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# Degree-anonymization using edge rotations

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ABSTRACT

The Min Anonymous-Edge-Rotation problem asks for an input graph $G$ and a positive integer $k$ to find a minimum number of edge rotations that transform $G$ into a graph such that for each vertex there are at least $k-1$ other vertices of the same degree (a $k$-degree-anonymous graph). In this paper, we prove that the Min Anonymous-Edge-Rotation problem is NP-hard even for $k = n/q$, where $n$ is the order of a graph and $q$ any positive integer, $q \geq 3$. We argue that under some constrains on the number of edges in a graph and $k$, Min Anonymous-Edge-Rotation is polynomial-time 2-approximable. Furthermore, we show that the problem is solvable in polynomial time for any graph when $k = n$ and for trees when $k = \theta(n)$. Additionally, we establish sufficient conditions for an input graph and $k$ such that a solution for Min Anonymous-Edge-Rotation exists.



## 1. Introduction

In recent years huge amounts of personal data has been collected on various networks as e.g. Facebook, Instagram, Twitter or LinkedIn. Ensuring the privacy of network users is one of the main research tasks. One possible model to formalise these issues was introduced by Liu and Terzi [18] who transferred the $k$-degree-anonymity concept from tabular data in databases [11] to graphs which are often used as a representation of networks. Following this study a graph is called $k$-degree-anonymous if for its each vertex there are at least $k-1$ other vertices with the same degree. The parameter $k$ represents the number of vertices that are mixed together and thus the increasing value of $k$ increases the level of anonymity. In [21], Wu et al. presented a survey of different anonymization models and some of their weaknesses. Casas Roma et al. [4] proposed a survey of several graph-modification techniques for privacy-preserving on networks. In this paper we consider the $k$-degree-anonymous concept of Liu and Terzi [18].

The main study problem related to $k$-degree anonymous graphs is to find a minimum number of graph operations to transform an input graph to a $k$-degree anonymous graph.

Different graph operations of transforming a graph into a $k$-degree-anonymous one are considered in research papers where the operations maybe the following: delete vertex/edge, add vertex/edge, or add/delete edge (see more details later). One advantage in the approaches based on vertex/edge deletion/adding is that a solution always exists since in the worst case scenario one can consider the empty or the complete graph that is $k$-degree-anonymous for any $k$ (at most the number of vertices of the graph). However, the basic graph parameters as the number of vertices and edges could be modified with such transformations.

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Vertex/edge modification versions associated to $k$-degree-anonymity have been relatively well studied. Hartung et al. [15,16] studied the edge adding modification as proposed by Liu and Terzi [18]. For this type of modification Chester et al. [8] established a polynomial time algorithm for bipartite graphs.

The variant of adding vertices instead of edges was studied by Chester et al. in [7] where they presented an approximation algorithm with an additive error. Bredereck et al. [2] investigated the parameterized complexity of several variants of vertex adding which differ in the way the inserted vertices can be adjacent to existing vertices. Concerning the vertex deletion variant, Bazgan et al. [1] showed the NP-hardness even on very restricted graph classes such as trees, split graphs, or trivially perfect graphs. Moreover, in [1] the vertex and edge deletion variants are proved intractable from the approximability and parameterized complexity point of view.

Several papers study the basic properties of edge rotations, including some bounds for the minimum number of edge rotations between two graphs [5,6,10,13,17].

In this paper we consider the version of transforming a graph into a $k$-degree-anonymous one using edge rotations which don't modify the number of vertices/edges. It should be noticed that in such case a solution may not always exist, as we discuss in Section 3.

To the best of our knowledge the problem of transforming a graph to a $k$-degree anonymous graph using the edge rotations has not been fully explored. In some particular cases some research has been done in [19] where the authors study the edge rotation distance and various metric between the degree sequences to find a "closest" regular graph. In paper [3] the authors proposed an heuristic to compute the edge rotation distance to a $k$-degree anonymous graph.

*Our results.* In this paper we study the various aspects of the Min Anonymous-Edge-Rotation problem. An input to the problem is an undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges and an integer $k \le n$. The goal is to find a shortest sequence of edge rotations that transforms $G$ into a $k$-degree-anonymous graph, if such a sequence exists. We first show that when $\frac{n}{2} \le m \le \frac{n(n-3)}{2}$ and $k \le \frac{n}{4}$ a solution always exists. Moreover for trees a solution exists if and only if $\frac{2m}{n}$ is an integer. We prove that Min Anonymous-Edge-Rotation is NP-hard even when $k = \frac{n}{q}$ and $q \ge 3$ is a fixed positive integer. On the positive side we provide a polynomial-time 2-approximable algorithm under some constraints. Finally, we demonstrate that Min Anonymous-Edge-Rotation is solvable in polynomial time for trees when $k = \theta(n)$ and for any graph when $k = n$.

Our paper is organized as follows. Some preliminaries about edge rotations and our formal definitions are given in Section 2. The study of feasibility is initiated in Section 3. Section 4 presents the NP-hardness proof. In Section 5 we study properties of the specific $k$-degree anonymous degree sequences that are used in Section 6 to present a polynomial-time 2-approximation algorithm and in Section 7 to establish a polynomial time algorithm for trees. Moreover in Section 7 we consider the case $k = n$ in general graphs. Some conclusions are given at the end of the paper.

## 2. Preliminaries

In this paper we assume that all graphs are undirected, without loops and multiple edges, and not necessary connected graphs.

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $deg_G(v)$ be the degree of $v$ in $G$, and $\Delta_G$ be the maximum degree of $G$. A vertex $v$ with degree $deg_G(v) = |V| - 1$ is called a universal vertex. The neighbourhood of $v$ in $G$ is denoted by $\mathcal{N}_G(v) = \{u \in V : uv \in E\}$ and $Inc_G(v)$ is the set of all edges incident to $v$, $Inc_G(v) = \{e \in E : v \in e\}$. If the underlying graph $G$ is clear from the context, we omit the subscript $G$.

**Definition 1.** Given a graph $G = (V, E)$ of order $n$, the degree sequence $S_G$ of $G$ is the non-increasing sequence of its vertex degrees, $S_G = (deg(v_1), \dots, deg(v_n))$, where $deg(v_1) \ge deg(v_2) \ge \dots \ge deg(v_n)$.

**Definition 2.** A sequence $D$ of non-negative integers $D = (d_1, d_2, \dots, d_n)$ is **graphic** if there exists a graph $G$ such that its degree sequence coincides with $D$.

As follows from Erdős-Gallai theorem (see e.g. [9]) the necessary and sufficient conditions for a non-increasing sequence $D = (d_1, d_2, \dots, d_n)$ to be graphic are:

$$\sum_{i=1}^{n} d_i \text{ is even} \tag{1}$$

$$\sum_{i=1}^{\ell} d_i \le \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min(d_i, \ell) \text{ holds for any } 1 \le \ell \le n. \tag{2}$$

Furthermore, it is an easy exercise to prove that a sequence of integers $D = (d_1, d_2, \dots, d_n)$ corresponds to a degree sequence of a tree on $n$ vertices if and only if each $d_i \ge 1$ and $\sum_{i=1}^{n} d_i = 2(n-1)$.

Let $\mathbf{G}(n, m)$ be the set of all graphs with $n$ vertices and $m$ edges.

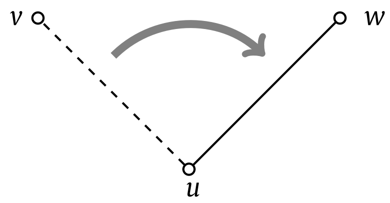


**Fig. 1.** An edge rotation $(uv, uw)$ from $uv$ to $uw$.

**Definition 3.** Let $G, G' \in \mathbf{G}(n, m)$. We say that $G'$ can be obtained from $G$ by an **edge rotation** $(uv, uw)$ if $V(G) = V(G')$ and there exist three distinct vertices $u$, $v$ and $w$ in $G$ such that $uv \in E(G)$, $uw \notin E(G)$, and $E(G') = (E(G) \setminus \{uv\}) \cup \{uw\}$, see Fig. 1.

**Remark 1.** Let $G$ be a graph. For the vertices $u$, $v$, $w$ in $G$ the edge rotation $(uv, uw)$ modifies $G$ into the graph $G'$ such that $deg_{G'}(v) = deg_G(v) - 1$, $deg_{G'}(w) = deg_G(w) + 1$, and the degree of the other vertices is not changed. Let define a $(+1, -1)$-degree modification of the degree sequence $D = (d_1, \ldots, d_n)$ in such a way that $d_i := d_i + 1$, $d_j := d_j - 1$ for any two indices $i, j$ such that $i, j \in \{1, \ldots, n\}$. Note that each edge rotation corresponds to a $(+1, -1)$-degree modification, but not opposite.

