On the Loebl-Komlós-Sós conjecture

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Abstract

The Loebl-Komlós-Sós conjecture says that any graph G on n vertices with at least half of vertices of degree at least k contains each tree of size k. We prove that the conjecture is true for paths as well as for large values of k ($k \geq n - 3$).

1 Introduction

We shall use standard graph theory notation. We consider only finite, undirected graphs of order $n = |V(G)|$ and size $e(G) = |E(G)|$. All graphs will be assumed to have neither loops nor multiple edges.

The below conjecture was firstly formulated by Loebl in 1994 in the case $k = \frac{n}{2}$ and next generalized by Komlós and Sós.

Conjecture 1 (Loebl-Komlós-Sós [3]) *If G is a graph on n vertices and at least $\frac{n}{2}$ vertices have degrees at least k, then G contains all trees of size at most k.*

The Loebl-Komlós-Sós conjecture has some similarity with the well known Erdős-Sós conjecture.

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Conjecture 2 (Erdős-Sós) *If G is a graph on n vertices and the number of edges of G is $e(G) > \frac{n(k-1)}{2}$ then G contains all trees of size at most k.*

As remarked in [5], the condition that the average degree of the graph G is greater than $k-1$ from the Erdős-Sós conjecture is replaced in Loebl-Komlós-Sós conjecture by the condition that the medium degree of G is greater than k (for some special cases of the Erdős-Sós conjecture see for example [8] as well as [2] and [7]).

For a graph satisfying the hypothesis of the Loebl-Komlós-Sós conjecture we define $B = \{v \in V(G) \mid d_G(v) \geq k\}$ and $S = V(G) - B$. The vertices of B and S will be also referred as B-vertices and S-vertices, respectively. Some additional definitions and notations will be given in next sections.

Observe first that the Loebl-Komlós-Sós conjecture is true for stars. For, if G is a graph satisfying the hypothesis of the conjecture, we can identify the center of a star with one B-vertex of G.

The Loebl-Komlós-Sós conjecture holds also for double-stars with at most k edges. Indeed, let G be a graph satisfying the hypothesis of the conjecture. It is easy to see that the set B contains at least one edge. For, otherwise, if B is independent, then there are more than $k|B|$ edges between B and S and also there are less than $(k-1)|S|$ between S and B. Thus

$$k|B| \leq (k-1)|S| \leq (k-1)|B|.$$

The contradiction proves that B cannot be an independent set. Now, let v, w be two vertices of B such that $vw \in E(G)$. It suffices to identify the two centers of a double star with v and w.

In this paper we shall consider some others special cases of the Loebl-Komlós-Sós conjecture. In particular we shall prove that it holds for paths (Section 2). We can also show that the conjecture holds for large values of parameter k, namely $k \geq n-3$ (the proof of this result can be find in [1]).

Mention that using the Regularity Lemma, Ajtai, Komlós and Szemerédi proved the following approximate form of the Loebl-Komlós-Sós conjecture (see [5]).

Theorem 3 *For every $\epsilon > 0$ there is a threshold n_0 such that for all $n \geq n_0$, if G is a graph on n vertices and it has at least $\frac{(1+\epsilon)n}{2}$ vertices of degree at least $\frac{(1+\epsilon)n}{2}$, then G contains all trees with at most $\frac{n}{2}$ edges.*

2 Paths

Recall first that P_r denote the path on r vertices (*i.e.* of length $r-1$) and C_r denote the cycle on r vertices (*i.e.* of length r).

For a given cycle C denote by $\overrightarrow{C}$ one of its orientations. Then the opposite orientation is denoted by $\overleftarrow{C}$. For $v, w \in V(C)$ we denote by $v\overrightarrow{C}w$ the path starting in v and ending in w which contains all vertices of C between v and w following the orientation $\overrightarrow{C}$. Similarly, we denote by $v\overleftarrow{C}w$ the path which contains all vertices of C between v and w following the opposite orientation.

If C is a cycle with a given orientation and v a vertex of C we denote by v^+ and v^- the successor and the predecessor, respectively, of the vertex v on the cycle C with respect to this orientation.

