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Graphs without a partition into two proportionally dense subgraphs

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ABSTRACT

A *proportionally dense subgraph* (PDS) is an induced subgraph of a graph such that each vertex in the PDS is adjacent to proportionally as many vertices in the subgraph as in the rest of the graph. In this paper, we study a partition of a graph into two proportionally dense subgraphs, namely a 2-*PDS partition*, with and without additional constraint of connectivity of the subgraphs. We present two infinite classes of graphs: one with graphs without a 2-PDS partition, and another with graphs that only admit a disconnected 2-PDS partition. These results answer some questions proposed by Bazgan et al. (2018) [3].

1. Introduction

The problems of partitioning a graph into two parts have been intensively studied with various objective functions and constraints. Let's mention at least two such NP-hard problems. The Satisfactory Partition problem [2] asks whether a graph can be partitioned into two parts such that every vertex is adjacent to more vertices in its own part than in the other. In the Maximally Balanced Connected Partition problem, the task is to partition a graph into two connected subgraphs such that the size of the smallest subgraph is maximised [5].

The notion of *proportionally dense subgraph* is closely related to the notion of *community* as introduced in [10]. Olsen defines a *community structure* as a partition of the vertices into communities, where a part, *i.e.* an induced subgraph (with at least 2 vertices), is a *community* if and only if each vertex has proportionally as many neighbours in its community than in any other community. In [3], the authors investigate the notion of 2-community structure as a community structure with exactly two parts. We use the same definition (up to the special case where a community is of size one) to define a 2-PDS partition.

So far, only few results are known about the existence of a 2-PDS partition in a graph, and the complexity of finding one. It has been proved in [6] that deciding if a graph contains a 2-PDS partition with both PDS's of the same size is NP-complete. On trees [3,6] and graphs with maximum degree 3 or minimum degree $n-3$, (n the order of the graph) a connected 2-PDS partition always exists and can be found in polynomial time [3]. The results extensively use the connectivity of the PDS's. To find a connected 2-PDS partition in a tree, one can prove that there exists an edge such that its removal yields two connected PDS's. If a graph has a maximum degree at most 3, a greedy algorithm keeps decreasing the size of a cut under some constraints and the removal of the final cut describes two connected PDS's.

Another problem related to the notion of PDS is the Max PDS problem. In this problem, the goal is to determine the size of a maximum PDS (with regard to the number of vertices) in a given graph. Hence, only the vertices in-

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side the PDS must be satisfied. In [4], the authors prove that Max PDS is NP-hard on bipartite and split graphs, and propose a polynomial-time $(2 - \frac{2}{\Delta+1})$-approximation algorithm, where Δ is the maximum degree of the graph. They also show that deciding if a subset of vertices can be a (proper) subset of the vertices of a PDS is co-NP-complete on bipartite graphs.

Our contributions. In Section 2, we formally define the concepts of proportionally dense subgraphs and PDS partitions, and outline the known results about the 2-PDS partition problem. Then, we construct an infinite family of graphs without a 2-PDS partition in Section 3.1. As far as we know, these are the first negative results regarding the existence of a 2-PDS partition. We also give examples of graphs without a 2-PDS partition that do not belong to the family. In Section 3.2 we present another infinite family of graphs without a connected 2-PDS, but with a disconnected one.

2. Proportionally dense subgraphs

All graphs in this paper are simple. Given a graph $G = (V, E)$ and a subset of vertices $S \subset V$, $\bar{S}$ refers to the set $V \setminus S$. For a vertex $u \in V$, $N(u)$ represents the set of neighbours of u, $d(u) := |N(u)|$ is the degree of u, and $d_S(u) := |N(u) \cap S|$ denotes the degree of u in S. We say that a vertex $u \in V$ is *universal* if it is connected to all other vertices of the graph, that is, $d(u) = |V| - 1$.

The density of a subgraph on a vertex set $S \subseteq V$ is usually defined as $\frac{|E(S)|}{|S|}$, where $E(S)$ is the set of edges in the subgraph. The problem of finding a subgraph of maximum density can be solved in polynomial time [8], but it becomes NP-hard when at least, or exactly, k vertices must belong to the subgraph [1,7,9].