**Definition 4.** A sequence of integers $D = (d_1, d_2, \ldots, d_n)$ is called ***k*-anonymous** where $k \in \{1, \ldots, n\}$, if for each element $d_i$ from $D$ there are at least $k - 1$ other elements in $D$ with the same value. A graph $G$ is called ***k*-degree-anonymous** if its degree sequence is ***k*-anonymous**. The vertices of the same degree correspond to a **degree class**.

In this paper we study the following anonymization problem:

Min Anonymous-Edge-Rotation
**Input:** $(G, k)$ where $G = (V, E)$ is an undirected graph and $k$ a positive integer, $k \in \{1, \ldots, |V|\}$.
**Output:** If there is a solution, find a sequence of a minimum number $\ell + 1$ of graphs $G_0 = G, G_1, G_2, \ldots, G_\ell$ such that $G_{i+1}$ can be obtained from $G_i$ by one edge rotation, and $G_\ell$ is $k$-degree-anonymous.

Note that a solution to the Min Anonymous-Edge-Rotation problem may not exist for all instances. For example, if $G$ is a complete graph without an edge, $K_n \setminus \{e\}$, $n \geq 6$, then there is no solution for such graph $G$ and $k = 3$. Therefore, we are only interested in studying of **feasible instances** $(G, k)$ defined as an instance for which there exists a solution to Min Anonymous-Edge-Rotation. Our initial study of sufficient conditions for feasibility is presented in Section 3.

Obviously, since all graphs are 1-degree-anonymous, we are only interested in cases where $k \geq 2$.

The decision version associated to Min Anonymous-Edge-Rotation is defined as follows for a feasible instance $(G, k)$:

Anonymous-Edge-Rotation
**Input:** $(G, k, r)$ where $G = (V, E)$ is an undirected graph, $k \in \{1, \ldots, |V|\}$, and $r$ be a positive integer.
**Question:** Is there a sequence of $\ell + 1$ graphs $G_0 = G, G_1, G_2, \ldots, G_\ell$ such that $\ell \leq r$, $G_{i+1}$ can be obtained from $G_i$ by one edge rotation, and $G_\ell$ is $k$-degree-anonymous?

We also consider the Min Anonymous-Edge-Rotation problem in restricted graph classes, e.g. trees. In that case we require that all graphs in the sequence $G_0, \ldots, G_\ell$ must be from the same graph class. Note that the problem can also be studied without this requirement, but the results may be different.

The following theorem shows important properties about the edge rotations. The result was already proved in [6], but due to the simplicity of our approach, we present another proof here.

**Theorem 1.** *For any two graphs $G, G' \in \mathbf{G}(n, m)$, we can transform $G$ to $G'$ using a sequence of edge rotations.*

**Proof.** Let $E_1 = E(G) \setminus (E(G) \cap E(G'))$ be the set of edges that are in $G$ and not in $G'$ and $E_2 = E(G') \setminus (E(G) \cap E(G'))$ be the set of edges that are in $G'$ and not in $G$. For all $u$, $v$ and $w$ such as $uv \in E_1$ and $uw \in E_2$, we add one edge rotation $(uv, uw)$. In all other cases, let $uv \in E_1$ and $u'v' \in E_2$, where all vertices $u, v, u', v'$ are distinct. There are two case: 1) $uu'$, $uv'$, $vu'$ and $vv' \in E(G)$ or 2) at least one of these four edges is missing.

In the first case we can make the following two edge rotations to move $uv$ from $G$ to $u'v'$ in $G'$: $(v'v, v'u')$ and $(vu, vv')$ (see Fig. 2). In the second case, if for example $vv'$ is missing, we can use the following two rotations $(vu, vv')$ and then $(v'v, v'u')$ (see Fig. 3) and similarly if another edge is missing. □

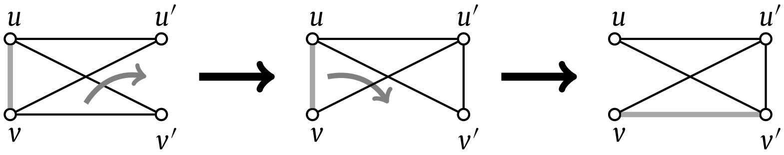


**Fig. 2.** Case 1.

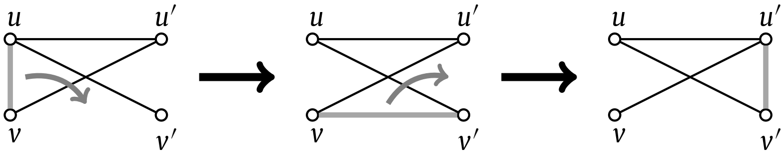


**Fig. 3.** Case 2.

**Corollary 1.** *For any two graphs $G, G' \in \mathbf{G}(n,m)$, the edge distance between $G$ and $G'$ is bounded by $2m$.*

## 3. Feasibility study

As it was discussed in Section 2, the Min Anonymous-Edge-Rotation problem does not have a solution for every input instance. It is not difficult to see that if a graph is 'almost' complete or 'almost' empty, then there are only restricted options on the number of different degree classes and therefore a solution may not exist.

First we present some sufficient conditions for an instance to be feasible showing that if a graph is not 'almost' complete or an empty graph, then a solution of the problem exists for all $k \le \frac{n}{4}$, where $n$ is the order of the graph.

**Theorem 2.** *Let $G \in \mathbf{G}(n,m)$ such that $\frac{n}{2} \le m \le \frac{n(n-3)}{2}$ and $n \ge 4$. Then there exists a feasible solution for the* Min Anonymous-Edge-Rotation *problem, hence a $k$-degree-anonymous graph $G' \in \mathbf{G}(n,m)$, for any $k \le \frac{n}{4}$.*

**Proof.** Let $m$, $n$, $k$ be fixed. Any graph $G \in \mathbf{G}(n,m)$ is a 1-degree-anonymous graph, hence we can suppose $k \ge 2$.

In the first part of the proof we describe a construction of a $k$-anonymous sequence $D = (d_1, d_2, \ldots, d_n)$ with property $\sum_{i=1}^{n} d_i = 2m$ for any $m$, $n$, $k$ satisfying the restriction of the theorem. In the second part we show that the sequence $D$ is graphic, hence that the sequence satisfies the conditions (1) and (2) from Section 2.

As $\sum_{i=1}^{n} d_i = 2m$ is the condition for a constructed sequence, the property (1) trivially holds.

Now we construct three distinct $k$-anonymous sequences Type 1, 2, 3 of integers based on the values of $k$ and $s \equiv 2m \bmod n$. Denote by $d$ the average degree of the graph $G$ defined as $d = \left\lfloor \frac{2m}{n} \right\rfloor$.

**Type 1:** $k \le s \le n-k$

Let $D_1 = (d_1^1, d_2^1, \ldots, d_s^1, d_1^2, d_2^2, \ldots, d_{n-s}^2)$ be a sequence of positive integers where for all $i$, $1 \le i \le s$, $d_i^1 = d+1$ and for all $j$, $1 \le j \le n-s$, $d_j^2 = d$ (see Fig. 4). The sequence contains $n$ elements and it is easy to see that $\sum_{i=1}^{s}(d+1) + \sum_{j=1}^{n-s} d = 2m$.

Following the assumptions $s \ge k$ and $n-s \ge k$, therefore $D_1$ is a $k$-anonymous sequence.

**Type 2:** $s < k$

Let $D_2 = (d_1^1, d_2^1, \ldots, d_{s+k}^1, d_1^2, d_2^2, \ldots, d_{n-s-2k}^2, d_1^3, d_2^3, \ldots, d_k^3)$ be a sequence of positive integers where for all $i$, $1 \le i \le s+k$, $d_i^1 = d+1$; for all $r$, $1 \le r \le n-s-2k$, $d_r^2 = d$; for all $j$, $1 \le j \le k$, $d_j^3 = d-1$ (see Fig. 5). The sequence contains $n$ elements and $\sum_{i=1}^{s+k}(d+1) + \sum_{j=1}^{k}(d-1) + \sum_{\ell=1}^{n-s-2k} d = 2m$.

Since $n \ge 4k$ and $s < k$, $n-s-2k \ge k$, $D_2$ is a $k$-anonymous sequence.