We shall use analogous notations for paths with given orientation.

The aim of this section is to prove the Loebl-Komlós-Sós conjecture for paths *i.e.* the following theorem.

Theorem 4 *If G is a graph on n vertices and it has at least $\frac{n}{2}$ vertices with the degrees at least k, then G contains a path of length at least k.*

Observe that evidently the above theorem improves the well-known Dirac's result (1952) saying that if G is a graph of minimum degree k, then G contains a path of length at least k. There is also the result of Posa [6] that if in a graph G for each subset X with $|X| \leq k$, $|N(X) - X| \geq 2|X| - 1$, then G has a path of length $3k-2$. However, this result and the other known conditions implying the existence of such a path (for instance the conditions concerning the average degree or the sum of degrees of nonadjacent vertices), could not be compared with Theorem 4.

The rest of this section is devoted to the proof of the theorem. Let n be the smallest integer such that there is an integer k and a graph on n vertices, say G, such that G satisfies the hypothesis of Theorem 4 but the conclusion does not hold. Subject to this choice we assume also that k is as small as possible. Since the claim of Theorem 4 is true for $k \leq 2$, we have $k \geq 3$. Moreover, without loss of generality we can choose a graph G of the size as small as possible.

We can suppose that G is connected. Otherwise, suppose that G has q connected components $Q_1, \ldots, Q_q$ with $n_1, \ldots, n_q$ vertices, respectively. Denote by p_i the number of vertices with the degree at least k in Q_i. Observe that if in one of q components of G the hypothesis of the theorem is satisfied (that is $p_i \geq \frac{n_i}{2}$) we can find a path with k edges in this components. Otherwise, for all i we have $p_i < \frac{n_i}{2}$ which implies $\sum_{i=1}^{q} p_i < \frac{n}{2}$, a contradiction.

We can also suppose that S is an independent set, for otherwise the graph obtained from G by removing the edges between the vertices of S also would satisfy the hypothesis of the theorem.

Finally, we can also suppose that each B-vertex of G has at most one S-neighbor with the degree one. Otherwise, let v_1, v_2 be two S-neighbors with the degree one of a vertex $b \in B$. The graph $G_1 = G - \{v_1, v_2\}$ satisfies the hypothesis of the Loebl-Komlós-Sós conjecture because G_1 has $n-2$ vertices and at least $\frac{n}{2} - 1$ vertices with the degree no less than k.

This last remark can be generalized in the following way.

Lemma 5 *Let $X \subseteq S$. Then $|X| < 2|N_G(X)|$.*

Proof. Suppose that there is a set of S-vertices X such that

$|X| \geq 2|N_G(X)|$. We will prove that in this case $G - X$ (and also G) contains a path with k edges.

Let us consider the graph $G' = G - X$ obtained from G by removing the vertices X from G. In $G - X$ the vertices of $N_G(X)$ could have the degree less than k. The number of vertices of $G - X$ with the degree at least k is at least $|B| - |N_G(X)|$.

If $|X| \geq 2|N_G(X)|$ then $2(|B| - |N_G(X)|) \geq 2|B| - |X| \geq |B| + |S| - |X|$ and thus, by the minimality of G, $G - X$ contains a path of length k. ■

Lemma 6 *There is no path P_k in G of length $k-1$ with one extremity in B and there is no path P_{k-1} of length $k-2$ with both extremities in B.*

Proof. Suppose that $P = x_1, \ldots, x_k$ is a path of G with the orientation from x_1 to x_k and such that $x_1 \in B$. Then x_1 has at least one neighbor v that is not on P and then $vx_1\overrightarrow{P}x_k$ is a path with k edges, a contradiction. The second affirmation can be deduced from the first one. ■

Lemma 7 *G contains no cycle of length k or $k-1$.*

Proof. Suppose first that C is a cycle with k vertices in G and denote by $\overrightarrow{C}$ one of its orientations. Since G is connected there is a vertex $v \in V(G)$, $v \notin V(C)$ and a vertex $w \in V(C)$ such that $vw \in E(G)$. Then $vw\overrightarrow{C}w^-$ is a path of size k in G, a contradiction.

