A note on the approximability of the toughness of graphs

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Abstract

We show that, if $NP \neq ZPP$, for any $\varepsilon > 0$, the toughness of a graph with n vertices is not approximable in polynomial time within a factor of $\frac{1}{2}(\frac{n}{2})^{1-\varepsilon}$. We give a 4-approximation for graphs with toughness bounded by $\frac{1}{3}$ and we show that this result cannot be generalized to graphs with a bounded toughness. More exactly we prove that there is no constant approximation for graphs with bounded toughness, unless *P*=*NP*.

Keywords: toughness, graph, approximation algorithm, complexity.

1 Introduction and preliminaries

We consider only finite, non-complete, undirected and connected graphs without loops or multiple edges. The maximum size of an independent set of G is denoted by $\alpha(G)$. For a set S of vertices of G, $c(G \setminus S)$ is the number of connected components of the graph $G \setminus S$ which is obtained by removing S from G. The connectivity of G, denoted by $k(G)$, is the minimum size of a set of vertices S such that $c(G \setminus S) \geq 2$. We denote by $\Delta^*(G)$ the minimum of the maximum degree of a spanning tree of G.

The notion of toughness was introduced by Chvátal in [3]. A graph G is *t-tough* if $c(G \setminus S) \leq \frac{|S|}{t}$ for every set of vertices S of G with the property that $c(G \setminus S) \geq 2$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough. We observe that $\tau(G) = min\{\frac{|S|}{c(G \setminus S)} : S \subseteq V, c(G \setminus S) \geq 2\}$ and in fact we will use this equivalent definition of toughness.

Bauer, Hakimi, Schmeichel proved in [1] that for any fixed positive rational k, it is *coNP*-complete to decide whether a graph is k-tough. This implies that computing the toughness of a graph is *NP*-hard. At the EIDMA Workshop on Hamiltonicity of 2-Tough Graphs in 1995 ([2]) Brandt asked the question of the difficulty of approximating the toughness of a graph. In this paper we answer this question.

We consider the following minimization problem:
Min Toughness
Input: A graph $G = (V, E)$.
Output: A set of vertices S of G with the property $c(G \setminus S) \geq 2$ such that the ratio $\frac{|S|}{c(G \setminus S)}$ is minimized.

An algorithm is a $f(n)$-*approximation algorithm* for a maximization (respectively minimization) problem if for any instance x of the problem of size n, it returns a solution y of value $m(x, y)$ such that $m(x, y) \geq \frac{opt(x)}{f(n)}$ (respectively $m(x, y) \leq f(n) \times opt(x)$). An algorithm

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is a constant approximation algorithm if $f(n)$ is a constant. An optimization problem is $f(n)$-*approximable* if there exists a polynomial time $f(n)$-approximation algorithm for it.

2 Results

In this section we show first that if $NP \neq ZPP$, for any $\varepsilon > 0$, the toughness of a graph with n vertices is not approximable in polynomial time within a factor of $\frac{1}{2}(\frac{n}{2})^{1-\varepsilon}$. Secondly, we give a 4-approximation for graphs with toughness bounded by $\frac{1}{3}$ and we prove that there is no constant approximation for graphs with bounded toughness, unless $P= NP$.

We use in the following a result of Chvátal:

Lemma 1 ([3]) *For a graph G on n vertices, $\frac{k(G)}{\alpha(G)} \leq \tau(G) \leq \frac{n-\alpha(G)}{\alpha(G)}$.*

Theorem 2 *If $NP \neq ZPP$, for any $\varepsilon > 0$,* Min Toughness *is not $\frac{1}{2}(\frac{n}{2})^{1-\varepsilon}$ approximable in polynomial time where n is the number of vertices of the graph.*

Proof : We construct a reduction between Max Independent Set and Min Toughness. Given a graph G instance of Max Independent Set on n vertices, we construct a graph H from G by adding a clique C of size n and making each vertex of C adjacent to each vertex in G. By Lemma 1, $\tau(H) \leq \frac{2n-\alpha(H)}{\alpha(H)}$ and thus $\alpha(H) \leq \frac{2n}{\tau(H)}$. Since $\alpha(G) = \alpha(H)$ we have $\alpha(G) \leq \frac{2n}{\tau(H)}$.