**Type 3**: $s > n-k$

Let $D_3 = (d_1^1, d_2^1, \ldots, d_k^1, d_1^2, d_2^2, \ldots, d_{s-2k}^2, d_1^3, d_2^3, \ldots, d_{k+n-s}^3)$ be a sequence of positive integers where for all $i$, $1 \le i \le k$, $d_i^1 = d+2$; for all $r$, $1 \le r \le s-2k$, $d_r^2 = d+1$; for all $j$, $1 \le j \le k+n-s$, $d_j^3 = d$ (see Fig. 6). The sequence has $n$ elements and $\sum_{i=1}^{k}(d+2) + \sum_{j=1}^{k+n-s} d + \sum_{\ell=1}^{s-2k}(d+1) = 2m$.

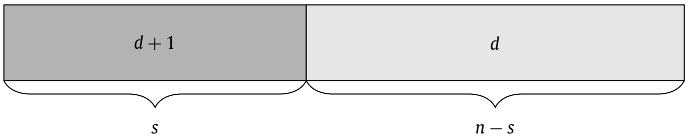


**Fig. 4.** The sequence $D_1$.

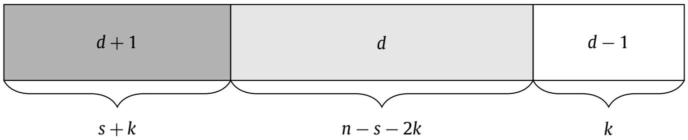


**Fig. 5.** The sequence $D_2$.

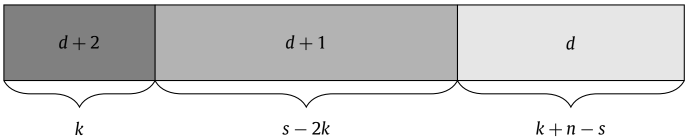


**Fig. 6.** The sequence $D_3$.

Because $n > s$, the number $d$ appears more than $k$-times in $D_3$. Due to the assumptions $n \geq 4k$ and $s > n-k$, also $s-2k \geq k$. Hence $D_3$ is a $k$-anonymous sequence.

Now we show that all three sequences are graphic, therefore that the condition (2) is true for any $\ell$. We split the proof into several subcases depending on the value of $\ell$ and the type of the sequence.

From our assumptions $\frac{n}{2} \leq m \leq \frac{n(n-3)}{2}$, it follows $1 \leq d \leq n-3$.

*Case A.* $\ell = 1$
Because $d \geq 1$, (2) trivially holds.

*Case B.* $\ell = 2$, due to $n \geq 8$, $\sum_{j=\ell+1}^{n} \min(d_j, \ell) \geq 6d-2$

$$\sum_{i=1}^{\ell} d_i \leq \ell(d+2) = 2(d+2) \leq 2 + (6d-2) \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$$

*Case C.* $3 \leq \ell < d$

$$\sum_{i=1}^{\ell} d_i \leq \ell(d+2) \leq \ell(n-1) = n\ell - \ell = \ell^2 - \ell + n\ell - \ell^2 = \ell(\ell-1) + (n-\ell)\ell \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$$

*Case D.* $3 \leq \ell = d$,
Type 1 & 3:

$$\sum_{i=1}^{\ell} d_i \leq \ell(d+2) \leq \ell(n-1) = n\ell - \ell = \ell^2 - \ell + n\ell - \ell^2 = \ell(\ell-1) + (n-\ell)\ell \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$$

Type 2, following our assumptions also $\ell = d \leq n-3$

$$\sum_{i=1}^{\ell} d_i \leq \ell(d+1) = \ell(\ell+1) = \ell(\ell-1) + 2\ell \leq \ell(\ell-1) + 3\ell - 3 = \ell(\ell-1) + 3(\ell-1) = \ell(\ell-1) + (\ell+3-\ell)(\ell-1) \leq \ell(\ell-1) + (n-\ell)(\ell-1) \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$$

*Case E.* $3 \leq \ell = d+1$. Furthermore, $\ell = d+1 \leq n-2$.
Type 1 & 2, $\ell \geq 4$:

$$\sum_{i=1}^{\ell} d_i \leq \ell(d+1) = \ell^2 = \ell(\ell-1) + \ell \leq \ell(\ell-1) + 2\ell - 4 = \ell(\ell-1) + 2(\ell-2) = \ell(\ell-1) + (\ell+2-\ell)(\ell-2) \leq \ell(\ell-1) + (n-\ell)(\ell-2) = \ell(\ell-1) + (n-\ell)(d-1) \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$$

Type 1 & 2, $\ell = 3$:
Due to $n \geq 8$, $\sum_{j=\ell+1}^{n} \min(d_j, \ell) \geq 5d - 4 \geq \ell$

Therefore $\sum_{i=1}^{\ell} d_i \leq \ell(d+1) = \ell^2 = \ell(\ell-1) + \ell \leq \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

Type 3, $3 \le \ell \le n-3$

$\sum_{i=1}^{\ell} d_i \le \ell(d+2) = \ell(\ell+1) = \ell(\ell-1) + 2\ell \le \ell(\ell-1) + 3\ell - 3 = \ell(\ell-1) + 3(\ell-1) = \ell(\ell-1) + (\ell+3-\ell)(\ell-1) \le \ell(\ell-1) + (n-\ell)(\ell-1) \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

Type 3, $\ell = n-2$

$\sum_{i=1}^{\ell} d_i = k(d+2) + (s-2k)(d+1) + (k+n-s-2)d = nd - 2d + s \le d(n-2) + n - 1 \le (n-3)(n-2) + n - 1 \le \ell(\ell-1) + 2d = \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$.

*Case F.* $3 \le \ell = d+2$. Furthermore, $\ell = d+2 \le n-1$.

Type 1 & 2:

$\sum_{i=1}^{\ell} d_i \le \ell(d+1) \le \ell(\ell-1) \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

Type 3, $\ell = 3$:

Due to $n \ge 8$, $\sum_{j=\ell+1}^{n} \min(d_j, \ell) \ge 5 \ge \ell$. Then

$\sum_{i=1}^{\ell} d_i \le \ell(d+2) = \ell^2 = \ell(\ell-1) + \ell \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

Type 3, $4 \le \ell \le n-2$:

$\sum_{i=1}^{\ell} d_i \le \ell(d+2) = \ell^2 = \ell(\ell-1) + \ell \le \ell(\ell-1) + 2\ell - 4 = \ell(\ell-1) + 2(\ell-2) = \ell(\ell-1) + (\ell+2-\ell)(\ell-2) \le \ell(\ell-1) + (n-\ell)(\ell-2) \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

Type 3, $\ell = n-1$:

$\sum_{i=1}^{\ell} d_i = k(d+2) + (s-2k)(d+1) + (k+n-s-1)d = s + nd - d \le n - 1 + (n-1)(n-3) = (n-1)(n-2) = \ell(\ell-1) \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$.

*Case G.* $d+2 < \ell < n$

$\sum_{i=1}^{\ell} d_i \le \ell(d+2) \le \ell(\ell-1) \le \ell(\ell-1) + \sum_{j=\ell+1}^{n} \min(d_j, \ell)$

*Case H.* $\ell = n$

$\sum_{i=1}^{\ell} d_i \le \ell(d+2) \le \ell(\ell-1)$

Therefore, we have proved that there exists a $k$-degree-anonymous graph $G' \in \mathbf{G}(n,m)$ and the graph $G$ can be transformed to $G'$ using a sequence of edge rotations due to Theorem 1. □

Now we extend the feasibility study to the case $k = n$ for which we get necessary and sufficient conditions.

**Theorem 3.** *Let $G \in G(n,m)$ for some positive integers n and m. Then $(G,n)$ is a feasible instance of* Min Anonymous-Edge-Rotation *if and only if $\frac{2m}{n}$ is an integer.*

**Proof.** Since $k = n$ in Min Anonymous-Edge-Rotation, every vertex has to be in the same degree class, so if there is a solution, the resulting graph has to be regular. Moreover, a necessary and sufficient condition for a $p$-regular graph with $n$ vertices to exist is that $n \ge p+1$ and $np$ must be even [20].

If $\frac{2m}{n}$ is not an integer then obviously there is no regular graph in $G(n,m)$ and therefore $(G,n)$ is not a feasible instance.