In this paper, we introduce the notion of *proportionally dense subgraph* (PDS), which captures both the size of the subset and the number of neighbours.

Definition 1. For a graph $G = (V, E)$, a *proportionally dense subgraph* of G is an induced subgraph on a vertex set $S \subset V$ such that each vertex $u \in S$ is *satisfied* in S, that is,

$$|\bar{S}| \cdot d_S(u) \geq (|S| - 1) \cdot d_{\bar{S}}(u),$$

or, equivalently, $(|V| - 1) \cdot d_S(u) \geq (|S| - 1) \cdot d(u)$.

Note that if $|S| \geq 2$, then we can rewrite the inequalities as

$$\frac{d_S(u)}{|S| - 1} \geq \frac{d_{\bar{S}}(u)}{|\bar{S}|} \text{ or, equivalently, } \frac{d_S(u)}{|S| - 1} \geq \frac{d(u)}{|V| - 1}.$$

The proof of the equivalence can be found in [3]. Note that a subgraph containing a single vertex is also a PDS, but obviously a PDS cannot be the entire graph.

Definition 2. A 2-*PDS partition* of a graph $G = (V, E)$ is a partition $\Pi = \{S_1, S_2\}$ of V such that S_1 and S_2 induce two PDS's in G.

In this paper, we address the problem of deciding if a graph admits a 2-PDS partition. Notice that a PDS doesn't necessarily need to be connected. Therefore we also consider the problem of deciding if a graph has a *connected* 2-*PDS partition*, that is, a 2-PDS partition whose PDS's are connected subgraphs.

If a graph is disconnected, both problems become trivial, hence we assume that all graphs are connected.

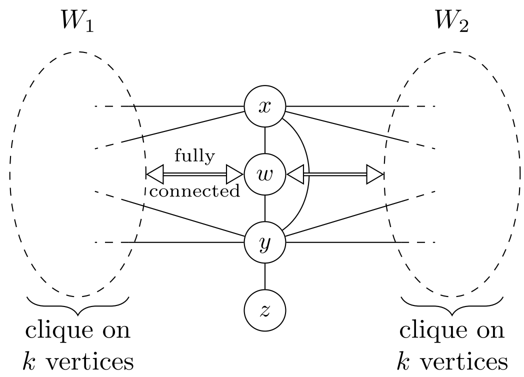

Fig. 1. A schematic representation of a graph in $\mathcal{G}$.

3. Infinite classes of graphs

3.1. Graphs without 2-PDS partition

The question about the existence of graphs without a 2-PDS was left open in [3]. To the best of our knowledge, no graphs without a 2-PDS partition were known. In this section we present an infinite class $\mathcal{G}$ (see Definition 3) of graphs with even number of vertices without a 2-PDS partition.

Definition 3. Let $\mathcal{G}$ be the class of graphs such that, if $G = (V, E) \in \mathcal{G}$, then

- $V = W_1 \cup W_2 \cup \{w, x, y, z\}$, where W_1, W_2 are cliques of the same size k, $k \geq 3$, and $\{w, x, y\}$ is a clique of size 3;
- w is adjacent to all vertices in $W_1 \cup W_2$, and z is only adjacent to y
- $1 \leq d_{W_1}(x) = d_{W_2}(x) \leq k - 1$ and $2 \leq d_{W_1}(y) = d_{W_2}(y) \leq k - 1$;
- $|W_i \cap (N(x) \cup N(y))| > \frac{3k}{k+3}$ for each $i \in \{1, 2\}$;
- there exist vertices $\alpha, \beta \in W_1 \cup W_2$ such that $\alpha \in N(y) \setminus N(x)$, and $\beta \in N(x) \cap N(y)$;
- there is no edge between the vertex sets W_1 and W_2.

Note that the smallest graphs in $\mathcal{G}$ have 10 vertices, and one of them is planar (see Fig. 2).

