

Introduction to Tournaments

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- Voting

Input: Preference of agents over a set of candidates or outcomes

Output: one candidate or outcome (or a set)

- Tournament

Input: Binary relation between outcomes or candidates

Output: One candidate or outcome (or a set)

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Input: Binary relation between outcomes or candidates

Output: One candidate or outcome (or a set)

When no ties are allowed between any two alternatives.

Either x beats y or y beats x .

which are the best outcomes?

Notations

- X is a *finite* set of alternatives.
- T is a relation on X , i.e, $T \subset X^2$.
- notation: $(x, y) \in T \Leftrightarrow xTy \Leftrightarrow x \rightarrow y \Leftrightarrow x$ “beats” y
- $\mathcal{T}(X)$ is the set of tournaments on X
- $T^+(x) = \{y \in X \mid xTy\}$: successors of x .
- $T^-(x) = \{y \in X \mid yTx\}$: predecessors of x .
- $s(x) = \#T^+(x)$ is the **Copeland** score of x .

Definition (Tournament)

The relation T is a **tournament** iff

- 1 $\forall x \in X (x, x) \notin T$
- 2 $\forall (x, y) \in X^2 x \neq y \Rightarrow [((x, y) \in T) \vee ((y, x) \in T)]$
- 3 $\forall (x, y) \in X^2 (x, y) \in T \Rightarrow (y, x) \notin T$.

A **tournament** is a complete and asymmetric binary relation

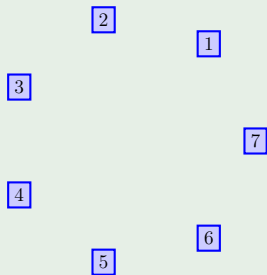
Majority voting and tournament:

- I finite set of individuals. The **preference** of an individual i is represented by a complete order P_i defined on X .
- The outcome of majority voting is the binary relation $M(P)$ on X such that $\forall (x, y) \in X, xM(P)y \Leftrightarrow \#\{i \in I | xP_iy\} > \#\{i \in I | yP_ix\}$
If initial preferences are strict and number of individual is odd,
 $M(P)$ is a tournament.

Example (cyclone of order n)

Z_n set of integers modulo n .

$$xC_ny \Leftrightarrow y - x \in \left\{1, \dots, \frac{n-1}{2}\right\}$$

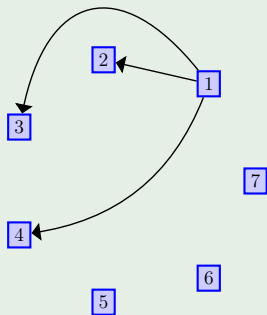


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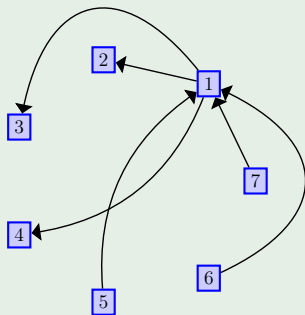
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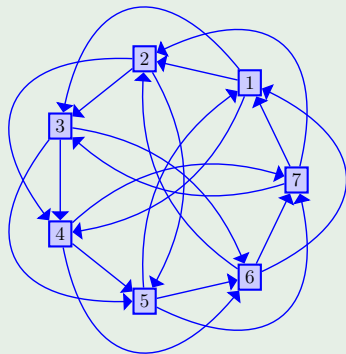
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Definition (isomorphism)

Let X and Y be two sets, $T \in \mathcal{T}(X)$, $U \in \mathcal{T}(Y)$ two tournaments on X and Y .

A mapping $\phi : X \rightarrow Y$ is a **tournament isomorphism** iff

- ϕ is a bijection
- $\forall (x, y) \in X^2, xTx' \Leftrightarrow \phi(x)U\phi(x')$

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On a set X of cardinal n , there are $2^{\frac{n \cdot (n-1)}{2}}$ tournaments, but many of them are isomorphic.

n	$2^{\frac{n(n-1)}{2}}$	number of non-isomorphic tournaments
8	268,435,456	6,880
10	35,184,372,088,832	9,733,056

Outline

- 1 Introduction: Reasoning about pairwise competition
- 2 Desirable properties of solution concepts
- 3 Solution based on scoring and Ranking
- 4 Solutions based on Covering
- 5 Solution based on Game Theory
- 6 Contestation Process
- 7 Knockout tournaments
- 8 Notes on the size of the choice set

Condorcet principle

Definition (Condorcet winners)

Let $T \in \mathcal{T}(X)$. The set of Condorcet winners of T is

$$\mathcal{C}ondorcet(T) = \{x \in X \mid \forall y \in X, y \neq x \Rightarrow xTy\}$$

Property

Either $\mathcal{C}ondorcet(T) = \emptyset$ or $\mathcal{C}ondorcet(T)$ is a singleton.

