On the differential approximation of MIN SET COVER

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Abstract

We present in this paper differential approximation results for MIN SET COVER and MIN WEIGHTED SET COVER. We first show that the differential approximation ratio of the natural greedy algorithm for MIN SET COVER is bounded below by $1.365/\Delta$ and above by $4/(\Delta+1)$, where Δ is the maximum set-cardinality in the MIN SET COVER-instance. Next we study another approximation algorithm for MIN SET COVER that computes 2-optimal solutions, i.e., solutions that cannot be improved by removing two sets belonging to them and adding another set not belonging to them. We prove that the differential approximation ratio of this second algorithm is bounded below by $2/(\Delta+1)$ and that this bound is tight. Finally, we study an approximation algorithm for MIN WEIGHTED SET COVER and provide a tight lower bound of $1/\Delta$. Our results identically hold for MAX HYPERGRAPH INDEPENDENT SET in both the standard and the differential approximation paradigms.

1 Introduction

Given a family $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of a ground set $C = \{c_1, c_2, \ldots, c_n\}$ (we assume that $\cup_{S_i \in \mathcal{S}} S_i = C$), a set-cover of C is a sub-family $\mathcal{S}' \subseteq \mathcal{S}$ such that $\cup_{S_i \in \mathcal{S}'} S_i = C$; MIN SET COVER is the problem of determining a minimum-size set-cover of C. MIN WEIGHTED SET COVER consists of considering that sets of $\mathcal{S}$ are weighted by positive weights; the objective becomes then to determine a minimum total-weight cover of C.

Given $I = (\mathcal{S}, C)$ and a cover $\hat{\mathcal{S}}$, the sub-instance $\hat{I}$ of I induced by $\hat{\mathcal{S}}$ is the instance $(\hat{\mathcal{S}}, C)$. For simplicity, we identify in what follows a feasible (resp., optimal) cover $\mathcal{S}'$ (resp., $\mathcal{S}^*$) by the set of indices N' (resp., N^*) of the sets of the cover, i.e., $\mathcal{S}' = \{S_i : i \in N'\}$ (resp., $\mathcal{S}^* = \{S_i : i \in N^*\}$). So, for the i-th set of $\mathcal{S}$, we use the term $i \in N'$ (resp., $i \in N^*$) when dealing with the set itself, and the term S_i to denote the set of elements contained in this set.

Definition 1. Given an instance $(\mathcal{S}, C)$ of MIN SET COVER, its characteristic graph $B = (L, R; E)$ is a bipartite graph B with color-classes $L = \{1, \ldots, m\}$, corresponding to the members of the family $\mathcal{S}$ and $R = \{c_1, \ldots, c_n\}$, corresponding to the elements of the ground set C; the edge-set E of B is defined as $E = \{(i, c_j) : c_j \in S_i\}$. ∎

A cover $\mathcal{S}'$ of C is said to be minimal (or minimal for the inclusion) if removal of any set $S \in \mathcal{S}'$ results in a family that is not anymore a cover for C.

Remark 1. Consider an instance $(\mathcal{S}, C)$ of MIN SET COVER and a minimal set-cover $\mathcal{S}'$ for it. Then, for any $S_i \in \mathcal{S}'$, there exists $c_j \in C$ such that S_i is the only set in $\mathcal{S}'$ covering c_j. Such a c_j will be called *non-redundant with respect to* $S_i \in \mathcal{S}'$; furthermore, S_i itself will be called *non-redundant for* $\mathcal{S}'$. With respect to the characteristic bipartite graph B' corresponding to the sub-instance I' of I induced by $\mathcal{S}'$ (it is the subgraph B' of B induced by $L' \cup R$ where $L' = N'$), for any $i \in L'$, there exists a $c_j \in R$ such that $d(c_j) = 1$, where, for a vertex v of a graph G, $d(v)$ denotes the degree of v. In particular, there exists at least $|N'|$ non-redundant elements, one for each set. ∎

As previously, we simplify notations considering only one non-redundant element with respect to $S_i \in \mathcal{S}'$. Moreover, we assume that this element is c_i for the set $i \in N'$. Thus, the set of non-redundant elements with respect to $\mathcal{S}'$ considered here is $C_1 = \{c_i,\ i \in N'\}$.

In this paper we study differential approximability for MIN SET COVER in both unweighted and weighted versions. Differential approximability is analyzed using the so-called differential approximation ratio defined, for an instance I of an **NPO** problem Π (an optimization problem is in **NPO** if its decision version is in **NP**) and an approximation algorithm $\mathtt{A}$ computing a solution S for Π in I, as $\delta_{\mathtt{A}}(I, S) = |\omega(I) - m_{\mathtt{A}}(I, S)|/|\omega(I) - \mathrm{opt}(I)|$, where $\omega(I)$ is the value of the worst Π-solution for I, $m_{\mathtt{A}}(I, S)$ is the value of S and $\mathrm{opt}(I)$ is the value of an optimal Π-solution for I. For an instance $I = (\mathcal{S}, C)$ of MIN SET COVER, $\omega(I) = m$, the size of the family $\mathcal{S}$. Obviously, this is the maximum-size cover of I. Finally, standard approximability is analyzed using the standard approximation ratio defined as $m_{\mathtt{A}}(I, S)/\,\mathrm{opt}(I)$.