Theorem 1. *All graphs in $\mathcal{G}$ do not have a 2-PDS partition.*

Proof. Let $G = (V, E)$ be a graph in $\mathcal{G}$. Firstly, notice that there is no 2-PDS partition $\{A, B\}$ in G such that $|A| = 1$ or $|B| = 1$. Without loss of generality, suppose by contradiction that $A = \{v\}$ for some vertex $v \in V$, and notice that the neighbour of v in B must be a universal vertex in order to be satisfied. Since G does not contain a universal

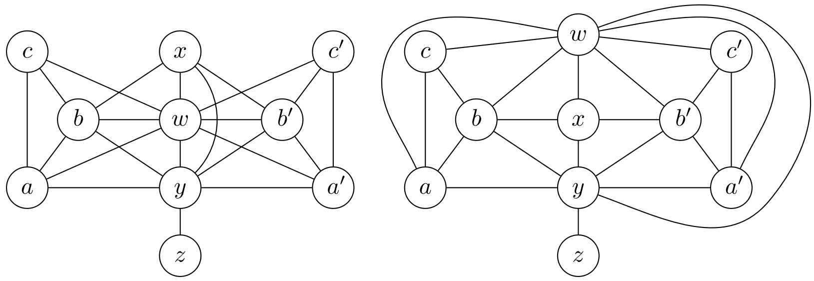

Fig. 2. A planar graph from $\mathcal{G}$ with 10 vertices without a 2-PDS partition. On the left, its schematic representation as in Fig. 1; on the right, its planar representation.

vertex, there is no 2-PDS partition $\{A, B\}$ in G with $|A| = 1$ or $|B| = 1$. Hence, assume that $|A|, |B| \geq 2$.

Observe that the vertex z is satisfied if and only if it belongs to the same PDS as the vertex y. Hence, without loss of generality, we assume that $y, z \in B$. In addition, the vertex w has degree $|V| - 2$ and is not connected to $z \in B$. Hence, necessarily $w \in A$.

Now we prove that for any partition $\{A, B\}$ of V, where $w \in A$ and $y, z \in B$, there is at least one vertex which is not satisfied, hence there is no 2-PDS partition in G. For any partition $\{A, B\}$ of V, we denote by A_i and B_i the sets $A \cap W_i$ and $B \cap W_i$, respectively, for $i \in \{1, 2\}$. We split the proof into two cases: In the first case, we suppose that B_1 or B_2 is empty; in the second case, we assume that B_1 and B_2 are not empty.

Case 1: $B_1 = \emptyset$ or $B_2 = \emptyset$
Suppose first that $B_1 = \emptyset$ and $B \subseteq \{x, y, z\} \cup W_2$.