Definition (Tournament solution)

A tournament solution \mathcal{S} associates to any tournament $\mathcal{T}(X)$ a subset $\mathcal{S}(T) \subset X$ and satisfies

- $\forall T \in \mathcal{T}(X), \mathcal{S}(T) \neq \emptyset$
- For any tournament isomorphism $\phi, \phi \circ \mathcal{S} = \mathcal{S} \circ \phi$ (anonymity)
- $\forall T \in \mathcal{T}(X), \text{Condorcet}(T) \neq \emptyset \Rightarrow \mathcal{S}(T) = \text{Condorcet}(T)$

For $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ tournament solutions.

- $\mathcal{S}_1 \circ \mathcal{S}_2(T) = \mathcal{S}_1(T / \mathcal{S}_2(T)) = \mathcal{S}_1(\mathcal{S}_2(T))$
- $\mathcal{S}^1 = \mathcal{S}, \mathcal{S}^{k+1} = \mathcal{S} \circ \mathcal{S}^k, \mathcal{S}^\infty = \lim_{k \rightarrow \infty} \mathcal{S}^k$
- solutions may be finer/more selective:
$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \forall T \in \mathcal{T}(X) \mathcal{S}_1(T) \subset \mathcal{S}_2(T) \text{ than } \mathcal{S}_2.$$
- solutions may be different:
$$\mathcal{S}_1 \not\subset \mathcal{S}_2 \Leftrightarrow \exists T \in \mathcal{T} \mid \mathcal{S}_1(T) \cap \mathcal{S}_2(T) = \emptyset$$
- solution may have common elements:
$$\mathcal{S}_1 \cap \mathcal{S}_2 \Leftrightarrow \forall T \in \mathcal{T} \mid \mathcal{S}_1(T) \cap \mathcal{S}_2(T) \neq \emptyset$$

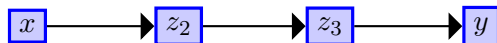
A first solution: the Top Cycle (TC)

Definition (Top Cycle)

The top cycle of $T \in \mathcal{T}(X)$ is the set TC defined as

$$TC(T) = \left\{ x \in X \mid \forall y \in X, \exists k > 0 \left| \begin{array}{l} \exists (z_1, \dots, z_k) \in X^k, \\ z_1 = x, z_k = y, \\ \text{and} \\ 1 \leq i < j \leq k \Rightarrow z_i T z_j \end{array} \right. \right\}$$

The top cycle contains outcomes that beat directly or indirectly every other outcomes.



Properties of Solutions

- Regular
- Monotonous
- Independent of the losers
- Strong Superset Property
- Idempotent
- Aizerman property
- Composition-consistent and weak composition-consistent

Definition (Regular tournament)

A tournament is **regular** iff all the points have the same Copeland score.

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Definition (Monotonous)

A solution \mathcal{S} is **monotonous** iff $\forall T \in \mathcal{T}(X), \forall x \in \mathcal{S}(T), \forall T' \in \mathcal{T}(X)$

such that
$$\begin{cases} T'/X \setminus \{x\} = T/X \setminus \{x\} \\ \forall y \in X, xTY \Rightarrow xT'y \end{cases}$$

one has $x \in \mathcal{S}(T')$

“Whenever a winner is reinforced, it does not become a loser.”

Definition (Independence of the losers)

A solution \mathcal{S} is **independent of the losers** iff $\forall T \in \mathcal{T}(X), \forall T' \in \mathcal{T}(X)$ such that $\forall x \in \mathcal{S}(T), \forall y \in X, xTy \Leftrightarrow xT'y$ one has $\mathcal{S}(T) = \mathcal{S}(T')$.

“the only important relations are $\left\{ \begin{array}{l} \text{winners to winners} \\ \text{winners to losers} \end{array} \right.$ *”*

“What happens between losers do not matter.”

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“the only important relations are $\begin{cases} \text{winners to winners} \\ \text{winners to losers} \end{cases}$ ”
“What happens between losers do not matter.”

Definition (Strong Superset Property (SSP))

A solution \mathcal{S} satisfies the **Strong Superset Property (SSP)** iff $\forall T \in \mathcal{T}(X), \forall Y \mid \mathcal{S}(T) \subset Y \subset X$ one has $\mathcal{S}(T) = \mathcal{S}(T/Y)$

“We can delete some or all losers, and the set of winners does not change”

Definition (Idempotent)

A solution \mathcal{S} is **idempotent** iff $\mathcal{S} \circ \mathcal{S} = \mathcal{S}$.