If $\frac{2m}{n}$ is an integer, since $n \times \frac{2m}{n} = 2m$ is even, $n \ge \frac{2m}{n} + 1$ there is a $\frac{2m}{n}$-regular graph in $G(n,m)$ as it was mentioned before. By Theorem 1 we conclude that there exists a sequence of edge rotations that leads to a $\frac{2m}{n}$-regular graph starting from $G$. □

## 4. Hardness of Min Anonymous-Edge-Rotation

In this section we show that the decision version of Min Anonymous-Edge-Rotation, the problem Anonymous-Edge-Rotation, is NP-hard. The proof is based on a reduction from the restricted version of a cover set problem, Exact Cover By 3-Sets, which is known to be NP-complete ([12]).

| |
|---|
| Exact Cover By 3-Sets (X3C) |
| **Input:** A set $X$ of elements with $|X| = 3m$ and a collection $C$ of 3-elements subsets of $X$ where each element appears in exactly 3 sets. |
| **Question:** Does $C$ contain an exact cover for $X$, i.e. a subcollection $C' \subseteq C$ such that every element occurs in exactly one member set of $C'$? |

**Remark 2.** Note that $|C| = 3m$ and we can suppose that $m$ is even and larger than 6. If $m$ is odd, we consider the instance $I_{even}$ defined as follows: $X_{even} = X \cup \{x' \mid x \in X\}$ and $C_{even} = C \cup \{c_{x'y'z'} \mid c_{xyz} \in C\}$, and thus in the new instance $I_{even}$ the set has $6m$ elements and the collection has $6m$ 3-elements subsets.

We define a polynomial-time reduction and then prove the NP-hardness of Anonymous-Edge-Rotation.

**Reduction.** Let $I = (X, C)$ be an instance of $X3C$ with $|X| = |C| = 3m$ and $m$ even and $q \geq 3$ a given constant. We describe the construction $\sigma$ transforming an instance $I$ into the graph $G := \sigma(I)$ where $G = (V, E)$ is defined as follows:

- For each element $x \in X$, we add a vertex $v_x$ to the set $V_{elem} \subset V$ and a vertex $u_x$ to the set $V_{hub} \subset V$.
- For each 3-element set $\{x, y, z\}$ of the collection $C$, we add 4 vertices $c^1_{xyz}$, $c^2_{xyz}$, $c^3_{xyz}$ and $c^4_{xyz}$ to the set $V_{set} \subset V$.
- For each $i \in \{1, \ldots, 5m\}$ we add a vertex $w_i$ to the set $V_{reg} \subset V$ and for each $j \in \{1, \ldots, 10m\}$ we add a vertex $t_j$ to $V_{single} \subset V$.

Let $V^- = V_{elem} \cup V_{hub} \cup V_{set} \cup V_{reg} \cup V_{single}$ and $|V^-| = 3m + 3m + 12m + 15m = 33m$. If $q = 3$, then let $V = V^-$. If $q \geq 4$, then for each $i$, $4 \leq i \leq q$, add a set of $11m$ vertices denoted $V^i_{dummy}$. Let $V_{dummy} = V^4_{dummy} \cup \cdots \cup V^q_{dummy}$ and define $V = V^- \cup V_{dummy}$. Obviously, $|V| = 33m + (q - 3)11m$.

Now we define the set $E$ of the edges in $G$.

- For all $x, y \in X$, such that $x \neq y$, we add the edge $v_x u_y$ between the vertex $v_x \in V_{elem}$ and $u_y \in V_{hub}$, to $E_X \subset E$.
- For each 3-element set $\{x, y, z\}$ of the collection $C$, $\forall i \in \{1, 2, 3, 4\}$, we add the edges $c^i_{xyz} u_x$, $c^i_{xyz} u_y$ and $c^i_{xyz} u_z$ to the set $E_C \subset E$.
- We add the set of edges $E' \subset E$ to the vertex set $V_{elem}$ such that $(V_{elem}, E')$ is a 11-regular graph. Since the number of vertices in the set $|V_{elem}| = 3m$ is even ($m$ is even) and $11 < 3m$ such a regular graph exists [20]. Furthermore, such a graph can be constructed in polynomial time using Havel-Hakimi algorithm [14].
- We add the set of the edges $E'' \subset E$ to the vertex set $V_{reg}$ such that $(V_{reg}, E'')$ is a $(3m + 11)$-regular graph. Since the number of vertices of $V_{reg}$ is even and $3m + 11 < 5m$, similarly to the previous case such a regular graph exists and can be constructed in polynomial time.

Finally, let $E^- = E_X \cup E_C \cup E' \cup E''$. If $q = 3$, then let $E = E^-$. If $q \geq 4$, then the set $E$ contains $E^-$ and for any $i$, such that $4 \leq i \leq q$, we add the set of edges $E^i_{dummy} \subseteq E$ to the vertex set $V^i_{dummy}$ such that $(V^i_{dummy}, E^i_{dummy})$ is a $(9m + 12)$-regular. Since the number of vertices of $V^i_{dummy}$ is even ($m$ is even) and $9m + 12 \leq 11m$, similarly to the previous case such a regular graph exists and can be constructed in polynomial time.

Obviously, the graph $G = (V, E)$ has the following properties: (i) $10m$ vertices of degree 0 (the vertices of the set $V_{single}$), (ii) $12m$ vertices of degree 3 (the vertices of the set $V_{set}$), (iii) $8m$ vertices of degree $3m + 11$ (the vertices of the set $V_{reg}$ and $V_{hub}$), (iv) $3m$ vertices of degree $3m + 10$ (the vertices of the set $V_{elem}$), (v) $(q - 3)11m$ vertices of degree $(9m + 12)$ (the vertices of the set $V_{dummy}$).

**Example.** Fig. 7 represents the transformation $\sigma$ for $q = 3$. Let $I_1$ be the following instance of X3C: $m = 2$, $X = \{1, 2, 3, 4, 5, 6\}$, and $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{1, 5, 6\}, \{1, 2, 6\}\}$. To simplify the figure, we only consider $m = 2$, but for the construction $m$ must be at least 6 (due to an $(3m + 11)$-regular graph on the vertex set of $V_{reg}$).

**Theorem 4.** Anonymous-Edge-Rotation *is NP-hard even in case* $k = \frac{n}{q}$ *where* $n$ *is the order of the graph* $G$ *for an input instance* $(G, k, r)$ *and* $q$ *is a fixed number greater than or equal to* 3.

**Proof.** Let $C' \subseteq C$ be an exact cover for $X$ of size $m$. Now we define $3m$ rotations which are independent from each other: for every 3-element set $\{x, y, z\} \in C'$, we replace the edge $u_x c^1_{xyz}$ by the edge $u_x v_x$, and similarly $u_y c^1_{xyz}$ by $u_y v_y$ and $u_z c^1_{xyz}$

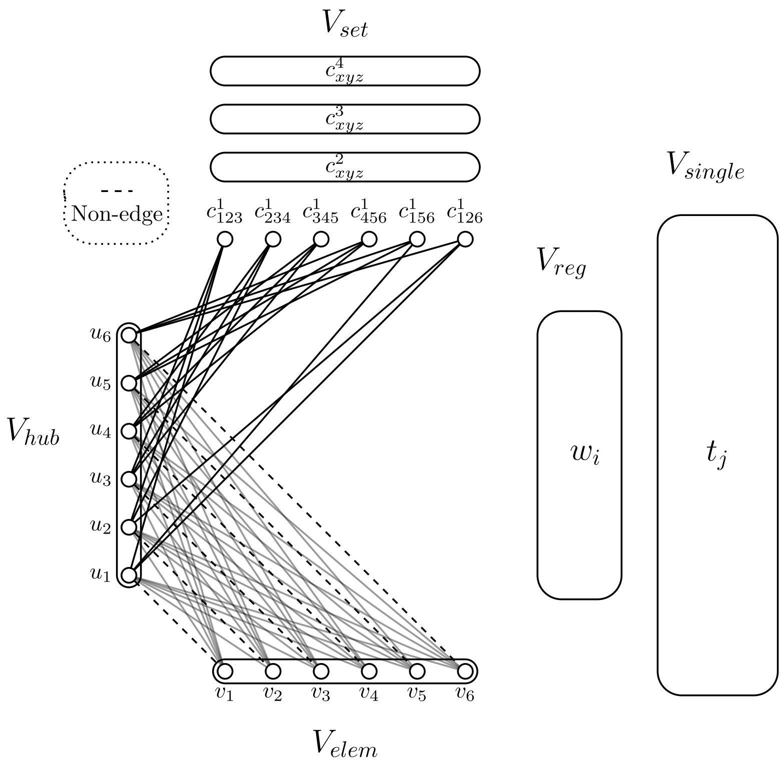


**Fig. 7.** $\sigma(I_1)$.

by $u_z v_z$. Since $C'$ is of size $m$, we define exactly $3m$ rotations. Let $G'$ be the graph obtained from $G$ after applying all $3m$ rotations. Since $C'$ is an exact cover of size $m$: (i) there are $m$ vertices of type $c^1_{xyz}$ that lost all 3 neighbours and become of degree 0 in $G'$, (ii) all $3m$ vertices of type $v_x$ are attached to a new neighbour, so they become of degree $3m+11$ in $G'$.