Surprisingly enough, differential approximation, although introduced in [3] since 1977, has not been systematically used until the 90's ([4, 1, 5, 16] are, to our knowledge, the most notable uses of it) when a formal framework for it and a more systematic use started to be drawn ([7, 8]). In general, no apparent links exist between standard and differential approximations in the case of minimization problems, in the sense that there is no evident transfer of a positive, or negative, result from one paradigm to the other. Hence a "good" differential approximation result does not signify anything for the behavior of the approximation algorithm studied when dealing with the standard framework and vice-versa. As already mentioned, the differential approximation ratio measures the quality of the computed feasible solution according to both optimal value and the value of a worst feasible solution. The motivation for this measure is that it gives the position of the computed feasible solution in the interval between an optimal solution and a worst-case one. Even if differential approximation ratio is not as popular as the standard one, it is interesting enough to be investigated for some fundamental problems as MIN SET COVER, in order to observe how they behave under several approximation criteria. Such joint investigations can significantly contribute to a deeper apprehension of the approximation mechanisms for the problems dealt. A further motivation for the study of differential approximation is the stability of the differential approximation ratio under affine transformations of the objective function. This stability often serves in order to derive differential approximation results for minimization (resp., maximization) problems by analyzing approximability of their maximization (resp., minimization) equivalents under affine transformations. We will apply such transformation in Sections 3 and 4.

We first study in this paper the performance of two approximation algorithms for (unweighted) MIN SET COVER. The first one is the classical greedy algorithm studied, for the unweighted case and for the standard approximation ratio, in [13, 14] and, more recently, in [15]. For this algorithm, we provide a differential approximation ratio bounded below by $1.365/\Delta$, when $\Delta = \max_{S_i \in \mathcal{S}}\{|S_i|\}$ is sufficiently large, and an upper bound of $4/(\Delta + 1)$. The second approximation algorithm studied for MIN SET COVER, called 2_OPT in the sequel, computes a 2-optimal set-cover for C. A set-cover is 2-optimal if and only if it cannot be reduced by removing two sets from it and by adding one new set to it. We prove that algorithm 2_OPT achieves differential approximation ratio $2/(\Delta + 1)$. Finally, we deal with MIN WEIGHTED SET COVER and analyze the differential approximation performance of a simple greedy algorithm that starts from the whole $\mathcal{S}$ considering it as solution for MIN WEIGHTED SET COVER and then it reduces it by removing the heaviest of the remaining sets of $\mathcal{S}$ per time until the cover becomes minimal. We show that this algorithm achieves differential approximation ratio $1/\Delta$.

Differential approximability for both MIN SET COVER and MIN WEIGHTED SET COVER have already been studied in [6] and discussed in [11]. The differential approximation ratios provided there are $1/\Delta$, for the former, and $1/(\Delta+1)$, for the latter. Our current work improves (quite significantly for the unweighted case), these old results. Note also that an approximation algorithm for MIN SET COVER has been analyzed also in [8] under the assumption $m \geqslant n$, the size of the ground set C. It has been shown that, under this assumption, MIN SET COVER is approximable within differential approximation ratio $1/2$. Under the same assumption (i.e., $m \geqslant n$), analyses for MIN WEIGHTED SET COVER and for MIN SET COVER have been recently performed in [12] (for the former) and in [11] (for the latter). In the cost model of [12], any element $c_i \in C$ has some non-negative weight w_i and the weight $w(S_j)$ of a set S_j is some convex function of the weights of its elements (for example, $\max_{c_i \in S_j}\{w_i\}$, or $\min_{c_i \in S_j}\{w_i\}$, or $\sum_{c_i \in S_j} w_i/|S_j|, \ldots$). In [11] MIN SET COVER has been proved approximable within differential approximation ratio $289/360$.

The following inapproximability result, proved in [8], holds for MIN SET COVER and shows that approximation ratios of the same type as in standard approximation (for example, $O(1/\ln\Delta)$, or $O(1/\log n)$) are extremely unlikely for MIN SET COVER in differential approximation. Consequently, differential approximation results for MIN SET COVER cannot be trivially achieved by simply transposing the existing standard approximation results to the differential framework. This is a further motivation of our work.

Proposition 1. *([8]) If $\mathbf{P} \neq \mathbf{NP}$, then inapproximability bounds for standard (and differential) approximation for* MAX INDEPENDENT SET *hold as differential inapproximability bounds for* MIN SET COVER. *Consequently, unless $\mathbf{P} = \mathbf{NP}$,* MIN SET COVER *is not differentially approximable within $O(n^{\epsilon-(1/2)})$, for any $\epsilon > 0$.*

Observe first that, for MAX INDEPENDENT SET, standard and differential approximation ratios coincide[1]. What can be deduced from the proof of Proposition 1 in [8] is that, in differential paradigm, any approximation ratio for MIN SET COVER can be immediately shifted to MAX INDEPENDENT SET. In particular, any differential ratio, function of Δ, for the former, would be transformed into a ratio of the same form and function of the maximum degree $\Delta(G)$ of the input graph for the latter. The best known ratio, expressed only in terms of $\Delta(G)$, for MAX INDEPENDENT SET in general graphs, is asymptotically (i.e., when $\Delta(G) \to \infty$) bounded above by $k/\Delta(G)$, for any $k \in \mathbb{N}$ ([9]). Consequently, the only one can reasonably hope for differential approximation ratio of MIN SET COVER, is to increase factor 1 in the expression $1/\Delta$.

Another interesting conclusion of our paper is that contrary to what is observed in standard paradigm, the greedy algorithm does not seem to guarantee the best differential approximation ratio for MIN SET COVER. However, we present its analysis both for its own mathematical interest and because this algorithm is the most known one for our problem.

In what follows, we deal with non-trivial instances of (unweighted) MIN SET COVER. An instance I is non-trivial for unweighted MIN SET COVER if the two following conditions hold simultaneously:

- no set $S_i \in \mathcal{S}$ is a proper subset of a set $S_j \in \mathcal{S}$;
- no element in C is contained in I by only one subset of $\mathcal{S}$ (i.e., there is no non-redundant set for $\mathcal{S}$).