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$$\mathcal{S}(T)$$

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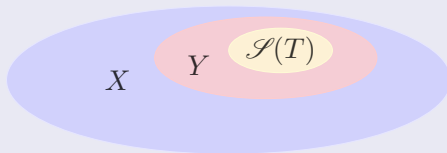
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$\mathcal{S}(T)$

Definition (Aizerman property)

A solution \mathcal{S} satisfies the Aizerman property iff $\forall T \in \mathcal{T}(X), \forall Y \subset X$
 $\mathcal{S}(T) \subset Y \subset X \Rightarrow \mathcal{S}(T/Y) \subset \mathcal{S}(T)$



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Y $\mathcal{S}(T)$

Solution Concepts

- Copeland solution (C)
- the Long Path (LP)
- Markov solution (MA)
- Slater solution (SL)
- Uncovered set (UC)
- Iterations of the Uncovered set (UC^∞)
- Dutta's minimal covering set (MC)
- Bipartisan set (BP)
- Bank's solution (B)
- Tournament equilibrium set (TEQ)

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method for ranking

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- Uncovered set (UC)
- Iterations of the Uncovered set (UC^∞) based on the notion of covering
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Based on Contestation

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 - Tournament equilibrium set (TEQ)
- method for ranking
- based on the notion of covering
- Game theory based
- Based on Contestation

	TC	UC	UC [∞]	MC	BP	B	TEQ	SL	C
Monotonicity	✓	✓	✗	✓	✓	✓	?	✓	✓
Independence of the losers	✓	✗	✗	✓	✓	✗	?	✗	✗
Idempotency	✓	✗	✓	✓	✓	✗	?	✗	✗
Aïzerman property	✓	✓	✗	✓	✓	✓	?	✗	✗
Strong superset property	✓	✗	✗	✓	✓	✗	?	✗	✗
Composition-consistency	✗	✓	✓	✓	✓	✓	✓	✗	✗
Weak Comp.-consist.	✓	✓	✓	✓	✓	✓	✓	✓	✗
Regularity	✓	✓	✓	✓	✓	✗	✗	✓	✗
Copeland value	1	1	1/2	1/2	1/2	≤ 1/3	≤ 1/3	1/2	1
Complexity	$O(n^2)$	$O(n^{2.38})$	\mathcal{P}			\mathcal{NP} -hard	\mathcal{NP} -hard	\mathcal{NP} -hard	$O(n^2)$

	TC	UC	UC [∞]	MC	BP	B	TEQ	C
UC	⊂							
UC [∞]	⊂	⊂						
MC	⊂	⊂	⊂					
BP	⊂	⊂	⊂	⊂				
B	⊂	⊂	∩	∩	a			
TEQ	⊂	⊂	⊂	b	a	⊂		
C	⊂	⊂	∅	∅	∅	∅	∅	
SL	⊂	⊂	∅	∅	∅	∅	∅	∅

- a $\exists T \in \mathcal{T}_{29} \mid B(T) \subset BP(T)$ and $B(T) \neq BP(T)$
 $\exists T' \in \mathcal{T}_6 \mid BP(T') \subset B(T')$ and $B(T') \neq BP(T')$.
 It is unknown if $B \cap BP$ can be empty.
 Same for TEQ and BP.
- b TEQ \subset MC is a conjecture

Outline

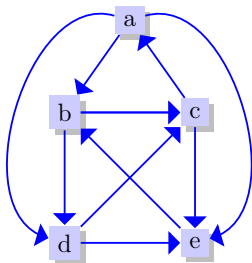
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Recall: Copeland score $s(x) = |T^+(x)| = |\{y \in X \mid xTy\}|$
 $s(x)$ is the number of alternatives that x beats.

Definition (Copeland solution (C))

Copeland winners of $T \in \mathcal{T}(X)$ is

$$C(T) = \{x \in X \mid \forall y \in X, s(y) = s(x)\}$$

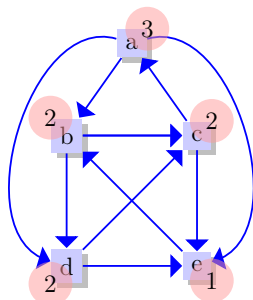


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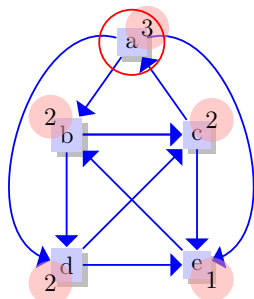


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Definition (Slater, Kandall, or Hamming distance)

Let $(T, T') \in \mathcal{T}(X)$

$$\Delta(T, T') = \frac{1}{2} \# \{ (x, y) \in X^2 \mid xTy \wedge yT'x \}$$

How many arrows are flipped in the tournament graph?