Then $G'$ has $10m + m = 11m$ vertices of degree 0, $12m - m = 11m$ of degree 3 vertices, $8m + 3m = 11m$ of degree $3m + 11$ vertices and it contains $q - 3$ disconnected $(9m + 12)$-regular subgraphs of size $11m$, hence we conclude that $G'$ is the $11m$-anonymous graph.

Let $I'$ be a yes-instance of Anonymous-Edge-Rotation. Then there exists a sequence of $3m$ rotations such that the graph $G' = (V, E')$ obtained after applying the rotations to $G$ is a $11m$-anonymous graph. Since $|V| = 33m + (q - 3)11m$, there must be only $q$ different degrees classes in $G'$. Note that with one rotation, we can change the degree of two vertices, therefore the degree at most $6m$ vertices can be changed by $3m$ rotations. Since the graph $G$ has more than $6m$ vertices of the degrees $3m + 11$, 3, 0 and $9m + 12$, all these degree classes must be in $G'$. Furthermore, due to the number of vertices of $G$, these are the only degree classes in $G'$. This means that in $G'$ the number of vertices of degree $3m + 11$ must be increased by $3m$, the number of vertices of degree 0 must be increased by $m$, the number of vertices of degree 3 must be decreased by $m$, there are no vertices of degrees $3m + 10$ in $G'$ and the other degree classes keep the same amount of vertices.

A single rotation can increase or decrease the degree of a vertex by 1 therefore using $3m$ rotations no vertex of degree $3m + 10$ in $G$ can have degree 0 in $G'$ and similarly, no vertex of degree 3 in $G$ can have degree $3m + 11$ in $G'$. Therefore the $3m$ new vertices of degree $3m + 11$ in $G'$ must have degree $3m + 10$ in $G$. This is only possible if the degree of each vertex $v_x$ from the set $V_{elem}$ is increased by 1. Similarly, the $m$ new vertices of degree 0 in $G'$ must have degree 3 in $G$, let $C_{G'}$ be the set of such vertices. Obviously, $C_{G'}$ must be a subset of $V_{set}$, in which the vertices have the form $c^\ell_{xyz}$ with $x, y, z \in X$, for any set $\{x, y, z\} \in C$, and $\ell \in \{1, 2, 3, 4\}$. For the same reasons, vertices of degree greater than $9m + 12$ cannot be degree less than $3m + 12$.

To reach the requested degree configuration in $G'$ with exactly $3m$ edge rotations, in each rotation the degree of each vertex from $V_{elem}$ must be increased by 1 and the degree of each vertex from the set $C_{G'}$ must be decrease by 1. To achieve that, for each vertex $v_x$ from $V_{elem}$, the only possible rotation is to add the edge $u_x v_x$ where $u_x \in V_{hub}$ and remove the edge $u_x c^\ell_{xyz}$ where $c_{xyz} \in C_{G'}$. To fulfil the condition about the degree classes and the number of the rotations, the only way to achieve that is that $C'' = \{\{x, y, z\} \mid c^\ell_{xyz} \in C_{G'}\}$ is an exact cover of $X$. $\square$

## 5. Characterization of the "closest" $k$-anonymous degree sequence

In this section we suppose that $(G,k)$ is a feasible instance. For any such instance we define a $k$-anonymous degree sequence $S_{bound}$ that can be computed in polynomial time if $k=\theta(n)$. We show that with the $(+1,-1)$-degree modifications (Remark 1) the graph $G$ can be transformed into a $k$-degree-anonymous graph $G'$ with degree sequence $S_{bound}$ using at most double of edge rotations as in an optimal solution of Min Anonymous-Edge-Rotation for $(G,k)$.

Note that in general a $(+1,-1)$-degree modification doesn't correspond to an edge rotation, but as we show later in Section 7.1, it is true for trees.

Now in the following steps we show how to define the degree sequence $S_{bound}$.

*Step 1: Compute every available target sequence.*
Let $S=(s_1,\dots,s_n)$ be a non-increasing sequence of non-negative integers, $r\in\{1,\dots,n\}$. Any partition of $S$ into $r$ contiguous subsequences (i.e. if $S[a]$ and $S[b]$ are in one part, then all $S[i]$, $a\le i\le b$ must be in the same part) is called a contiguous $r$-partition. The number of contiguous $r$-partitions of $S$ is $\binom{n-1}{r-1}$, therefore bounded by $(n-1)^{r-1}$. Then the number of contiguous partitions of $S$ with at most $r$ parts can be bounded by $\sum_{i=0}^{r-1}(n-1)^i\le 2n^{r-1}$.

For each contiguous $\ell$-partition $p$, $1\le\ell\le r$, we use notation $p=[p_1,\dots,p_\ell]$, where $p_i$ denotes the number of elements in part $i$, $1\le i\le\ell$. Note that at this stage important is the number of elements in each part, not which elements from $S$ are in it.

Let $G$ be a graph of order $n$ and $k$ an integer, $k\ge 2$. If $G$ is a $k$-degree-anonymous graph, then the vertices of $G$ can be partitioned into at most $c=\lfloor\frac{n}{k}\rfloor$ parts where the vertices in each part have the same degree. Let $P$ be the set of all such contiguous partitions with at most $c$ parts. As it follows from the initial discussion, the number of such partitions is bounded by $2n^{c-1}$.

Now for each contiguous partition $p=[p_1,p_2,\dots,p_\ell]\in P$, $\ell\in\{1,\dots,c\}$, we compute all non-increasing sequences $(d_1,d_2,\dots,d_\ell)$ of $\ell$ integers $d_i$ such that $0\le d_i<|V|$. Let $\hat{P}_p$ be the set of all feasible $k$-anonymous degree sequences for $p$, i.e.

$$S=(\underbrace{d_1,\dots,d_1}_{p_1-\text{times}},\underbrace{d_2,\dots,d_2}_{p_2-\text{times}},\dots,\underbrace{d_\ell,\dots,d_\ell}_{p_\ell-\text{times}})=(d_1^{p_1},d_2^{p_2},\dots,d_\ell^{p_\ell})\in\hat{P}_p$$

if and only if $\sum_{i=1}^{\ell}p_id_i=2|E|$, $S$ is graphic and $k$-anonymous.

For each contiguous partition $p$ with $\ell$ parts, $1\le\ell\le c$, there are at most $n$ possibilities for a degree on each position. The test whether the generated sequence is graphic and $k$-anonymous can be done in $O(n)$ operations. Since $|P|=O(n^{c-1})$, there are at most $O(n^{c-1}\times n^\ell\times n)\le O(n^{2c})$ operations to compute all feasible degree sequences of every partition, where $c=\lfloor\frac{n}{k}\rfloor$. Obviously, if $c$ is a constant, such number of operations is polynomial.

*Step 2: Find the best one.*
Now based on the previous analysis we can define the degree sequence $S_{bound}$ and prove some basic properties.

**Definition 5.** Let $G$ be a graph with the degree sequence $S_G$. Then define $S_{bound}$ for $G$ as a degree sequence for which the sum $\sum_{i=1}^{n}|S_G[i]-S[i]|$ achieves the minimum for all elements $S\in\hat{P}_p$ and $p\in P$.

**Remark 3.** Similarly to a $k$-anonymous sequence $S_{bound}$ defined in Definition 5 for a graph, we can define a $k$-anonymous sequence $S_{Tbound}$ for a tree. The only difference is that in the set $\hat{P}_p$, every feasible solution must have $d_i\ge 1$, which would be a subset of $\hat{P}_p$. Also for the testing, we don't need to check whether $S$ is graphic, the condition $\sum_{i=1}^{\ell}p_id_i=2|E|$, is enough for the degree sequence of a tree.

**Lemma 1.** *Let $S$ be a $n$-sequence of non-negative integers and denote by $S'$ the sequence $S$ sorted in non-increasing order. Let $S_s$ be another $n$-sequence of non-negative integers sorted in non-increasing order. Then*

$$\sum_{i=1}^{n}|S_s[i]-S'[i]|\le\sum_{i=1}^{n}|S_s[i]-S[i]| \tag{3}$$

**Proof.** If $S$ is already in non-increasing order then (3) holds. If not then there exist positive integers $a$, $b$ such that $a<b$ and $S[a]<S[b]$. Let $S_1$ be the sequence defined swapping the values $S[a]$, $S[b]$, hence: $S_1[a]=S[b]$, $S_1[b]=S[a]$, and $S_1[i]=S[i]$ otherwise. We denote

$$\begin{aligned}
A &= \sum_{i=1}^{n} |S_s[i] - S[i]| - \sum_{i=1}^{n} |S_s[i] - S_1[i]| \\
&= |S_s[a] - S[a]| - |S_s[a] - S_1[a]| + |S_s[b] - S[b]| - |S_s[b] - S_1[b]| \\
&= |S_s[a] - S_1[b]| - |S_s[a] - S_1[a]| + |S_s[b] - S_1[a]| - |S_s[b] - S_1[b]|
\end{aligned}$$

In order to follow easier six different cases, let $x_1 = S_s[a]$, $x_2 = S_s[b]$, $x_3 = S_1[a]$, $x_4 = S_1[b]$, and thus $A = |x_1 - x_4| - |x_1 - x_3| + |x_2 - x_3| - |x_2 - x_4|$. Following our assumptions $x_1 \geq x_2$ and $x_3 > x_4$.