- If $B_2 = \emptyset$, we have two possibilities:
  - if $x \in B$, then $B = \{x, y, z\}$ and $\beta \in A$ is not satisfied since $\frac{d_A(\beta)}{|A|-1} = \frac{k}{2k} < \frac{2}{3} = \frac{d_B(\beta)}{|B|}$;
  - if $x \in A$, then $B = \{y, z\}$ and $\alpha \in A$ is not satisfied since $\frac{d_A(\alpha)}{|A|-1} = \frac{k}{2k+1} < \frac{1}{2} = \frac{d_B(\alpha)}{|B|}$.
- If $B_2 \neq \emptyset$ and $B_2 \neq W_2$:
  - Case $x \in B$.
    * If there exists $u \in A_2$ such that $u \in N(x) \cup N(y)$ and u is satisfied, then we have:
$$\frac{|A_2|}{k + |A_2|} = \frac{d_A(u)}{|A| - 1} \geq \frac{d(u)}{|V| - 1} \geq \frac{k+1}{2k+3},$$
      which implies that $|A_2| \cdot (k+2) \geq k \cdot (k+1)$, hence that $|A_2| > k - 1$. A contradiction since $|A_2| \leq k - 1$.
    * Otherwise, for all $u \in A_2$, $u \notin N(x) \cup N(y)$. Hence, for any $u \in A_2$, if u is satisfied then:
$$\frac{|A_2|}{k + |A_2|} = \frac{d_A(u)}{|A| - 1} \geq \frac{d(u)}{|V| - 1} = \frac{k}{2k+3},$$
      which implies that $|A_2| \cdot (k+3) \geq k^2$, hence that $|A_2| \geq \frac{k^2}{k+3}$. Due to our assumptions about the graph, $|W_2 \cap (N(x) \cup N(y))| > \frac{3k}{k+3}$. Thus, $k - \frac{3k}{k+3} > |W_2 \setminus (N(x) \cup N(y))| \geq |A_2| \geq \frac{k^2}{k+3}$ which implies $k > k$, a contradiction.
  - Case $x \in A$. Let $u \in A_2$.
    * If $u \in N(y) \cap N(x)$ and u is satisfied, then we have:
$$\frac{|A_2| + 1}{k + |A_2| + 1} = \frac{d_A(u)}{|A| - 1} \geq \frac{d(u)}{|V| - 1} = \frac{k+2}{2k+3},$$
      which implies that $|A_2| \geq k - \frac{1}{k+1}$, and then $|A_2| \geq k$, a contradiction since $B_2 \neq \emptyset$.
    * If $u \in N(y) \setminus N(x)$, then $d_A(u) = |A_2|$ and $d(u) = k+1$. Therefore, similarly to the previous case, we obtain that $|A_2| \geq k + \frac{1}{k+2}$ and so $|A_2| > k$, a contradiction.
    * If $u \in N(x) \setminus N(y)$, then:
$$\frac{|A_2| + 1}{k + |A_2| + 1} = \frac{d_A(u)}{|A| - 1} \geq \frac{d(u)}{|V| - 1} = \frac{k+1}{2k+3},$$
      which implies that $|A_2| \cdot (k+2) \geq k^2 - 2$, hence $|A_2| \geq \frac{k^2-2}{k+2} > k - 2$. Since assuming that there is a vertex in $A_2 \cap N(y)$ leads to a contradiction (see previous cases), we can assume that $A_2 \cap N(y) = \emptyset$. Then, since $d_{W_2}(y) \geq 2$, then $|W_2 \setminus N(y)| \leq k - 2$. Thus $k - 2 \geq |A_2| > k - 2$, a contradiction.
    * If $u \notin N(x) \cup N(y)$, then $d_A(u) = |A_2|$ and $d(u) = k+1$. Thus, we obtain $|A_2| > k$, a contradiction since $|B_2| \neq \emptyset$.
- If $B_2 = W_2$, then either $B = \{x, y, z\} \cup W_2$, and we have $|A| + 2 = |B|$ but $d_A(x) = d_B(x)$ thus x is not satisfied, or $B = \{y, z\} \cup W_2$, and since $|A| = |B|$ we have: $\frac{d_B(y)}{|B|-1} < \frac{d_B(y)+1}{|B|} = \frac{d_A(y)}{|B|} = \frac{d_A(y)}{|A|}$, thus y is not satisfied.

We conclude that if there is a 2-PDS partition in G, then $B_1 \neq \emptyset$. The case $B_2 = \emptyset$ is similar, therefore if there is a 2-PDS partition in G, then $B_2 \neq \emptyset$.

Case 2: $B_1, B_2 \neq \emptyset$.
Without loss of generality, we suppose $|B_1| \leq |B_2|$. Let $u \in B_1$ and suppose that u is satisfied in the partition $\{A, B\}$. We prove that in all cases, if u is satisfied then it implies a contradiction with $|B_1| \leq |B_2|$.

- If $x \in A$:
  * If $u \in N(x) \cap N(y)$ is satisfied, then:
$$\frac{|B_1|}{|B_1| + |B_2| + 1} = \frac{d_B(u)}{|B| - 1} \geq \frac{d(u)}{|V| - 1} = \frac{k+2}{2k+3},$$
    which implies that $|B_1| \cdot (k+1) \geq (|B_2| + 1) \cdot (k+2)$, hence that $|B_1| > |B_2|$. A contradiction with the assumption that $|B_1| \leq |B_2|$, hence u is not satisfied.

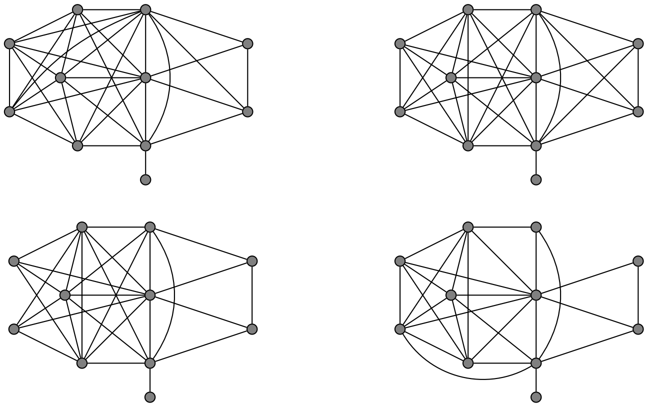

Fig. 3. Four graphs with 11 vertices which do not have a 2-PDS partition.