2 The natural greedy algorithm for MIN SET COVER

Let us first note that a lower bound of $1/\Delta$ can be easily proved for the differential ratio of any algorithm computing a minimal set cover. We analyze in this section the differential approxi-

[1] The worst independent set in a graph is the empty set.

mation performance of the following very classical greedy algorithm for MIN SET COVER, called `SCGREEDY` in the sequel, computing a minimal set-covering in $O(nm)$ steps:

1. set $N'' = \emptyset$;
2. compute $S_i \in \text{argmax}_{S \in \mathcal{S}}\{|S|\}$;
3. set $N'' := N'' \cup \{i\}$;
4. update I setting: $\mathcal{S} = \mathcal{S} \setminus \{S_i\}$, $C = C \setminus S_i$ and, for any $S_j \in \mathcal{S}$, $S_j := S_j \setminus S_i$;
5. repeat Steps 2 to 4 until $C = \emptyset$;
6. range N'' in the order sets have been chosen and assume $N'' = \{i_1, i_2, \dots, i_k\}$;
7. Set $N' = N''$; for $j = k$ downto 1: if $N' \setminus \{i_j\}$ is a cover then $N' := N' \setminus \{i_j\}$;
8. output N' the minimal cover computed in Step 7.

Consider N'' and the sets $\mathcal{S}'' = \{S'_{i_1}, S'_{i_2} \dots, S'_{i_k}\}$, computed in Step 6 with their residual cardinalities, i.e., as they have been chosen during Steps 2 and 4; remark that, so-considered, the set $\mathcal{S}''$ forms a partition on C. On the other hand, consider solution N' output by the algorithm `SCGREEDY` and remark that family $\{S'_i : \ i \in N'\}$ does not necessarily cover C.

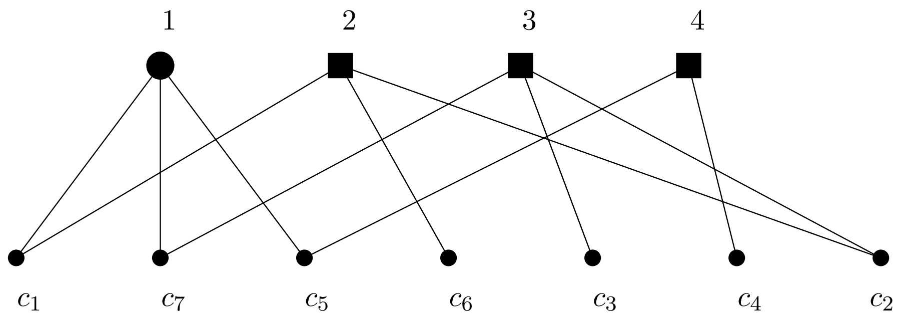

Figure 1: An example of application of Step 7 of `SCGREEDY`.

For example, assume some MIN SET COVER-instance $(\mathcal{S}, C)$ with $C = \{c_1, \dots, c_7\}$ and suppose that execution of Steps 2 to 6 has produced a cover $N'' = \{1, 2, 3, 4\}$ (given by the sets $\{S_1, S_2, S_3, S_4\}$). Figure 1 illustrates characteristic graph B', i.e., the subgraph of $B = (L, R; E)$ (see Definition 1) induced by $L' \cup R$ where L' and R correspond to the sets N'' and C, respectively. It is easy to see that N'' is not minimal and application of Step 7 of `SCGREEDY` drops the set S_1 out of N''; hence, $N' = \{2, 3, 4\}$. The residual parts of S_2, S_3 and S_4 are $S'_2 = \{c_2, c_6\}$, $S'_3 = \{c_3\}$ and $S'_4 = \{c_4\}$, respectively. Note that Step 7 of `SCGREEDY` is important for the solution returned in Step 8, since solution N'' computed in Step 6 may be a worst solution (see the previous example) and then, $\delta(I, N'') = (\omega(I) - |N''|)/(\omega(I) - \text{opt}(I)) = 0$.

Theorem 1. *For Δ sufficiently large, algorithm `SCGREEDY` achieves differential approximation ratio* $1.365/\Delta$.

Proof. The proof that follows is quite technical. For legibility, we will first sketch its basic idea. Fix an optimal sub-family $\mathcal{S}^*$ of $\mathcal{S}$ covering C, and denote by N^* the indices of its sets. Let C'_1 be the set of the non-redundant elements with respect to $N' \setminus N^*$. Let $N^*_1 \subset N^* \setminus N'$ be the set of the indices of the sets of $\mathcal{S}^*$ that cover C'_1. Finally, denote by $\bar{N}$, the set of the indices that are neither in N^*, nor in N' (in other words, $\bar{N}$ corresponds to sets of $\mathcal{S}$ that are taken neither in the fixed optimal solution nor in the one constructed by `SCGREEDY`. The proof can be split into three consecutive parts.