Now for all possible arrangements of $x_1$, $x_2$, $x_3$, $x_4$ we discuss the value $A$:

- $x_1 \geq x_2 \geq x_3 > x_4$: $A = x_1 - x_4 - x_1 + x_3 + x_2 - x_3 - x_2 + x_4 = 0$
- $x_3 > x_4 \geq x_1 \geq x_2$: $A = x_4 - x_1 - x_3 + x_1 + x_3 - x_2 - x_4 + x_2 = 0$
- $x_1 \geq x_3 > x_4 \geq x_2$: $A = x_1 - x_4 - x_1 + x_3 + x_3 - x_2 - x_4 + x_2 = 2x_3 - 2x_4 > 0$
- $x_3 \geq x_1 \geq x_2 \geq x_4$: $A = x_1 - x_4 - x_3 + x_1 + x_3 - x_2 - x_2 + x_4 = 2x_1 - 2x_2 \geq 0$
- $x_1 \geq x_3 \geq x_2 \geq x_4$: $A = x_1 - x_4 - x_1 + x_3 - x_2 - x_1 - x_2 + x_4 = 2x_3 - 2x_2 \geq 0$
- $x_3 \geq x_1 \geq x_4 \geq x_2$: $A = x_1 - x_4 - x_3 + x_1 + x_3 - x_2 - x_4 + x_2 = 2x_1 - 2x_4 \geq 0$

We can conclude that in all cases $A \geq 0$, therefore $\sum_{i=1}^{n} |S_s[i] - S_1[i]| \leq \sum_{i=1}^{n} |S_s[i] - S[i]|$.

If the sequence $S_1$ is still not in non-increasing order, we can repeat the process of swapping for the next two unsorted elements on $S_1$ until we obtain the non-increasing sequence $S'$. Each process can be repeated independently, therefore

$$\sum_{i=1}^{n} |S_s[i] - S'[i]| \leq \sum_{i=1}^{n} |S_s[i] - S[i]|. \quad \square$$

**Theorem 5.** *Let $(G, k)$ be a feasible instance for the* Min Anonymous-Edge-Rotation *problem. Let OPT be an optimum solution that is a minimum set of rotations that transform G to a k-degree-anonymous graph $G'$. Then $\sum_{i=1}^{n} |S_G[i] - S_{bound}[i]| \leq 2|OPT|$, where the degree sequence $S_{bound}$ is defined in Definition 5.*

**Proof.** Let $S_{G'}$ be the degree sequence of $G'$ sorted in the same order as $S_G$ (i.e. for every $v \in V$, if $deg_G(v)$ is in the position $i$ in $S_G$ then $deg_{G'}(v)$ is in the position $i$ in $S_{G'}$). Let $S'_{G'}$ be the degree sequence $S_{G'}$ sorted in non-increasing order. As in the definition of $S_{bound}$ we considered all the options, there must exist $p \in P$ and $S \in \hat{P}_p$ such that $S = S'_{G'}$, and

$$\sum_{i=1}^{n} |S_G[i] - S_{bound}[i]| \leq \sum_{i=1}^{n} |S_G[i] - S'_{G'}[i]|.$$

Since the degree sequence $S'_{G'}$ is sorted in non-increasing order, then

$$\sum_{i=1}^{n} |S_G[i] - S'_{G'}[i]| \leq \sum_{i=1}^{n} |S_G[i] - S_{G'}[i]|$$

by Lemma 1. One rotation from the graph $G_j$ to $G_{j+1}$ in the sequence of the graphs from $G$ to $G'$ can only decrease the degree of a vertex by one and increase the degree of another one by one, hence $\sum_{i=1}^{n} |S_{G_j}[i] - S_{G'}[i]| \leq \sum_{i=1}^{n} |S_{G_{j+1}}[i] - S_{G'}[i]| + 2$. This means by one rotation the value $\sum_{i=1}^{n} |S_G[i] - S_{G'}[i]$ decreases by at most 2. After $|OPT|$ rotations, the last graph $G_{j+1}$ in the sequence is $G'$, therefore $\sum_{i=1}^{n} |S_G[i] - S_{G'}[i]| \leq 2|OPT|$ and the theorem follows. $\square$

## 6. Approximation of Min Anonymous-Edge-Rotation

In this section we show that under some constraints on the number of edges and $k$, there exists a polynomial time 2-approximation algorithm for the Min Anonymous-Edge-Rotation problem for all feasible inputs $(G, k)$.

**Remark 4.** Let $S = (x_1, x_2, \ldots, x_n)$ be a non-increasing sequence of $n$ non-negative integers. Denote by $R = x_1 - x_n$, $A_0 = \frac{x_1 + x_n}{2}$, and let $A = \frac{\sum_{i=1}^{n} x_i}{n}$.

The standard deviation of $S$ is defined as $\sigma(S)=\sqrt{\frac{\sum(x_i-A)^2}{n}}$. It can be shown that

$$\sum_{i=1}^{n}(x_i-A)^2 \le \sum_{i=1}^{n}(x_i-A_0)^2 \le \frac{nR^2}{4},$$

hence $\sigma(S)\le \frac{R}{2}$.

The mean absolute derivation of $S$ is defined as $MAD[S]=\frac{1}{n}\sum_{i=1}^{n}|x_i-A|$. It is well known (e.g. applying Jensen's inequality) that $MAD[S]\le\sigma(S)$.

Based on the correlation mentioned in Remark 4, we calculate an upper bound on the values in the degree sequence $S_{bound}$ in the following lemma.

**Lemma 2.** *Let $(G,k)$ be an instance of the* Min Anonymous-Edge-Rotation *problem where $G$ is the graph with $n$ vertices and $m$ edges. Suppose that $\frac{n}{2}\le m\le\frac{n(n-3)}{2}$, $k\le\frac{n}{4}$, and let the constant $c$ be defined as $c=\lfloor\frac{n}{k}\rfloor$, hence $k=\theta(n)$. Let $S_{bound}$ be the $k$-anonymous degree sequence associated with $G$ defined following Definition 5. Then for every $i$, $S_{bound}[i]\le\min\{(1+\frac{n}{4k}+\frac{n}{k\Delta})\Delta, n-1\}$, $1\le i\le n$.*

**Proof.** Let $S_G$ be the degree sequence of $G$ sorted in non-increasing order and $D$ the $k$-anonymous degree sequence constructed following Theorem 2. Denote the unrounded average degree as $A=\frac{\sum_{i=1}^{n}S_G[i]}{n}$. Then using Remark 4, the standard deviation of $S_G$, $\sigma[S_G]\le\frac{\Delta}{2}$, and $MAD[S_G]\le\sigma(S_G)$. Hence

$$\begin{aligned}\sum_{i=1}^{n}|S_G[i]-D[i]| &\le \sum_{i=1}^{n}\max(|S_G[i]-\lfloor A+2\rfloor|, |S_G[i]-\lfloor A-1\rfloor|)\\ &\le \sum_{i=1}^{n}|S_G[i]-A|+\sum_{i=1}^{n}2 = nMAD[S_G]+2n\\ &\le n\sigma[S_G]+2n\le n\frac{\Delta}{2}+2n=\frac{n(\Delta+4)}{2}\end{aligned}$$

Let $\Delta'$ be the maximum value of $S_{bound}$. If $\Delta'\le\Delta$, then the condition from Lemma holds. If $\Delta'>\Delta$, then the distance between the $k$ first elements of $S_{bound}$ and the $k$ first elements of $S_G$ is at least $k(\Delta'-\Delta)$ since $S_{bound}$ is $k$-anonymous and sorted in non-increasing order. Because $\sum_{i=1}^{n}S_{bound}[i]=\sum_{i=1}^{n}S_G[i]$, if the value of some elements is increased of a certain amount, the value of some others have to be decreased by the same amount, so $\sum_{i=1}^{n}|S_G[i]-S_{bound}[i]|\ge 2k(\Delta'-\Delta)$.