Definition (Slater order)

Let $T \in \mathcal{T}(X)$.

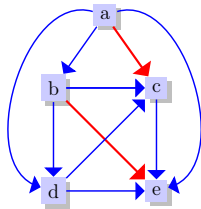
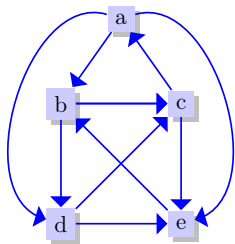
A **Slater order** for T is a **linear order** $U \in \mathcal{L}(X)$ such that

$$\Delta(T, U) = \min_{V \in \mathcal{L}(X)} \{ \Delta(T, V) \}$$

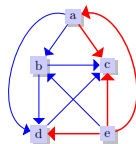
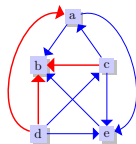
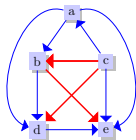
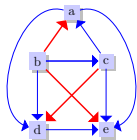
where $\mathcal{L}(X)$ is the set of linear order over X .

The set of **Slater winners of T** , noted $SL(T)$, is the set of alternatives in X that are Condorcet winner of a Slater order for T .

idea: approximate the tournament by a linear order.



$a \succ b \succ d \succ c \succ e$



$b \succ c \succ a \succ d \succ e$ $c \succ a \succ b \succ d \succ e$ $d \succ c \succ a \succ e \succ b$ $e \succ a \succ b \succ d \succ c$

to make b, c, d a Condorcet winner, it needs “3 flips”

to make e a Condorcet winner, it needs “4 flips”

Theorem

Computing a Slater ranking is \mathcal{NP} -hard.

Noga Alon. Ranking tournaments. *SIAM Journal of Discrete Mathematics*, 20(1):137-142, 2006

Vincent Conitzer, Computing Slater Rankings using similarities among candidates, AAI, 2006

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Definition (Covering)

Let $T \in \mathcal{T}(X)$ and $(x, y) \in X^2$

x **covers** y in X iff [xTy and $(\forall z \in X, yTz \Rightarrow xTz)$]

We note $x \triangleright y$

Definition (Equivalent definition of covering)

- $x \triangleright y$ iff xTy and $\forall z \in X, T/\{x,y,z\}$ is transitive.
- $x \triangleright y$ iff $x \neq y$ and $T^+(y) \subset T^+(x)$
- $x \triangleright y$ iff $x \neq y$ and $T^-(x) \subset T^-(y)$

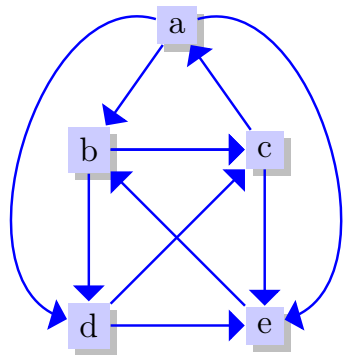
Definition (Uncovered Set (UC))

The **uncovered set** of T is $UC(T) = \{x \in X \mid \nexists y \in X \mid y \triangleright x\}$

Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803, 1977

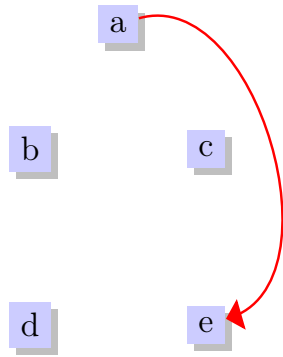
Fishburn. Condorcet social choice functions. *SIAM Journal of Applied Mathematics*, 33:469-489, 1977

Any outcome x in the Uncovered Set either beats y , or beats some z that beats y (x beats any other outcome it at most two steps).



tournament

$$UC(T) = \{a, b, c, d\}$$

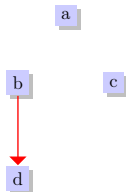
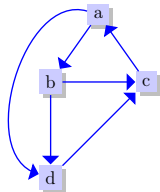


covering relation \triangleright

Proposition

$$\forall x \in X \setminus UC(X), UC^\infty(X) = UC^\infty(X \setminus \{x\})$$

Find a covered alternative, remove it, continue...