- ∗ If $u \in N(x) \setminus N(y)$, we have $d_B(u) = |B_1| - 1$ and $d(u) = k+1$ and similarly we obtain $|B_1| \cdot (k+2) \geq |B_2| \cdot (k+1) + (3k+4) \geq |B_2| \cdot (k+1) + (|B_2|+4) > |B_2| \cdot (k+2)$, a contradiction since $|B_1| \leq |B_2|$.
- ∗ If $u \in N(y) \setminus N(x)$, we have $d_B(u) = |B_1|$ and $d(u) = k+1$ and similarly we obtain $|B_1| \cdot (k+2) \geq |B_2| \cdot (k+1) + (k+1) \geq |B_2| \cdot (k+1) + (|B_2|+1) > |B_2| \cdot (k+2)$, a contradiction since $|B_1| \leq |B_2|$.
- ∗ If $u \notin N(x) \cup N(y)$, we have $d_B(u) = |B_1| - 1$ and $d(u) = k$ and similarly we obtain $|B_1| \cdot (k+3) \geq |B_2| \cdot k + 3(k+1) \geq |B_2| \cdot k + 3(|B_2|+1) > |B_2| \cdot (k+3)$, a contradiction since $|B_1| \leq |B_2|$.

- If $x \in B$:
  - ∗ If $u \in N(x) \cap N(y)$ is satisfied, then:

$$\frac{|B_1|+1}{|B_1|+|B_2|+2} = \frac{d_B(u)}{|B|-1} \geq \frac{d(u)}{|V|-1} = \frac{k+2}{2k+3},$$

which implies that $|B_1| \cdot (k+1) \geq |B_2| \cdot (k+2) + 1$, thus that $|B_1| > |B_2|$. A contradiction with the assumption that $|B_1| \leq |B_2|$, hence u is not satisfied.
  - ∗ If $u \in N(x) \setminus N(y)$ or $u \in N(y) \setminus N(x)$, we have $d_B(u) = |B_1|$ and $d(u) = k+1$ and similarly we obtain $|B_1| \cdot (k+2) \geq |B_2| \cdot (k+1) + 2(k+1) \geq |B_2| \cdot (k+1) + 2(|B_2|+1) > |B_2| \cdot (k+3)$, a contradiction since $|B_1| \leq |B_2|$.
  - ∗ If $u \notin N(x) \cup N(y)$, we have $d_B(u) = |B_1| - 1$ and $d(u) = k$ and similarly we obtain $|B_1| \cdot (k+3) \geq |B_2| \cdot k + 4k + 3 \geq |B_2| \cdot \frac{k}{k+3} + 4 \cdot |B_2| + 3 > |B_2| \cdot (k+4)$, a contradiction since $|B_1| \leq |B_2|$. □

In Fig. 3, we present four graphs with 11 vertices without a 2-PDS partition. These graphs have an odd number of vertices, hence they do not belong to $\mathcal{G}$. To prove that they do not have a 2-PDS partition, one can notice that, like the graphs in $\mathcal{G}$, they have a pendant vertex z connected to a vertex y, and a vertex w connected to all the vertices except the pendant vertex. As a result, the vertex z is satisfied if and only if it belongs to the same PDS as y, and thus w must be in the other PDS. The rest of the proof can be done by case distinction.

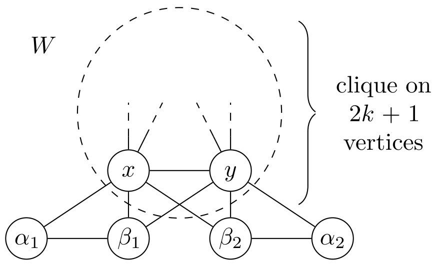

Fig. 4. A schematic representation of a graph in $\mathcal{H}$.

3.2. Disconnected 2-PDS partition

Now, we present an infinite family of graphs where each graph admits a disconnected 2-PDS partition, but not a connected one. The existence of such graphs was left as an open problem in [3].