If $\Delta'>(1+\frac{n}{4k}+\frac{n}{k\Delta})\Delta$ then $\sum_{i=1}^{n}|S_G[i]-S_{bound}[i]|>2k(\frac{n}{4k}+\frac{n}{k\Delta})\Delta=\frac{n(\Delta+4)}{2}\ge\sum_{i=1}^{n}|S_G[i]-D[i]|$, which is not possible due to minimality of $S_{bound}$. □

In the following two lemmas we prove that if a graph has 'sufficiently' many edges than edge rotations with the specific properties exist in a graph.

**Lemma 3.** *Let $G=(V,E)$ be a graph with $|E|>\Delta^2$, let $uv\in E$. Then there exists an edge $ab\in E$ such that both vertices $a$ and $b$ are different from $u$ and $v$ and at most one of the following edges $\{av, au, bv, bu\}$ is in $E$.*

**Proof.** For an edge $xy\in E$, let $N_x=\mathcal{N}_G(x)\setminus\{y\}$ and $N_y=\mathcal{N}_G(y)\setminus\{x\}$. For a contradiction suppose there exists an edge $uv\in E$ such that for every edge $ab\in E\setminus(Inc(u)\cup Inc(v))$ at least two of the edges $\{av, au, bv, bu\}$ are in $E$. Then at least one vertex from $\{a,b\}$ is incident to both vertices $u$, $v$, hence belongs to $N_u\cap N_v$, or both vertices $\{a,b\}$ are in $(N_u\cup N_v)\setminus(N_u\cap N_v)$. Moreover, every vertex in $N_u\cup N_v$ has at most $\Delta-1$ neighbours in $V\setminus\{u,v\}$. Hence,

$$\begin{aligned}|E\setminus(Inc(u)\cup Inc(v))| &\le (\Delta-1)\times(|N_u\cap N_v|+\frac{|(N_u\cup N_v)\setminus(N_u\cap N_v)|}{2})\\ &=(\Delta-1)\times\frac{|N_u\cap N_v|+|N_u\cup N_v)|}{2}\\ &=(\Delta-1)\times\frac{|N_u|+|N_v|}{2}\\ &\le(\Delta-1)^2\end{aligned}$$

Then $|E| \le |Inc(u) \cup Inc(v)| + |E \setminus (Inc(u) \cup Inc(v))| \le 1 + 2(\Delta - 1) + (\Delta - 1)^2 = \Delta^2$. This is in contradiction with hypothesis $|E| > \Delta^2$. □

**Lemma 4.** *Let $G = (V, E)$ be a graph and suppose $|E| > \Delta^2$. Let $v^+, v^- \in V$ such that $1 \le d_G(v^-) \le \Delta$ and $0 \le d_G(v^+) \le \Delta < |V| - 1$. Then there exists a sequence of at most two edge rotations that transform $G$ to $G'$ such that $d_{G'}(v^+) = d_G(v^+) + 1$, $d_{G'}(v^-) = d_G(v^-) - 1$ and degrees of other vertices in $G$ are not changed. These rotations can be found in $O(|E|^2)$ steps.*

**Proof.** Case 1: Suppose there exists a vertex $v \in V$ such that $v \in \mathcal{N}_G(v^-)$ and $v \notin \mathcal{N}_G(v^+)$. Let $G'$ be the graph obtained from $G$ removing the edge $v^-v$ and adding the edge $vv^+$, hence using rotation $(vv^-, vv^+)$. Obviously, $d_{G'}(v^+) = d_G(v^+) + 1$, $d_{G'}(v^-) = d_G(v^-) - 1$ and $G'$ is obtained by using a single rotation.
Case 2: $\mathcal{N}(v^-) \subseteq \mathcal{N}(v^+)$. Let $u \in \mathcal{N}_G(v^-)$. Since $|E| > \Delta^2$ and $uv^+ \in E$, by using Lemma 3 then there exists an edge $ab \in E$ such that at most one edge of the set $\{av^+, au, bv^+, bu\}$ is in $E$. If $au$ is in $E$, then the graph $G'$ obtained by two rotations $(ab, av^+)$ and $(uv^-, ub)$ has the required properties. If $av^+$ is in $E$, then the graph $G'$ obtained by two rotations $(ba, bv^+)$ and $(uv^-, ua)$ has the required properties. The remaining two cases if $bu$ or $bv^+$ are from $E$ are symmetrical to the above cases, it is enough to swap $a$ and $b$.

Obviously, such an edge $ab$ can be found in $O(|E|^2)$. □

**Theorem 6.** *The* Min Anonymous-Edge-Rotation *problem is polynomial time 2-approximable for all instances $(G, k)$, $k \le \frac{n}{4}$ where $k = \theta(n)$ and $G$ is the graph with $n$ vertices and $m$ edges, where $\max\{\frac{n}{2}, (1 + \frac{n}{4k} + \frac{n}{k\Delta})^2\Delta^2\} \le m \le \frac{n(n-3)}{2}$, and the constant $c$ is defined as $c = \lfloor \frac{n}{k} \rfloor$.*

**Proof.** Let $(G = (V, E), k)$ be an instance of Min Anonymous-Edge-Rotation and $S_G$ be the degree sequence of $G$. Let the constant $c$ be defined as $c = \lfloor \frac{n}{k} \rfloor$. Due to our assumptions about the number of edges and $k$, all such instances are feasible as follows from Section 3. First we compute a $k$-anonymous degree sequence $S_{bound}$ following Definition 5 in $O(n^{2c})$ steps. Due to the assumption $k = \theta(n)$ and consequently $c$ being a constant, such number of steps is polynomial. Furthermore, the condition on the number of edges ensures that we can always apply Lemma 4 and find suitable edge rotations.

If there exist two vertices $v^+$, $v^- \in V$ such that $0 \le S_G[v^+] < S_{bound}[v^+] \le (1 + \frac{n}{4k} + \frac{n}{k\Delta})\Delta < |V| - 1$ and $S_G[v^-] > S_{bound}[v^-]$ we apply Lemma 4 to transform $G$ to a graph $G_1$ with at most two rotations such that $d_{G_1}(v^+) = d_G(v^+) + 1$ and $d_{G_1}(v^-) = d_G(v^-) - 1$.

We'll be executing the above transformations while there are two vertices $v^+$, $v^- \in V$ with the required properties. In each such transformation we decrease the degree of one vertex by 1 and increase the degree of another one by 1 with at most two rotations. Hence we transform $G$ to a final graph $G'$ with degree sequence $S_{bound}$ by at most $\sum_{i=1}^{n} |S_G[i] - S_{bound}[i]|$ rotations. By Lemma 5 we know that $\sum_{i=1}^{n} |S_G[i] - S_{bound}[i]| \le 2|OPT|$, hence we use at most 2 times the numbers of rotations of an optimal solution. In each transformation loop searching for the vertices $v^+$ and $v^-$ can be done in time $O(n)$ and searching for an edge $ab$ in time $O(m^2)$ (Lemma 3). Due to the modifications in each transformation loop, there can be at most $O(n^2)$ loops. Therefore the time complexity is bounded by $O(n^{2c} + n^2 \times m^2 \times n)$. Since $c \ge 4$, $O(n^{2c} + m^2 \times n^3) \le O(n^{2c})$.

Finally, since $S_{bound}$ is $k$-anonymous, $G'$ is a $k$-degree-anonymous graph. □

## 7. Polynomial cases for Min Anonymous-Edge-Rotation

As follows from Section 4, the Min Anonymous-Edge-Rotation problem is NP-hard even for $k = \frac{n}{q}$ and $q \ge 3$ is a fixed constant where $n$ is the order of an input graph. In this section we show that the problem can be solved in polynomial time on trees when $k = \theta(n)$ or in case of any graph when $k = n$.

### 7.1. Trees

For a tree $T = (V, E)$ rooted in a vertex $r$, for any $v \in V$, $v \ne r$, $child(v)$ is a vertex that is a neighbour of $v$ not on the path from $r$ to $v$.