$T/\{a,b,c,d\}$

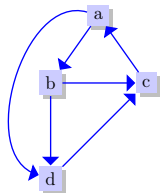
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$$UC(T/\{a,b,c,d\}) = \{a, b, c\}$$

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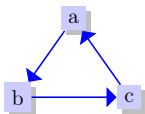
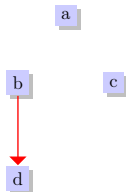
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$$T/\{a,b,c,d\}$$

$$UC(T/\{a,b,c,d\}) = \{a, b, c\}$$

covering relation \triangleright



$$T/\{a,b,c\}$$

covering relation \triangleright

Definition (Covering set)

Let $T \in \mathcal{T}(X)$ and $Y \subset X$.

Y is a **Covering set** for T iff $\forall x \in X \setminus Y, x \notin UC(Y \cup \{x\})$.

(x is covered by some elements in Y)

$C(T)$ is the family of covering sets for T .

Proposition

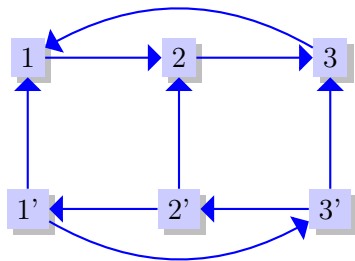
$\forall k \in (\mathbb{N} \cup \infty), UC^k(T)$ is a covering set for T .

proposition

The family $C(T)$ admits a minimal element (by inclusion) called the **minimal covering set** of T and denoted by $MC(T)$.

Dutta B. Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory* 44(1):63-80, 1988

$MC \subset UC^\infty$ and $MC \neq UC^\infty$



$$UC(T) = X = UC^\infty(T)$$

$$MC(T) = \{1, 2, 3\}$$

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Definition (tournament game)

A **tournament game** is a finite symmetric two-player game (X, g) such that, $\forall (x, y) \in X^2$

- $g(x, y) + g(y, x) = 0$ (zero-sum game)
- $x \neq y \Rightarrow g(x, y) \in \{-1, 1\}$

$T \in \mathcal{T}(X) \leftrightarrow$ tournament game (X, g)

with $\forall (x, y) \in X^2$, xTy iff $g(x, y) = +1$

Propositions

- y is a Condorcet winner $\Rightarrow \forall x \in X$, y is a best response to x .
- y is not a Condorcet winner $\Rightarrow \forall x \mid xTy$, x is a best response to y .
- (x, y) is a pure Nash equilibrium iff $\begin{cases} x = y \\ x \text{ is a Condorcet winner} \end{cases}$
- x dominates y in $(X, g) \Leftrightarrow x$ covers y
 - $UC(T)$ is the set of undominated strategies
 - $UC^\infty(T)$ is the set of strategies not sequentially dominated.

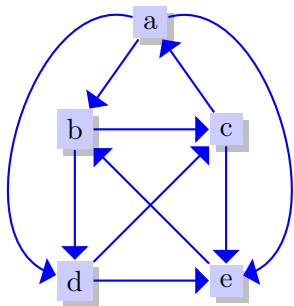
Theorem

A tournament game has a unique Nash equilibrium in mixed strategy, and this equilibrium is symmetric.

Definition (Bipartisan Set)

Let $T \in \mathcal{T}(X)$.

The **Bipartisan set** $BP(X)$ is the support of the unique mixed equilibrium of the tournament game associated with T .

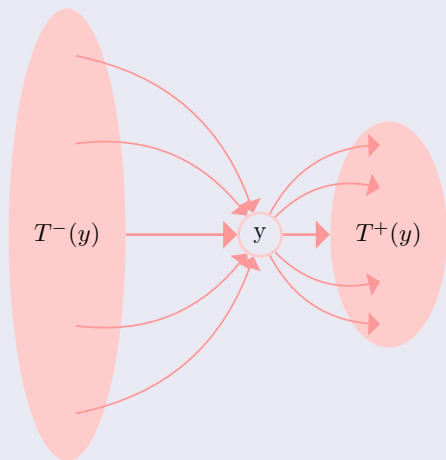


\vec{r}	a	b	c	d	e
a	0	1	-1	1	1
b	-1	0	1	1	-1
c	1	-1	0	-1	1
d	-1	-1	1	0	1
e	-1	1	-1	-1	0