1. We first prove that the differential approximation ratio of `SCGREEDY` can be expressed as:
$$\delta\left(I, N'\right) = \frac{|N_1^*| + |N^* \setminus (N' \cup N_1^*)| + \left|\bar{N}\right|}{|N' \setminus N^*| + \left|\bar{N}\right|} \tag{1}$$
2. We then prove that the greedy rule of `SCGREEDY` assures that there exists a set $C_2' \subseteq C \setminus C_1'$ covered (with respect to $\mathcal{S}^*$) by a set of indices $N_2^* \subseteq N^* \setminus N_1^*$, such that $|N_2^*| \geqslant |C_2'|/\Delta$.
3. Suppose that a portion of N_2^* belongs to $N^* \setminus N'$ (indeed, in $N^* \setminus (N' \cup N_1^*)$), the rest of N_2^* belonging to $N^* \cap N'$.
   (a) We prove that the former portion contributes to the value of the ratio given in (1).
   (b) Dealing with the latter part of N_2^*, i.e., the one in $N^* \cap N'$, we show that this part entails the existence of a subset $C_3' \subseteq C$, distinct from both C_1' and C_2', the elements of which are not covered by N_1^* and are contained in at least $|C_3'|/\Delta$ sets whose indices are either in $N^* \setminus N_1^*$, or in $\bar{N}$.
4. We finally prove that in both of the cases of Item (3b) these sets of indices contribute to the value of the ratio.

Figure 2 illustrates Items 1 through 4. Lines between an upper rectangle and a lower ellipsis show that elements of the corresponding subset of C (C_i', $i = 1, 2, 3$) are contained, or covered, by sets the indices of which belong to the corresponding thick rectangle.

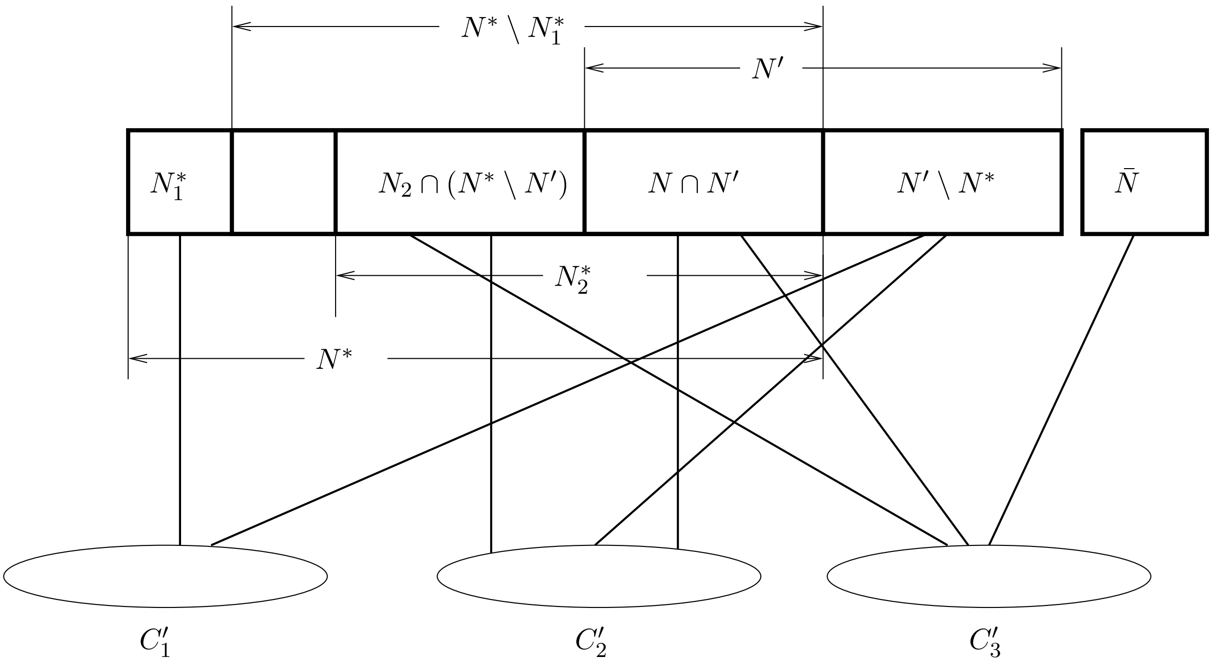

Figure 2: A rough illustration of the proof of Theorem 1.

We are ready now to undertake the formal proof of the theorem. Let $C_1' = \{c_i : i \in N' \setminus N^*\}$ be a set of non-redundant elements; obviously, by construction $|C_1'| = |N' \setminus N^*|$. Moreover, consider an optimal solution N^* given by the sets S_i, $i \in N^*$ and denote by $\{S_i^*\}$, $S_i^* \subseteq S_i$, $i \in N^*$, an arbitrary partition of C (if an element c is covered by more than one sets S_i, $i \in N^*$, then c is arbitrarily assigned to one of them). Let $N_1^* = \{j \in N^* : \exists c \in C_1', c \in S_j^*\}$. We deduce $N_1^* \subseteq N^* \setminus N'$, since any element $c \in C_1'$ is non-redundant for N' (otherwise, there would exist at least a $c \in C_1'$ covered twice: one time by a set in $N' \setminus N^*$ and one time by a set in $N' \cap N^*$, absurd by the construction of C_1'). Finally, set $\bar{N} = \{1, \ldots, m\} \setminus (N' \cup N^*)$. Observe that, using the notations just introduced, we have:

Consider the bipartite graph $B'' = (L'', R''; E'')$ with $L'' = N_1^* \cup (N' \setminus N^*)$, $R'' = C_1'$ and $(i, c_j) \in E''$ if and only if $c_j \in S_i^*$ or $i = j$. This graph is a partial graph of the characteristic bipartite graph B' induced by $L' = N_1^* \cup (N' \setminus N^*)$ and $R' = C_1'$. By construction, B'' is not necessarily connected and, furthermore, any of its connected components is of the form of Figure 3.