Suppose now that C is a cycle in G of size $k-1$. Denote by $\overrightarrow{C}$ one of the orientations of C. We shall consider two cases:

Case 1. There is a B-vertex, say b, outside of C.

Since G is connected, there is a path P in G between b and a vertex x lying on C with other vertices from $G-C$. By orientating P from b to x we see that $b\overrightarrow{P}x\overrightarrow{C}x^-$ is a path in G with at least $k-1$ edges having one extremity in B, a contradiction with Lemma 6.

Case 2. All B-vertices of G are on C.

If two consecutive vertices on C, say x and x^+ are B-vertices, then the path $x^+\overrightarrow{C}x$ is of size $k-2$ and has two extremities in B. Once again we get a contradiction with Lemma 6. Otherwise, all B-vertices on C are separated by S-vertices. Hence, all vertices of G are on C. So, G is Hamiltonian and contains the paths of all length, a contradiction. ■

Lemma 8 *G contains no cycle of length $k-2$.*

Proof. Suppose, contrary to the conclusion that G contains a cycle C of size $k-2$. Denote by $\overrightarrow{C}$ one of the orientations of C. Without loss of generality we can assume that the number of B-vertices on C is as large as possible.

We shall consider two main cases.

Case 1. There is a B-vertex, say b, outside of C.

Then, since G is connected, there is a path P in G, between b and a vertex x on C. In fact, this path is of length one, since otherwise a path beginning in b and having at least $k-1$ edges would be easy to find. Suppose first that x^- is a B-vertex. Then $bx\overrightarrow{C}x^-$ is a path in G with $k-2$ edges and with two extremities in B, which is impossible by Lemma 6. Similarly we can get a contradiction if x^+ is a B-vertex. So, we can assume that if $x \in V(C)$ and $bx \in E(G)$, then both vertices x^- and x^+ belong to S. Therefore we conclude that x must be in B.

Observe now that $b(x^+)^+ \notin E(G)$. For, otherwise $b(x^+)^+\overrightarrow{C}xb$ would be a cycle of the same size but with one B-vertex (namely $(x^+)^+$) more than C, which contradicts the choice of C. So, between two neighbors of b on C there are at least three vertices which are not neighbors of b. Hence $|N_G(b) \cap V(C)| \leq \frac{k-2}{4}$ and thus $|N_{G-C}(b)| \geq \frac{3k+2}{4} \geq 2$ for $k \geq 3$. Moreover all neighbors of b that are not on C are in S, otherwise we would have a path of size $k-1$ with one extremity in B.

By Lemma 5, at least one these neighbors, say s, is not of degree one and since it is a S-vertex, all its neighbors are on $C \cup \{b\}$. Denote by v one of its neighbors different from b. Then $v^-\overleftarrow{C}vsb$ is a path of length $k-1$ with one extremity in B. Once again we get a contradiction by Lemma 6.

Case 2. All B-vertices are on C.

In this case $k-2 \geq \frac{n}{2}$ *i.e.* $k \geq \frac{n}{2}+2$. Denote by S_C the set of S-vertices on C and by S_R the set of other S-vertices. We have

$$k-2 = |B| + |S_C| \tag{1}$$

Let A be the set of these B-vertices whose successors on C are also in B, *i.e.* $A = \{x \in B \cap V(C) |\ x^+ \in B\}$.

The cardinality of A is equal with the number of edges of C with both extremities in B and this number is exactly equal to $|E(C)| - 2|S_C| = k-2-2|S_C|$.

By (1) we have

$$|A| = |B| - |S_C| \tag{2}$$

Observe now that each B-vertex $b \in V(C)$ has at least three S-neighbors outside of C. Let b_1 and b_2 be two vertices of A and suppose that there exists a vertex s being the common neighbor of b_1 and b_2 outside of C (see Figure 1). Then $b_1^+ \overrightarrow{C} b_2 s b_1 \overleftarrow{C} b_2^+$ is a path with $k-2$ edges with two extremities in B which is impossible by Lemma 6.