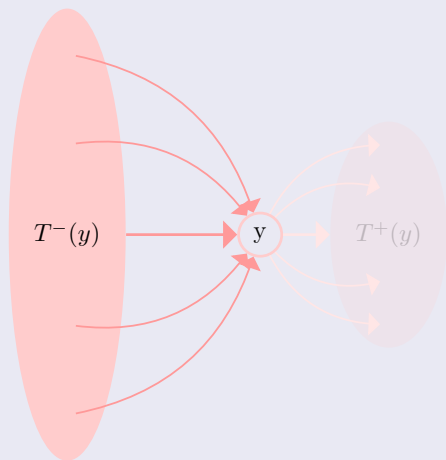
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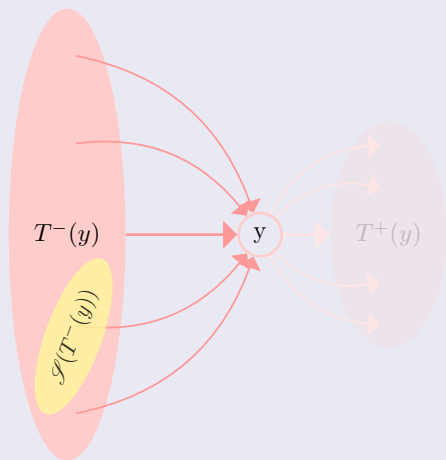
Is y a good outcome?



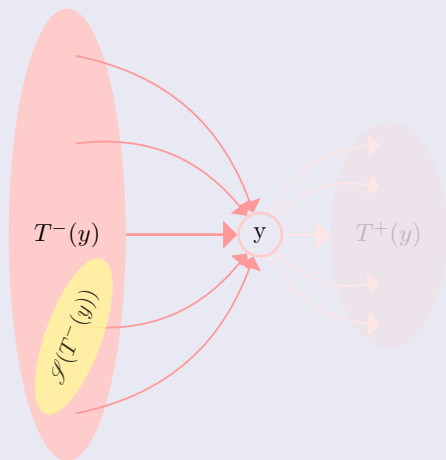
Is y a good outcome?



Is y a good outcome?



Is y a good outcome?



For a solution tournament \mathcal{S} and $T \in \mathcal{T}(X)$,
 $\forall (x, y) \in X^2 \quad xD(\mathcal{S}, T)y \Leftrightarrow x \in \mathcal{S}(T | T^-(y))$
 x is a **contestation** of y for T according to \mathcal{S} .

Bank's set

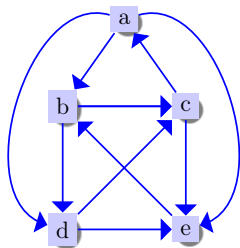
There exists a unique tournament solution B such that

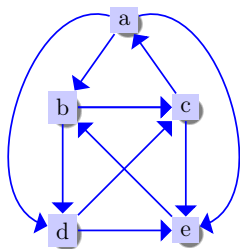
$$\forall T \in \mathcal{T}(X), o(T) \geq 2 \Rightarrow B(T) = D(B, T)^-(X)$$

$D(B, T)^-(X)$ is the set of points in X which are contestation of some point of X according to \mathcal{S} .

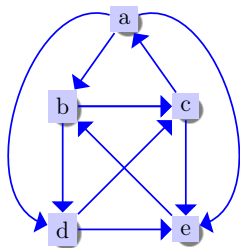
Proposition

$x \in B(T)$ iff $\exists Y \subset X$ such that $x \in Y$ and $T|_Y$ is an ordering for which x is the winner and no point of X beats all the points of Y .



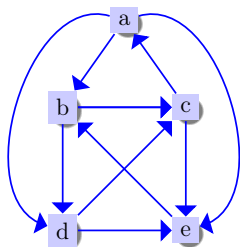


$a \ Y = \{d\}$, $a \succ d$ and aTb , dTc , aTe . ✓

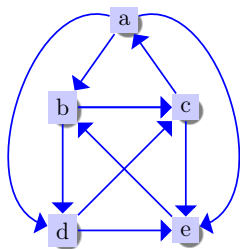


a $Y = \{d\}$, $a \succ d$ and aTb , dTc , aTe . ✓

b $Y = \{d, c\}$, $b \succ d \succ c$ and cTa , cTe . ✓



- a $Y = \{d\}$, $a \succ d$ and aTb , dTc , aTe . ✓
- b $Y = \{d, c\}$, $b \succ d \succ c$ and cTa , cTe . ✓
- c $Y = \{a\}$, $c \succ a$ and aTb , aTd , aTe . ✓
- d $Y = \{c, e\}$, $d \succ c \succ e$ and cTa , eTb . ✓



a $Y = \{d\}$, $a \succ d$ and aTb , dTc , aTe . ✓

b $Y = \{d, c\}$, $b \succ d \succ c$ and cTa , cTe . ✓

c $Y = \{a\}$, $c \succ a$ and aTb , aTd , aTe . ✓

d $Y = \{c, e\}$, $d \succ c \succ e$ and cTa , eTb . ✓

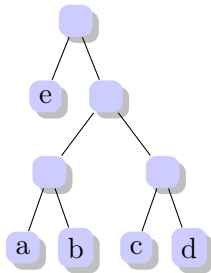
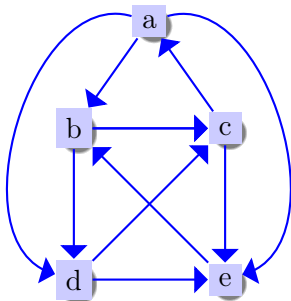
e $Y = \{b\}$ **no** because of aTb and aTe .