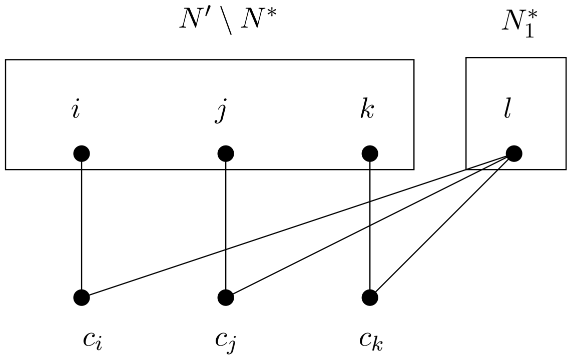

Figure 3: A connected component of B''.

For $i = 1, \ldots, \Delta$, denote by x_i the number of connected components of B'' corresponding to sets S_l^* of cardinality i. Then, by construction of this sub-instance, we have:

$$|N_1^*| = \sum_{i=1}^{\Delta} x_i \tag{2}$$

$$\left|N' \setminus N^*\right| = \left|C_1'\right| = \sum_{i=1}^{\Delta} i x_i \tag{3}$$

Consider $z \in [1, \Delta]$ such that $|C_1'| = i_0 |N_1^*|$ where $i_0 = \Delta / z$. One can easily see that i_0 is the average cardinality of sets in N_1^* (when we consider the sets S_i^*, $i \in N_1^*$, that form, by construction, a partition on C_1'). Indeed,

$$i_0 = \frac{1}{|N_1^*|} \sum_{i \in N_1^*} |S_i^*| = \frac{\sum_{i=1}^{\Delta} i x_i}{\sum_{i=1}^{\Delta} x_i} \tag{4}$$

We have immediately from (1), (2) and (3):

$$\delta\left(I, N'\right) \geqslant \frac{|N_1^*|}{|N' \setminus N^*|} = \frac{|N_1^*|}{|C_1'|} = \frac{1}{i_0} = \frac{z}{\Delta} \tag{5}$$

Consider once more the component of Figure 3, suppose that the set S_ℓ^* has cardinality i and denote it by $S_\ell^* = \{c_{\ell_1}, \ldots, c_{\ell_i}\}$ with $\ell_1 < \ldots < \ell_i$. By the greedy rule of `SCGREEDY`, we deduce that the sets $S'_{\ell_1}, \ldots, S'_{\ell_i}$ (recall that we only consider the residual part of the set) have been chosen in this order (cf. Steps 6 and 7 of SCGREEDY) and verify $|S'_{\ell_p}| \geqslant i+1-p$ for $p = 1, \ldots, i$. Indeed, by the greedy rule, when SCGREEDY takes the set S'_{ℓ_p} in the solution under construction, the residual part of this set is greater than, or equal to, the residual part of S^*_ℓ that is at equal to $i+1-p$. Consequently, there exist $(i-1)+(i-2)+\ldots+1 = i(i-1)/2$ elements of C not included in C'_1. Iterating this observation for any connected component of B'' we can conclude that there exists a set $C_2 \subseteq C$, outside set C'_1 (i.e., $C'_2 \subseteq C \setminus C'_1$), of size at least $|C_2| \geqslant \sum_{i=1}^{\Delta} i(i-1)x_i/2$. Elements of C_2 are obviously covered, with respect to N^*, by sets either from N^*_1, or from $N^* \setminus N^*_1$. Suppose that sets of N^*_1 of cardinality i (there exist x_i such sets), $i = 1, \ldots, \Delta$, cover a total of $k_i x_i$ elements of C_2. Therefore, there exists a subset $C'_2 \subseteq C_2$ of size at least:

$$\left|C'_2\right| \geqslant \sum_{i=1}^{\Delta}\left(\frac{i(i-1)}{2} - k_i\right) x_i \tag{6}$$

The elements of C'_2 are covered in N^* by sets in $N^* \setminus N^*_1$. Using (6), one can see that in order that C'_2 is covered, a family $N^*_2 \subseteq N^* \setminus N^*_1$ of size

$$|N^*_2| \geqslant \frac{1}{\Delta}\sum_{i=1}^{\Delta}\left(\frac{i(i-1)}{2} - k_i\right) x_i \tag{7}$$

is needed. Dealing with N^*_2, suppose that for a $y \in [0,1]$:

1. $(1-y)|N^*_2|$ sets of N^*_2 belong to $N^* \setminus N'$ (indeed, they belong to $N^* \setminus (N' \cup N^*_1)$) and
2. $y|N^*_2|$ sets of N^*_2 belong to $N^* \cap N'$;

We study the following two cases: $y \leqslant (\Delta-1)/\Delta$ and $y \geqslant (\Delta-1)/\Delta$.

$y \leqslant (\Delta - 1)/\Delta$

This case is equivalent to $(1-y) \geqslant 1/\Delta$ and then, taking into account that $k_i \leqslant \Delta - i$, we obtain:

$$(1-y)\left|N^*_2\right| \geqslant \frac{\left|N^*_2\right|}{\Delta} \geqslant \sum_{i=1}^{\Delta}\left(\frac{i(i-1)}{2\Delta^2} + \frac{i}{\Delta} - 1\right) x_i \tag{8}$$

Using (1), (2), (3), (7) and (8), we deduce:

$$\delta\left(I, N'\right) \geqslant \frac{|N^*_1| + \frac{|N^*_2|}{\Delta}}{|N' \setminus N^*|} \geqslant \frac{\sum_{i=1}^{\Delta}\left(\frac{i(i-1)}{2\Delta^2} + \frac{i}{\Delta}\right) x_i}{\sum_{i=1}^{\Delta} i x_i} = \frac{1}{\Delta} + \frac{\sum_{i=1}^{\Delta} f(i) x_i}{\sum_{i=1}^{\Delta} i x_i} \tag{9}$$

where $f(x) = x(x-1)/(2\Delta^2)$, with $1 \leqslant x \leqslant \Delta$. We will now show the following inequality (recall that, from (4), $i_0 = \sum_{i=1}^{\Delta} i x_i / \sum_{i=1}^{\Delta} x_i$):

$$\frac{\sum_{i=1}^{\Delta} f(i) x_i}{\sum_{i=1}^{\Delta} i x_i} \geqslant \frac{f\left(i_0\right)}{i_0} \tag{10}$$