**Lemma 5.** *Let $T = (V, E)$ be a tree and $v^-$, $v^+$ vertices from $V$ such that $v^-$ is not a leaf and $v^+$ is not a universal vertex. Then using one rotation we can transform $T$ into a tree $T'$ such that $d_{T'}(v^-) = d_T(v^-) - 1$ and $d_{T'}(v^+) = d_T(v^+) + 1$.*

**Proof.** Let $v^+$ be the root of $T$. Since $v^-$ is not a leaf, there exists a vertex $c \in child(v^-)$. Since $T$ is a tree, $cv^+ \notin E$. Therefore we can define the rotation $(cv^-, cv^+)$ (see Fig. 8). Let $T'$ be the graph obtained after a such rotation. Since there is no edge between the subtree of $c$ and other vertices, $T'$ is a tree. Moreover $d_{T'}(v^-) = d_T(v^-) - 1$ and $d_{T'}(v^+) = d_T(v^+) + 1$. □

**Theorem 7.** *The* Min Anonymous-Edge-Rotation *problem is polynomial-time solvable for any instance $(T, k)$ where $T$ is a tree of the order $n$, $k \le \frac{n}{4}$ and such that $c = \lfloor \frac{n}{k} \rfloor$ is a constant, hence $k = \theta(n)$.*

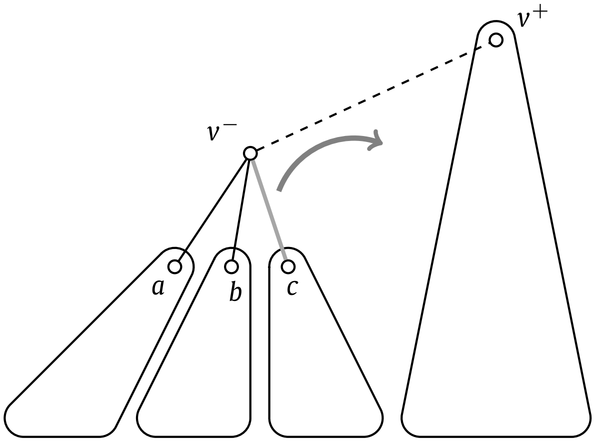


**Fig. 8.** Transformation $T$ to $T'$.

**Proof.** Let $T$ be a tree and $S_T = (d_1, d_2, \dots, d_n)$ its degree sequence sorted in non-increasing order. As it was mentioned in Section 2, for a degree sequence of a tree only the following conditions must hold $\sum_{i=1}^{n} d_i = 2(n-1)$ and $d_i \geq 1$ for all $i$, $1 \leq i \leq n$. Now based on $S_T$ define a $k$-anonymous sequence $S_{Tbound}$ as discussed in Section 5.

Let $x$ and $y$ be integers such that $S_T[x] > S_{Tbound}[x]$ and $S_T[y] < S_{Tbound}[y]$. Since $S_{Tbound}$ corresponds to a tree, $S_{Tbound}[x] \geq 1$ then $S_T[x] > 1$ and thus $v_x$ is not a leave in $T$. Moreover since $S_{Tbound}[y] \leq n-1$, $v_y$ is not a universal vertex in $T$.

By Lemma 5 there exists a tree $T_1$ such that $S_{T_1}[x] = S_T[x] - 1$ and $S_{T_1}[y] = S_T[y] + 1$. Repeat this operation until reaching a tree $T'$ with the degree sequence $S_{Tbound}$. The cost of one operation is $O(n)$ and we repeat it $\frac{\sum_{i=1}^{n} |S_T[i] - S_{Tbound}[i]|}{2} \leq n^2$ times. Since $S_{Tbound}$ is $k$-anonymous, $T'$ is a $k$-degree-anonymous tree. Since we use $\frac{\sum_{i=1}^{n} |S_T[i] - S_{Tbound}[i]|}{2} \leq |OPT|$ rotations (Lemma 5), the algorithm is optimal. The total cost of the algorithm is bounded by $O(n^{2c} + n^2) = O(n^{2c})$, where $c = \lfloor \frac{n}{k} \rfloor$ is a constant. $\square$

### 7.2. One degree class, $k = n$

In this part we show that Min Anonymous-Edge-Rotation is polynomial-time solvable for instances where $k$ coincides with the number of vertices of the graph, that means all vertices must be in the same degree class.

**Lemma 6.** *Let $G = (V, E)$ be a graph and $u, v \in V$. If $\mathcal{N}_G(u) \not\subseteq \mathcal{N}_G(v)$, then there is an edge rotation that leads to a graph $G'$ such that $d_{G'}(u) = d_G(u) - 1$ and $d_{G'}(v) = d_G(v) + 1$.*

**Proof.** Since $\mathcal{N}_G(u) \not\subseteq \mathcal{N}_G(v)$, there exists $w \in V$ such that $uw \in E$ and $vw \notin E$. Then we can do the following edge rotation $(uw, vw)$ and get the graph $G'$ with $E' = (E \setminus \{uw\}) \cup \{vw\}$. $\square$

**Remark 5.** Let $G = (V, E)$ be a graph, $\forall u, v \in V$, if $d_G(u) > d_G(v)$, then there is an edge rotation that leads to a graph $G'$ such that $d_{G'}(u) = d_G(u) - 1$ and $d_{G'}(v) = d_G(v) + 1$.

**Lemma 7.** *Let $(G, n)$ be an instance of* Min Anonymous-Edge-Rotation *where $G \in \mathbf{G}(n, m)$ for some positive integers $m$, $n$, and $\frac{2m}{n}$ is an integer. Then the optimum value of* Min Anonymous-Edge-Rotation *on $(G, n)$ is $\frac{\sum_{w \in V} |d_G(w) - 2m/n|}{2}$.*

**Proof.** As follows from Theorem 3, an instance $(G, n)$ from $\mathbf{G}(n, m)$ is a feasible instance of Min Anonymous-Edge-Rotation if and only if $\frac{2m}{n}$ is an integer. Let suppose that $(G, n)$ is such an instance. Obviously, if $G$ is a regular graph, then the degree of each vertex must be $\frac{2m}{n}$.

If $G$ is not a regular graph, then $\exists u, v \in V$ such that $d_G(u) > \frac{2m}{n}$ and $d_G(v) < \frac{2m}{n}$. By Remark 5, there is an edge rotation that leads to a graph $G'$ such that $d_{G'}(u) = d_G(u) - 1$ and $d_{G'}(v) = d_G(v) + 1$. Then obviously at least $\frac{\sum_{w \in V} |d_G(w) - 2m/n|}{2}$ rotations are necessary to have all the vertices of the same degree $\frac{2m}{n}$, therefore the optimum value of Min Anonymous-Edge-Rotation on the instance $(G, n)$ is at least $\frac{\sum_{w \in V} |d_G(w) - 2m/n|}{2}$.

Now suppose that the optimum value is $r$ strictly less than $\frac{\sum_{w \in V} |d_G(w) - 2m/n|}{2}$. Each rotation increases the degree of a vertex by one and decreases the degree of another vertex by one too. Obviously, each vertex $w$ has to be involved in at least $|d_G(w) - 2m/n|$ edge rotations to reach the degree $\frac{2m}{n}$. Hence if there are $r < \frac{\sum_{w \in V} |d_G(w) - 2m/n|}{2}$ edge rotations then in any graph $G'$ obtained from $G$ using $r$ edge rotations there exists $w' \in V$ such that $d_{G'}(w') > \frac{2m}{n}$ or $d_{G'}(w') < \frac{2m}{n}$. □

**Theorem 8.** *The* Min Anonymous-Edge-Rotation *problem is polynomial-time solvable for instances* $(G, k)$ *when* $k = n$*, where* $n$ *is the order of the graph* $G$.

**Proof.** In case $k = n$, we are looking for a $n$-degree-anonymous graph with only one degree class, hence for a regular graph. Due to Theorem 3, we can easily decide whether $(G, n)$ is a feasible instance of Min Anonymous-Edge-Rotation: if for $G \in \mathbf{G}(n, m)$ the fraction $\frac{2m}{n}$ is not an integer, $(G, n)$ is not a feasible input.

For a feasible input $(G, n)$, the result is based on Algorithm 1 and its correctness follows from Lemmas 6 and 7.

```
Input  : A graph G = (V, E)
Output: A sequence S of edge rotations if 2|E|/|V| is an integer
        NO otherwise

S = ∅ ;
d = 2|E|/|V| ;
if if d is not integer then
    return NO ;
else
    while ∃u, v ∈ V such that d_G(u) < d and d_G(v) > d do
        Let w ∈ N(v) \ N(u) ;
        E = E \ {vw};
        E = E ∪ {uw};
        S = S ∪ {(wv, wu)};
    end
end
```

**Algorithm 1:** Algorithm for $k = |V|$.

Obviously, the algorithm runs in polynomial time. □

## 8. Conclusion

In this paper we initiate the study of the complexity of Min Anonymous-Edge-Rotation problem in which the task is to transform a given graph to a $k$-degree anonymous graph using a minimum number of edge rotations. As we were able to prove NP-hardness in case where the number of vertices $k$ in each degree class is $\theta(n)$, further research could explore stronger hardness results or cases when $k$ is a constant. Our next research step includes relaxation of the condition on the number of the edges in the presented 2-approximation algorithm as well as extension of the graph classes in which the Min Anonymous-Edge-Rotation problem can be solved in polynomial time. As the problem doesn't have a solution for all graphs and all possible values of $k$, our initial feasibility study covers a large part of instances. The extensions of the results are still possible, in the sense of necessary and sufficient conditions.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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