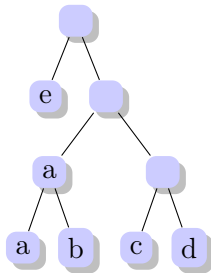
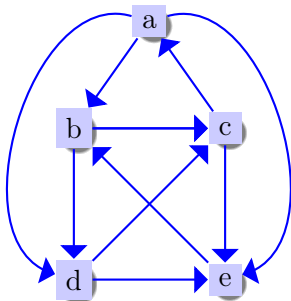
$Y = \{b, c\}$ **not** an ordering. ✗

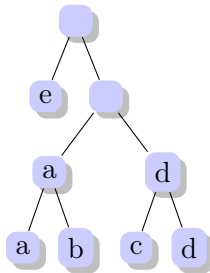
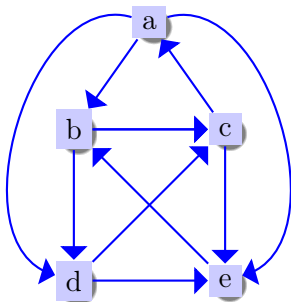
$B(T) = \{a, b, c, d\}$

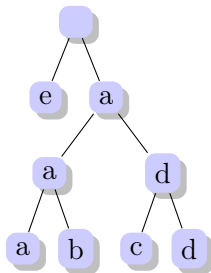
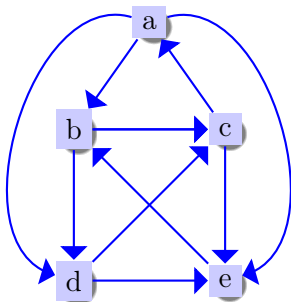
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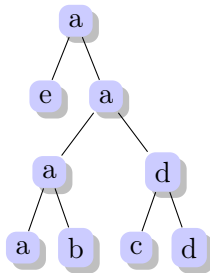
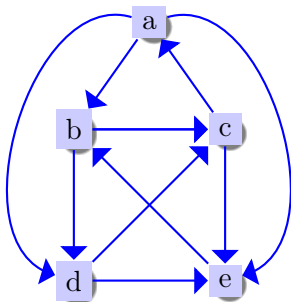
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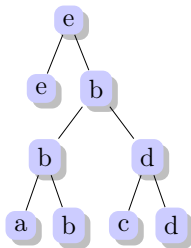
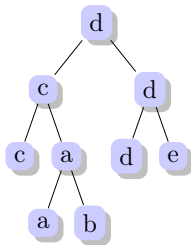
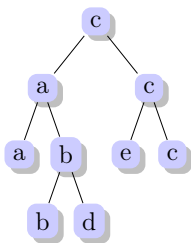
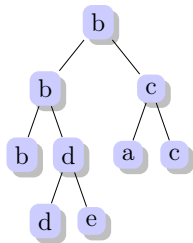
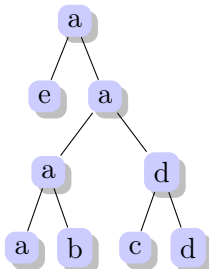
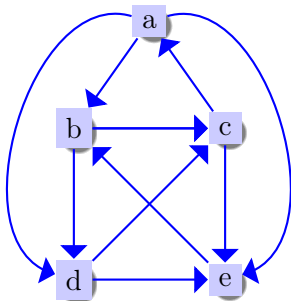












Definition (Algebraic solution)

A tournament solution \mathcal{S} is **computable by a binary tree** if, for any order n , there exists a labelled binary tree (N, A, i) of order n such that, for any tournament $T \in \mathcal{T}(X)$ of order n , $\mathcal{S}(T)$ is the set of winners of T along (N, T, i) for all drawing of X .