Suppose that Min Toughness is $\frac{1}{2}(\frac{n}{2})^{1-\varepsilon}$ approximable. Thus there is an algorithm that applied to H finds a set S of vertices such that $val = \frac{|S|}{c(G \setminus S)} \leq \frac{1}{2}n^{1-\varepsilon}\tau(H)$. We consider as solution for G an independent set that contains a vertex from each connected component of $c(G \setminus S)$. Thus the size of this independent set is $val' = c(G \setminus S) \geq \frac{c(G \setminus S) \times n}{|S|} = \frac{n}{val}$ since S contains at least the vertices of the clique C. Using the previous inequality we obtain $val' \geq \frac{2n}{n^{1-\varepsilon}\tau(H)} \geq \frac{\alpha(G)}{n^{1-\varepsilon}}$. Since Max Independent Set is not approximable within $n^{1-\varepsilon}$ for any $\varepsilon > 0$, unless NP=ZPP [5], the theorem is proved. □

In the following we restrict to graphs with bounded toughness. Computing $\Delta^*(G)$ of a graph G is a NP-hard problem. Fürer and Raghavachari gave in [4] an approximation algorithm that finds a spanning tree of G of degree at most $\Delta^*(G) + 1$. We use the following two results:

Theorem 3 ([4]) *Let G be a graph. Then $\Delta^*(G) - 3 < \frac{1}{\tau(G)} \leq \Delta^*(G)$.*

Lemma 4 *If a graph G has $\tau(G) < \frac{1}{k-1}$ for some integer $k \geq 2$ then $\Delta^*(G) \geq k$.*

Proof : If $\tau(G) < \frac{1}{k-1}$ for an integer $k \geq 2$, then $k - 1 < \frac{1}{\tau(G)} \leq \Delta^*(G)$ by Theorem 3, and so $\Delta^*(G) \geq k$. □

Theorem 5 Min Toughness *is 4-approximable for graphs with toughness less than* $1/3$*.*

Proof : Let G be a graph with $\tau(G) < \frac{1}{3}$. By Lemma 4 we have $\Delta^*(G) \geq 4$. By applying Fürer and Raghavachari's algorithm on G we obtain a spanning tree T with maximum degree d such that $\Delta^*(G) \leq d \leq \Delta^*(G) + 1$. We consider as solution for Min Toughness the set $S = S_d \cup S_{d-1}$ where S_d and S_{d-1} are respectively the set of vertices of G of degree d and $d-1$ in T. It is proved in [4] that the number of connected components of the graph $G \setminus S$ is $c(G \setminus S) \geq (d-2)|S_d| + (d-3)|S_{d-1}| + 2$. Thus $\frac{|S|}{c(G\setminus S)|} \leq \frac{1}{d-3}$ and using Theorem 3 we have

$$\frac{\frac{|S|}{c(G\setminus S)|}}{\tau(G)} \leq \frac{\frac{1}{d-3}}{\tau(G)} \leq \frac{1}{\Delta^*(G) - 3} \times \Delta^*(G) \leq 4.$$

□

In the following we use a result of [6] to prove that there is no polynomial time constant approximation algorithm for the toughness of graphs with a bounded toughness. An s-partitioned graph is a graph whose vertices are partitioned into s cliques.

Lemma 6 ([6]) *For each constant $g > 1$ there is a constant w such that it is NP-hard to distinguish if an s-partitioned graph G with the size of the cliques at most w has $\alpha(G) = s$ or $\alpha(G) < \frac{s}{g}$.*

Theorem 7 *For each constant $c > 1$ there is a constant k such that it is NP-hard to decide if a graph H with a bounded toughness has $\tau(H) \leq k$ or $\tau(H) > k \times c$.*

Proof : For a constant $c > 1$ we consider $g = 4c$. Let G be a s-partitioned graph with $m = w \times s$ vertices. We construct a graph H by adding to G an independent set S of size $\lceil \frac{s}{g} \rceil$ and a clique C of size $m - \lceil \frac{s}{g} \rceil$ and making adjacent each new vertex with each vertex of G and each vertex of C with each vertex of S. Thus $\alpha(H) = max\{\alpha(G), \lceil \frac{s}{g} \rceil\} \geq \frac{s}{g}$ and then the toughness of H is bounded by $\frac{2m}{\alpha(H)} - 1 \leq 2w \times g - 1$. If $\alpha(G) = s$ then $\tau(H) \leq \frac{2m-s}{s} = 2w - 1$ and if $\alpha(G) < \frac{s}{g}$ then $\tau(H) \geq \frac{k(H)}{\alpha(H)} \geq 2cw - 1$ since $k(H) \geq |C|$. Let $k = 2w - 1$. Thus $\alpha(G) = s$ if and only if $\tau(H) \leq 2w - 1$. The theorem is proved since if we can decide if H has toughness less than k or greater than $k \times c$ then we can decide if $\alpha(G) = s$ or $\alpha(G) < \frac{s}{g}$. □

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