Remark that (10) is equivalent to $\sum_{i=1}^{\Delta}(f(i)x_i)/\sum_{i=1}^{\Delta} x_i \geqslant f(i_0)$. On the other hand, since f is convex, we have by Jensen's theorem $\sum_{i=1}^{\Delta} z_i f(i) \geqslant f(\sum_{i=1}^{\Delta} i z_i)$, where $z_i \in [0,1]$, $\sum_{i=1}^{\Delta} z_i = 1$. Setting $z_i = x_i/\sum_{i=1}^{\Delta} x_i$, (10) follows.

Thus, since $i_0 = \Delta/z$ and we study an asymptotic ratio in Δ, (9) becomes

$$\delta\left(I, N'\right) \geqslant \frac{1}{\Delta} + \frac{1}{2\Delta^2}\left(\frac{\Delta}{z} - 1\right) \approx \frac{1}{\Delta} + \frac{1}{2\Delta z} \tag{11}$$

Expression (11) is decreasing with z, while (5) is increasing with z. Equality of both ratios is reached when $2z^2 - 2z - 1 = 0$, i.e., for $z = (2+\sqrt{12})/4 \approx 1.365$

$\boldsymbol{y \geqslant (\Delta - 1)/\Delta}$

Sub-family $N_2^* \cap N'$ (of size $y|N_2^*|$) is, by hypothesis, common to both N' (the cover computed by `SCGREEDY`) and N^*. Minimality of N' implies that, for any set $i \in N_2^* \cap N'$, there exists at least one element of C non-redundant with respect to S_i. So, there exist at least $|C_3| = |N_2^* \cap N'|$ elements of C outside C_1' and C_2.

Some elements of C_3 can be covered by sets in N_1^*. In any case, for the sets $\{j_1, \ldots, j_{x_i}\}$ of N_1^* of cardinality i with respect to the partition S_ℓ^*, there exist at most $(\Delta - (i+k_i))x_i$ elements of C_3 that can belong to them (so, these elements are covered by the residual set $S_{j_p} \setminus S_{j_p}^*$ for $p = 1, \ldots, x_i$). Thus, there exist at least

$$\left|C_3'\right| = |C_3| - \sum_{i=1}^{\Delta}\left(\Delta - (i+k_i)\right)x_i = y\left|N_2^*\right| - \sum_{i=1}^{\Delta}\left(\Delta - (i+k_i)\right)x_i \tag{12}$$

elements of C_3 not covered by sets in N_1^*. Since initial instance $(\mathcal{S}, C)$ is non-trivial, elements of C_3' are also contained in sets N_3 either from $N^* \setminus N_1^*$, or from $\bar{N}$. So, the family N_3 has size at least $|C_3'|/\Delta$. Moreover, using (7), (12) and $y \leqslant 1$, we get:

$$|N_3| \geqslant \frac{y\left|N_2^*\right|}{\Delta} - \sum_{i=1}^{\Delta}\frac{\left(\Delta - (i+k_i)\right)x_i}{\Delta} \geqslant y\sum_{i=1}^{\Delta}\frac{i(i-1)}{2\Delta^2}x_i + \sum_{i=1}^{\Delta}\left(\frac{i}{\Delta} - 1\right)x_i \tag{13}$$

We so deduce:

$$\begin{aligned}
\delta\left(I, N'\right) &\geqslant \frac{\left|N_1^*\right| + \left|N_3 \setminus \bar{N}\right| + \left|\bar{N}\right|}{\left|N' \setminus N^*\right| + \left|\bar{N}\right|} \overset{|\bar{N}| \geqslant |\bar{N} \cap N_3|}{\geqslant} \frac{\left|N_1^*\right| + \left|N_3\right|}{\left|N' \setminus N^*\right| + \left|\bar{N} \cap N_3\right|} \\
&\geqslant \frac{\left|N_1^*\right| + \left|N_3\right|}{\left|N' \setminus N^*\right| + \left|N_3\right|}
\end{aligned} \tag{14}$$

Note, furthermore, that function $(a+x)/(b+x)$ is increasing with x, for $a \leqslant b$ and $x > -b$. Therefore, using (2), (3), (13) and $y \geqslant (\Delta-1)/\Delta$, (14) becomes:

$$\delta\left(I, N'\right) \geqslant \frac{\sum_{i=1}^{\Delta}\left(\frac{(\Delta-1)i(i-1)}{2\Delta^3} + \frac{i}{\Delta}\right)x_i}{\sum_{i=1}^{\Delta}\left(i + \frac{(\Delta-1)i(i-1)}{2\Delta^3} + \frac{i}{\Delta} - 1\right)x_i} \tag{15}$$

Set now $f(x) = (\Delta - 1)(x(x-1)/2\Delta^3) + (x/\Delta)$. Then, (15) can be expressed as:

$$\delta\left(I, N'\right) \geqslant \frac{\sum_{i=1}^{\Delta} f(i)x_i}{\sum_{i=1}^{\Delta}\left(f(i) + i - 1\right)x_i} \tag{16}$$