Definition 4. We define the infinite class of graphs $\mathcal{H}$ such that, if $G = (V, E) \in \mathcal{H}$, then

- $V := W \cup \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, where W is a clique of size $2k+1$, $k \geq 3$;
- $\exists x, y \in W$ such that $\{x, \alpha_1\}, \{x, \beta_1\}, \{x, \beta_2\}, \{y, \alpha_2\}, \{y, \beta_2\}, \{y, \beta_1\} \in E$;
- $\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\} \in E$.

See Fig. 4 for a schematic representation of a graph in $\mathcal{H}$. Compared to the graphs in $\mathcal{G}$, each graph in $\mathcal{H}$ has an odd number of vertices (the smallest one has 11 vertices).

Theorem 2. *All graphs in $\mathcal{H}$ do not have a connected 2-PDS partition, but have a disconnected one.*

Proof. Let $G = (V, E) \in \mathcal{H}$. Suppose that G has a connected 2-PDS partition $\{A, B\}$. If $A \subseteq W$, then we have two cases: either $A = W$ but then $G[B]$ is disconnected, or $A \subset W$ but then a vertex in $W \setminus A$ is not satisfied in B. Hence, $A \not\subseteq W$ and similarly $B \not\subseteq W$. Consequently, to guarantee the connectivity of $G[A]$ and $G[B]$, the vertices x and y must be in different parts of the partition. Therefore, we assume without loss of generality that $x \in A$ and $y \in B$.

If $\alpha_1 \in B$, then y is not satisfied since it is connected to each vertex in A. Similarly, α_2 cannot belong to A since otherwise x is not satisfied. As a result, we only have to consider the possible cases for β_1 and β_2, knowing that $x, \alpha_1 \in A$ and $y, \alpha_2 \in B$.

If $\beta_1 \in A$ and $\beta_2 \in B$, then consider two vertices $a \in (W \setminus \{x\}) \cap A$ and $b \in (W \setminus \{y\}) \cap B$. The vertex a is satisfied in A if and only if

$$\frac{d_A(a)}{|A|-1} = \frac{|A|-3}{|A|-1} \geq \frac{|B|-2}{|B|} = \frac{d_B(a)}{|B|},$$

which implies that $|A| \geq |B| + 1$. Similarly, the vertex b is satisfied in B if and only if $|A| \leq |B| - 1$, which is a contradiction.

If $\beta_1, \beta_2 \in A$, then the vertex β_2 is satisfied in A if and only if

$$\frac{d_A(\beta_2)}{|A|-1} = \frac{1}{|A|-1} \geq \frac{2}{|B|} = \frac{d_B(\beta_2)}{|B|},$$

which implies that $|A| \leq \frac{|B|}{2} + 1$. Moreover, the vertex α_2 is satisfied in B if and only if

$$\frac{d_B(\alpha_2)}{|B|-1} = \frac{1}{|B|-1} \geq \frac{1}{|A|} = \frac{d_A(\alpha_2)}{|A|},$$

which implies that $|A| \geq |B| - 1$. We then obtain $|B| - 1 \leq |A| \leq \frac{|B|}{2} + 1$, and therefore $|B| \leq 4$. Thus, $|A| \leq 3$, which is not possible since $|V| \geq 11$. Similar arguments can be used to prove that β_1 and β_2 cannot both belong to B.

We conclude that G does not have a connected 2-PDS partition. However, it is easy to see that, if $A := \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and $B := V \setminus A$, then $\{A, B\}$ is a disconnected 2-PDS partition of G. □

4. Conclusion and further work

The definition of a proportionally dense subgraph is based on a combination of local and global properties, where each vertex has to satisfy a condition depending not only on its degree but also on the size of the subgraph. This property makes the problem complex from an algorithmic point of view and requires a novel approach.

Our infinite families of graphs bring a new insight into the existence of 2-PDS partitions in graphs, with and without constraint of connectivity. Further research may investigate the structural characterisations of graphs with or without a (connected) 2-PDS partition. These results can help to answer the following important question: what is the complexity of deciding whether a graph admits a (connected) 2-PDS partition?

Declaration of competing interest

There is no competing interest.

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