ is non-empty when $\mathcal{I}mp \neq \emptyset$, which is a great property as agents will always find an agreement. But there is more! No we need to prove that there is *one* agreement which is stable according to the nucleolus. To prove the unicity of the nucleolus, we again need to prove two results.

4.3.3. THEOREM. *Let A be a non-empty convex subset of $\mathbb{R}^m$. Then the set $\{x \in A \mid \forall y \in A\ \ x^{\blacktriangleright} \leq_{lex} y^{\blacktriangleright}\}$ has at most one element.*

Proof.

Let A be a non-empty convex subset of $\mathbb{R}^m$, and $M^{in} = \{x \in A \mid \forall y \in A\ \ x^{\blacktriangleright} \leq_{lex} y^{\blacktriangleright}\}$. We now prove that $|M^{in}| \leq 1$.

Towards a contradiction, let us assume M^{in} has at least two elements x and y, $x \neq y$. By definition of M^{in}, we must have $x^{\blacktriangleright} = y^{\blacktriangleright}$.

Let $\alpha \in (0, 1)$ and σ be a permutation of $\{1, \ldots, m\}$ such that $(\alpha x + (1-\alpha)y)^{\blacktriangleright} = \sigma(\alpha x + (1-\alpha)y) = \alpha\sigma(x) + (1-\alpha)\sigma(y)$. Let us show by contradiction that $\sigma(x) = x^{\blacktriangleright}$ and $\sigma(y) = y^{\blacktriangleright}$.

Let us assume that either $\sigma(x) <_{lex} x^{\blacktriangleright}$ or $\sigma(y) <_{lex} y^{\blacktriangleright}$, it follows that

$$\alpha\sigma(x) + (1-\alpha)\sigma(y) <_{lex} \alpha x^{\blacktriangleright} + (1-\alpha)y^{\blacktriangleright} = x^{\blacktriangleright}.$$

Since A is convex, $\alpha x + (1-\alpha)y \in A$. But this is a contradiction because by definition of M^{in}, $\alpha x + (1-\alpha)y \in A$ cannot be strictly smaller than $x^{\blacktriangleright}$, $y^{\blacktriangleright}$ in A. This proves $\sigma(x) = x^{\blacktriangleright}$ and $\sigma(y) = y^{\blacktriangleright}$.

Since $x^{\blacktriangleright} = y^{\blacktriangleright}$, we have $\sigma(x) = \sigma(y)$, hence $x = y$. This contradicts the fact that $x \neq y$. Hence, M^{in} cannot have at least two elements, and $|M^{in}| \leq 1$. □

4.3.4. THEOREM. *Let (N, v) be a TU game such that $\mathcal{I}mp \neq \emptyset$.*

(i) $\mathcal{I}mp$ is a non-empty and convex subset of $\mathbb{R}^{|N|}$

(ii) $\{e(x) \mid x \in \mathcal{I}mp\}$ *is a non-empty convex subset of* $\mathbb{R}^{2^{|N|}}$

Proof. Let (N, v) be a TU game such that $\mathcal{I}mp \neq \emptyset$ (in case $\mathcal{I}mp = \emptyset$, $\mathcal{I}mp$ is trivially convex). Let $(x, y) \in \mathcal{I}mp^2$, $\alpha \in [0, 1]$. Let us prove $\mathcal{I}mp$ is convex by showing that $u = \alpha x + (1-\alpha)y \in \mathcal{I}mp$, i.e., that u is individually rational and efficient.

Individual rationality: Since x and y are individually rational, for all agents i, $u_i = \alpha x_i + (1-\alpha)y_i \geq \alpha v(\{i\}) + (1-\alpha)v(\{i\}) = v(\{i\})$. Hence u is individually rational.

Efficiency: Since x and y are efficient, we have $\sum_{i \in N} u_i = \sum_{i \in N} \alpha x_i + (1-\alpha)y_i = \alpha \sum_{i \in N} x_i + (1-\alpha) \sum_{i \in N} y_i = \alpha v(N) + (1-\alpha)v(N)$. Hence we have $\sum_{i \in N} u_i = v(N)$ and u is efficient.

Thus, $u \in \mathcal{I}mp$. ✔

Let (N, v) be a TU game and $\mathcal{I}mp$ its set of imputations. We need to show that the set $\{e(z) \mid z \in \mathcal{I}mp\}$ is a non-empty convex subset of $\mathbb{R}^m$. Remember that $e(z)$ is the sequence of excesses of all coalitions for the payoff distribution z in a given order of the coalitions (i.e., it is a vector of size $2^{|N|}$). Since $\mathcal{I}mp$ is non-empty, $\{e(z) \mid z \in \mathcal{I}mp\}$ is trivially non non-empty. Then we just need to prove it is convex. Let $(x, y) \in \mathcal{I}mp^2$, $\alpha \in [0, 1]$, and $\mathcal{C} \subseteq N$. We consider the vector $\alpha e(x) + (1-\alpha)e(y)$ and we look at the entry corresponding to coalition $\mathcal{C}$.

$$\begin{aligned}
(\alpha e(x) + (1-\alpha)e(y))_{\mathcal{C}} &= \alpha e(\mathcal{C}, x) + (1-\alpha)e(\mathcal{C}, y) \\
&= \alpha(v(\mathcal{C}) - x(\mathcal{C})) + (1-\alpha)(v(\mathcal{C}) - y(\mathcal{C})) \\
&= v(\mathcal{C}) - (\alpha x(\mathcal{C}) + (1-\alpha)y(\mathcal{C})) \\
&= v(\mathcal{C}) - ([\alpha x + (1-\alpha)y](\mathcal{C})) \\
&= e(\alpha x + (1-\alpha)y, \mathcal{C})
\end{aligned}$$

Since the previous equality is valid for all $\mathcal{C} \subseteq N$, both sequences are equal and we can write

$$\alpha e(x) + (1-\alpha)e(y) = e(\alpha x + (1-\alpha)y).$$

Since $\mathcal{I}mp$ is convex, $\alpha x + (1-\alpha)y \in \mathcal{I}mp$, it follows that $e(\alpha x + (1-\alpha)y) \in \{e(z) \mid z \in \mathcal{I}mp\}$. Hence, $\{e(z) \mid z \in \mathcal{I}mp\}$ is convex. □

Now we finally are ready to prove that the nucleolus has at most one element.

Proof. Let (N, v) be a TU game, and $\mathcal{I}mp$ its set of imputations.

According to Theorem 4.3.4(ii), we have that $\{e(x) \mid x \in \mathcal{I}mp\}$ is a non-empty convex subset of $\mathbb{R}^{2^{|N|}}$. Applying theorem4.3.3 with $A = \{e(x) \mid x \in \mathcal{I}mp\}$ we obtain the following statement:

$B = \{e(x) \mid x \in \mathcal{I}mp \wedge \forall y \in \mathcal{I}mp\ \ e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright}\}$ has at most one element.

B is the image of the nucleolus under the function e. We need to make sure that an $e(x)$ corresponds to at most one element in $\mathcal{I}mp$. This is true since for $(x, y) \in \mathcal{I}mp^2$, we have $x \neq y \Rightarrow e(x) \neq e(y)$.

Hence $Nu(N, v) = \{x \mid x \in \mathcal{I}mp \wedge \forall y \in \mathcal{I}mp\ \ e(x)^{\blacktriangleright} \leq_{lex} e(y)^{\blacktriangleright}\}$ has at most one element! □

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