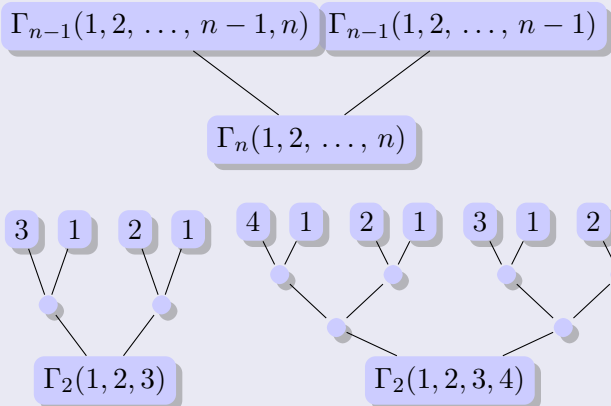
\mathcal{S} is computable by a binary tree iff \mathcal{S} is **algebraic**.

- Any algebraic tournament solution selects a winner in the top cycle.
- The Copeland and Markov solutions are not algebraic.
- Strengthening a winner can make her lose.
- There exists a non monotonous algebraic tournament solution.

Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803,1977

McKelvey, Niemi. A multistage game representation of sophisticated voting for binary procedures. *Journal of Economic Theory* 18:1-22,1978

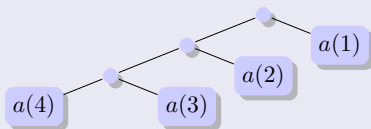
Multistage elimination tree or sophisticated agenda



Miller. Graph Theoretical approaches to the Theory of Voting. *American Journal of Political Sciences*, 21:769-803,1977

Hervé Moulin. Dominance Solvable Voting Schemes, *Econometrica*, 47(6):1337-1352,1979

Sophisticated voting on simple agendas



- $\Gamma_k(a)$: outcome of *strategic* voting on the simple agenda of order k with agenda a
- $a_{-n} = a(1) \cdot a(2) \dots a(n-2) \cdot a(n-1)$
- $a_{-(n-1)} = a(1) \cdot a(2) \dots a(n-2) \cdot a(n) \dots a(n)$

Voting for $a(n)$ or $a(n-1) \Rightarrow$ Comparing $\Gamma_{n-1}(a_{-n})$ and $\Gamma_{n-1}(a_{-(n-1)})$, i.e.,
$$\Gamma_n(a) = \Gamma_{n-1}(a_{-n}) \cdot \Gamma_{n-1}(a_{-(n-1)})$$

Sophisticated agenda and sophisticated voting

Strategic voting on a simple agenda results in choosing the winner of the associated sophisticated agenda.

Property

Let \mathcal{B} the set of all permutations of $X = \{1, \dots, n\}$

Let $a \in \mathcal{B}$, $w(\Gamma_n, T, a)$ is the winner of the tournament $T \in \mathcal{T}(X)$ along the sophisticated agenda Γ_n for the drawing a .

$$\{w(\Gamma_n, T, a), a \in \mathcal{B}\} = \text{Bank}(T)$$

Knockout tournaments

Definition (General Knockout Tournament)

Given a set N of players and a matrix P such that P_{ij} denotes the **probability** that player i wins against player j in a pairwise elimination match and $\forall (i, j) \in N^2$ $0 \leq P_{ij} = 1 - P_{ji} \leq 1$,

a **knockout tournament** $KTN = (T, S)$ is defined by:

- A tournament structure T : a binary tree with $|N|$ leaf nodes
- A seeding S : a bijection between the players in N and the leaf nodes of T

Theorem

It is \mathcal{NP} -complete to decide whether there exists a tournament structure KT with round placement R such that a target player $k \in N$ will win the tournament.

Thuc Vu, Alon Altman, Yoav Shoham, "On the Complexity of Schedule Control Problems for Knockout Tournaments", AAMAS 2009

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Properties

For Bipartisan set, minimal covering set, iterated uncovered set and the top cycle

- if \exists a Condorcet winner, the winner is unique (definition)
- if \nexists a Condorcet winner, the set of winners contains at least 3 alternatives.

Properties

If all tournaments are equiprobable, the top cycle is almost surely the whole set of alternatives.

Probability that every alternative is in the Banks set in a random tournament goes to one as the number of alternatives goes to infinity. (*every* alternative is in the Banks set in *almost all* tournaments).

Mark Fey. Choosing from a large tournament, *Social Choice and Welfare*, 31(2):301–309

Bibliography

- Jean Francois Laslier *Tournament Solution and Majority Voting*, Springer 1997.
- Thuc Vu, Alon Altman, Yoav Shoham, “*On the Complexity of Schedule Control Problems for Knockout Tournaments*”, AAMAS 2009.
- F. Brandt, F. Fischer, P. Harrenstein, and M. Mair. “*A computational analysis of the tournament equilibrium set*”. AAI-2008, COMSOC-2008.