Using the same arguments as previously about the convexity of f, we deduce from (16):

$$\delta\left(I, N'\right) \geqslant \frac{f\left(i_0\right)}{f\left(i_0\right)+i_0-1} = \frac{\frac{(\Delta-1)i_0(i_0-1)}{2\Delta^3}+\frac{i_0}{\Delta}}{i_0+\frac{(\Delta-1)i_0(i_0-1)}{\Delta^3}+\frac{i_0}{\Delta}-1} \tag{17}$$

Recall that we have fixed $i_0 = \Delta/z$. If one assumes that Δ is arbitrarily large, one can simplify calculations by replacing $i_0 - 1$ by i_0. Then, (17) becomes:

$$\delta\left(I, N'\right) \geqslant \frac{\frac{i_0^2}{2\Delta^2}+\frac{i_0}{\Delta}}{\frac{i_0^2}{2\Delta^2}+\frac{i_0}{\Delta}+i_0} \geqslant \frac{\frac{1}{2z^2}+\frac{1}{z}}{\frac{1}{2z^2}+\frac{1}{z}+\frac{\Delta}{z}} \approx \frac{1}{2z\Delta}+\frac{1}{\Delta} \tag{18}$$

Ratio given by (5) is increasing with z, while the one of (18) is decreasing with z. Equality of both ratios is reached when $2z^2 - 2z - 1 = 0$, i.e., for $z \approx 1.365$.

So, in any of the cases studied above, the differential approximation ratio achieved by `SCGREEDY` is greater than, or equal to, $1.365/\Delta$ and the proof of the theorem is now complete. ∎

Proposition 2. *There exist* MIN SET COVER-*instances where the differential approximation ratio of* `SCGREEDY` *is* $4/(\Delta+2)$ *for any* $\Delta \geqslant 3$.

Proof. Assume a fixed $t > 1$, a ground set $C = \{c_{ij} : i = 1, \ldots, t-1, j = 2, \ldots, t, j > i\}$ and a system $\mathcal{S} = \{S_1, \ldots, S_t\}$, where $S_i = \{c_{ji} : j < i\} \cup \{c_{ij} : j > i\}$, for $i = 1, \ldots, t$. Denote by $I_t = (\mathcal{S}, C)$ the instance of MIN SET COVER defined on C and $\mathcal{S}$.

Remark that the smallest cover for C includes at least $t-1$ sets of $\mathcal{S}$. Indeed, consider a family $\mathcal{S}' \subseteq \mathcal{S}$ of size less than $t-1$. Then, there exists $i_0 < j_0$ such that neither S_{i_0}, nor S_{j_0} belong to $\mathcal{S}'$. In this case element $c_{i_0j_0} \in C$ is not covered by N'. Note finally that, for I_t, the maximum size of the subsets of $\mathcal{S}$ is $\Delta = t-1$. Indeed, for any $i = 1, \ldots, t$, $|\{c_{ji} : j < i\}| = i-1$ and $|\{c_{ij} : j < i\}| = t-i$; so, $|S_i| = t-1$.

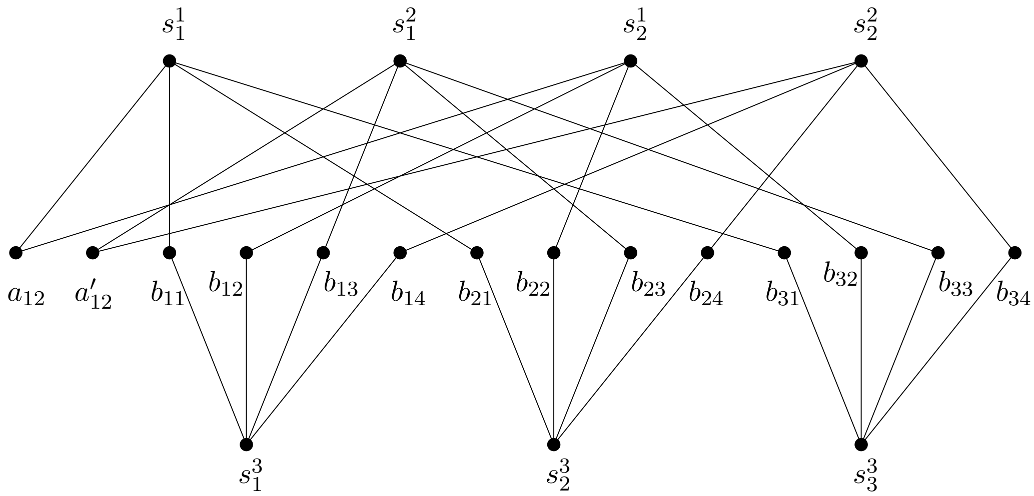

Figure 4: The characteristic graph of instance $\hat{I}$ for $\Delta = 4$.

Fix now an even Δ and construct the following instance $\hat{I} = (\mathcal{S}, C)$ for MIN SET COVER:

$$\begin{aligned} C &= \left\{a_{ij}, a'_{ij} : i = 1, \ldots, \frac{\Delta}{2}, j = 1, \ldots, \frac{\Delta}{2}, j > i\right\} \cup \left\{b_{ij} : i = 1, \ldots, \frac{\Delta+2}{2}, j = 1, \ldots, \Delta\right\} \\ S_i^1 &= \{a_{ji} : j < i\} \cup \{a_{ij} : j > i\} \cup \left\{b_{ji} : j = 1, \ldots, \frac{\Delta+2}{2}\right\} \quad i = 1, \ldots, \frac{\Delta}{2} \end{aligned}$$

$$
\begin{aligned}
S_i^2 &= \{a'_{ji} : j < i\} \cup \{a'_{ij} : j > i\} \cup \left\{b_{jk} : j = 1, \ldots, \frac{\Delta+2}{2}, k = i + \frac{\Delta}{2}\right\} \quad i = 1, \ldots, \frac{\Delta}{2} \\
S_i^3 &= \{b_{ij} : j = 1, \ldots, \Delta\} \quad i = 1, \ldots, \frac{\Delta+2}{2} \\
\mathcal{S}^j &= \left\{S_i^j : i = 1, \ldots, \frac{\Delta}{2}\right\} \quad j = 1, 2 \\
\mathcal{S}^3 &= \left\{S_i^3 : i = 1, \ldots, \frac{\Delta+2}{2}\right\} \\
\mathcal{S} &= \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3
\end{aligned}
$$