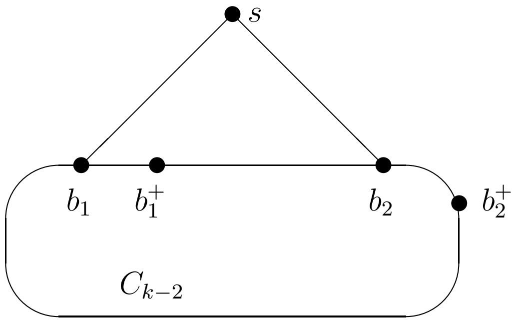

Figure 1

Denote by $\tilde{N}_G(x) = N_G(x) - V(C)$ i.e. the neighbors of x which are outside of the cycle C. So, for each $b_1, b_2 \in A$ we have: $\tilde{N}_G(b_1) \cap \tilde{N}_G(b_2) = \emptyset$. Thus

$$\tilde{N}_G(A) = \bigcup_{b \in A} \tilde{N}_G(b) \subseteq S_R.$$

Since $|\tilde{N}_G(b)| \geq 3$ for every B-vertex b we get $3|A| \leq |S_R|$.

Using (2) we obtain $3|B| - 3|S_C| \leq |S_R|$. Hence

$$3|B| \leq |S_R| + |S_C| + 2|S_C| = |S| + 2|S_C| = n - |B| + 2|S_C|$$

and we have

$$n \geq 4|B| - 2|S_C| \geq 2n - 2|S_C|.$$

Finally $|S_C| \geq \frac{n}{2}$. Since the cycle C contains also at least $n/2$ B-vertices, thus C is a Hamiltonian cycle and G has the paths of all lengths, a contradiction. ■

Proof of Theorem 4.

We can assume that G contains a path of length $k-1$ because of the choice of n and k. By Lemma 6 this path has its two extremities in S. By removing these two extremities we get a path we shall denote by P. Observe that P is a path of length $k-3$ and has its both extremities in B. Denote these vertices by b_1 and b_2, respectively and let $\overrightarrow{P}$ be the orientation of P from b_1 to b_2.

It is easy to see that both b_1 and b_2 have at least three neighbors outside of P because they have at most $k-3$ neighbors on the path P. Moreover, the neighbors of b_1 and b_2 outside of P are S-vertices. For, otherwise, if for example b_1 has a neighbor b in B, $b \notin V(P)$ then $bb_1\overrightarrow{P}b_2$ is a path with $k-2$ edges with both extremities in B.

Denote by W_1 and W_2 the set of these neighbors of b_1 and b_2, respectively, which are not vertices of the path P. By Lemma 8, $W_1 \cap W_2 = \emptyset$. As remarked above, these two sets contain only S-vertices. So, all the neighbors of the vertices belonging to W_1 or W_2 are in B, and, in consequence are on P. Denote by A_1 the set of all neighbors of vertices of W_1 except b_1, $A_1 = N_G(W_1) - \{b_1\}$ and, similarly, let $A_2 = N_G(W_2) - \{b_2\}$. Let us put $B_1 = A_1 \cup A_2$.

We claim that for each $b \in A_1$ its predecessor $b^- \in S$. Otherwise, suppose that $b \in N_G(v)$ where $v \in W_1$. Then $b^-\overleftarrow{P}b_1vb\overrightarrow{P}b_2$ is a path of length $k-2$ and having its both extremities in B, a contradiction. By the same argument we can show that for each $b \in A_2$ its successor b^+ is in S. We put $A_1^- = \{b^- \mid b \in A_1\}$, $A_2^+ = \{b^+ \mid b \in A_2\}$ and $S_1 = A_1^- \cup A_2^+$. In other words, each B-vertex belonging to B_1 generates one S-vertex belonging to S_1. We shall show that distinct vertices of B_1 generate distinct vertices of S_1. Suppose now that there exists a vertex s such that $s \in A_1^- \cap A_2^+$ where $s^+ \in N_G(v)$ with $v \in W_1$ and $s^- \in N_G(w)$ with $w \in W_2$ (see Figure 2).

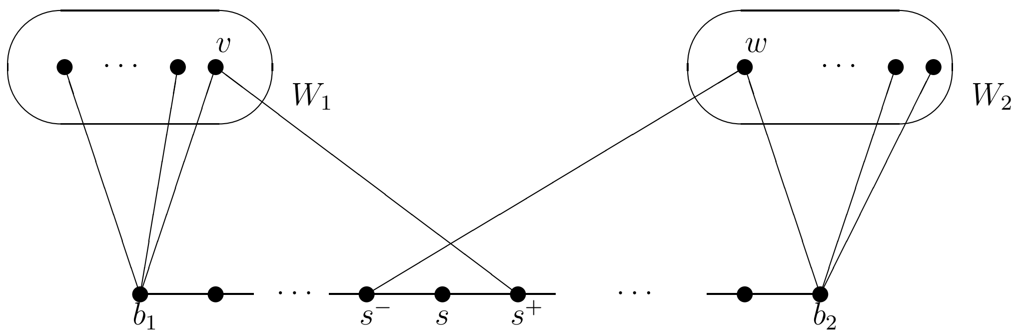

Figure 2

Then $s^+\overrightarrow{P}b_2ws^-\overleftarrow{P}b_1vs^+$ is a cycle of length $k-1$ contradicting Lemma 7. Hence $A_1^- \cap A_2^+ = \emptyset$. Thus we have

$$|A_1^-| + |A_2^+| = |S_1| = |B_1|$$

By Lemma 8 there is no vertex $v \in V(P)$ such that $b_1v \in E(G)$ and $v^-b_2 \in E(G)$, for, otherwise $v\overrightarrow{P}b_2v^-\overleftarrow{P}b_1v$ would be a cycle of size $k-2$. For the same reason $b_1b_2 \notin E(G)$.

Then $|N_G(b_1) \cap V(P)| + |N_G(b_2) \cap V(P)| \leq |P|-1 = k-3$. Since $b_1, b_2 \in B$ this implies that $|W_1| + |W_2| \geq k+3$. Using Lemma 5 with $X = W_1 \cup W_2$ we get $2 + |B_1| > \frac{|W_1|+|W_2|}{2} \geq \frac{k+3}{2}$.

Hence we obtain $|B_1| + |S_1| > k+3-4 = k-1$. Therefore, together with b_1 and b_2, the path P has at least $k+1$ vertices, a contradiction. ■.

Corollary 9 *Let n and k be two integers, $k \leq n-1$ and let G be a graph on n vertices with at least $\frac{n}{2}$ vertices of degree at least k. For any three integers p, q, r such that $p+q+r = k$ denote by $T(p,q,r)$ the tree obtained from the path $P = x_0, \ldots, x_p, x_{p+1}, \ldots, x_{p+q}$ of length $p+q$ by adding r new vertices $y_1, \ldots, y_r$ and r new edges x_py_i, $i = 1, \ldots, r$. Then G contains $T(p,q,r)$.*

Proof. Denote by G' the graph obtained from G by removing all edges between the vertices of the set S. By Theorem 4, G' contains a path of length k. Denote by $z_0, \ldots, z_k$ the vertices of this path. Since the set S is independent, either z_p or z_{p+1} must be in B. Suppose that $z_p \in B$ and consider the path $P = z_0, \ldots, z_{p+q}$. Since $z_p \in B$ it has at least $k-p-q = r$ neighbors outside of P. Now it is easy to define a subgraph of G' that is isomorphic to the tree $T(p,q,r)$.

If $z_{p+1} \in B$, we repeat the above reasoning with the path P defining by $P = z_1, \ldots, z_{p+q+1}$. ■

Remark. The following example, given in [5], shows that if the Loebl-Komlós-Sós conjecture is true, then, in general, the condition on the number of vertices in B is the best possible. We put $n = 2m+2$, $|B| = m$ and $|S| = m+2$. The graph G is defined as the join between the complete graph on m vertices with vertex set B and the set of independent vertices S. Then, each vertex of B is of degree $n-1$, however G contains no path of length $n-1$.

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