For example, the following instance $\hat{I} = (\mathcal{S}, C)$ can be constructed, according to above schema, with $\Delta = 4$ (its characteristic graph is illustrated in Figure 4):

$$
\begin{aligned}
C &= \{a_{12}, a'_{12}, b_{11}, b_{12}, b_{13}, b_{14}, b_{21}, b_{22}, b_{23}, b_{24}, b_{31}, b_{32}, b_{33}, b_{34}\} \\
S_1^1 &= \{a_{12}, b_{11}, b_{21}, b_{31}\} \\
S_1^2 &= \{a'_{12}, b_{13}, b_{23}, b_{33}\} \\
S_1^3 &= \{b_{11}, b_{12}, b_{13}, b_{14}\} \\
S_2^1 &= \{a_{12}, b_{12}, b_{22}, b_{32}\} \\
S_2^2 &= \{a'_{12}, b_{14}, b_{24}, b_{34}\} \\
S_2^3 &= \{b_{21}, b_{22}, b_{23}, b_{24}\} \\
S_3^3 &= \{b_{31}, b_{32}, b_{33}, b_{34}\} \\
\mathcal{S}^1 &= \{S_1^1, S_2^1\} \\
\mathcal{S}^2 &= \{S_1^2, S_2^2\} \\
\mathcal{S}^3 &= \{S_1^3, S_2^3, S_3^3\} \\
\mathcal{S} &= \{S_1^1, S_2^1, S_1^2, S_2^2, S_1^3, S_2^3, S_3^3\}
\end{aligned}
$$

It is easy to see that, $\forall S_i \in \mathcal{S}$, $|S_i| = \Delta$. Hence, during its first iteration, `SCGREEDY` can choose a set in $\mathcal{S}^3$. Such a choice does not reduce cardinalities of the remaining sets in this sub-family; so, during its first $(\Delta+2)/2$ iterations, `SCGREEDY` can exclusively choose all sets in $\mathcal{S}^3$. Remark that such choices entail that the surviving instance is the union of two disjoint instances $I_{\Delta/2}$ (i.e., instances of type I_t, as the ones defined at the beginning of this section, with $t = \Delta/2$), induced by the sub-systems $(\mathcal{S}^1, \{a_{ij}\})$ and $(\mathcal{S}^2, \{a'_{ij}\})$. According to what has been discussed at the beginning of the section, any cover for such instances uses at least $(\Delta/2) - 1$ sets. We so have, for a set-cover N' computed by `SCGREEDY` (remark that N' is minimal):

$$
m\left(\hat{I}, N'\right) \geqslant \frac{\Delta+2}{2} + 2\left(\frac{\Delta}{2} - 1\right) = \frac{3\Delta}{2} - 1 \tag{19}
$$

Furthermore, given that sub-family $\mathcal{S}^1 \cup \mathcal{S}^2$ is a cover for C, we have:

$$
\operatorname{opt}\left(\hat{I}\right) = \Delta \tag{20}
$$

$$
\omega\left(\hat{I}\right) = \frac{3\Delta}{2} + 1 \tag{21}
$$

Combination of (19), (20) and (21) concludes $\delta(\hat{I}, N') = 4/(\Delta+2)$.

Assume now that Δ is odd and consider the following instance $(\mathcal{S}, C)$ for MIN SET COVER:

$$
C = \left\{a_{ij} : i = 1, \ldots, \frac{\Delta-1}{2}, j = 2, \ldots, \frac{\Delta-1}{2}, j > i\right\}
$$

$$
\begin{array}{rcl}
 & & \cup \left\{ a'_{ij} : i = 1, \ldots, \frac{\Delta+1}{2}, j = 2, \ldots, \frac{\Delta+1}{2}, j > i \right\} \\
 & & \cup \left\{ b_{ij} : i = 1, \ldots, \frac{\Delta+1}{2}, j \in 1, \ldots, \Delta \right\} \\
S_i^1 & = & \{a_{ji} : j < i\} \cup \{a_{ij} : j > i\} \cup \left\{ b_{ji} : j = 1, \ldots, \frac{\Delta+1}{2} \right\} \quad i = 1, \ldots, \frac{\Delta-1}{2} \\
S_i^2 & = & \{a'_{ji} : j < i\} \cup \{a_{ij} : j > i\} \\
 & & \cup \left\{ b_{jk} : j = 1, \ldots, \frac{\Delta+1}{2}, k = i + \frac{\Delta-1}{2} \right\} \quad i = 1, \ldots, \frac{\Delta+1}{2} \\
S_i^3 & = & \{b_{ij} : j = 1, \ldots, \Delta\} \quad i = 1, \ldots, \frac{\Delta+1}{2}
\end{array}
$$

The same arguments conclude an upper bound of $4/(\Delta+1)$, for odd values of Δ. ▮

3 Local optimality and MIN SET COVER

Given a solution $N' \subseteq \{1, \ldots, m\}$ of an instance $I = (\mathcal{S}, C)$ of MIN SET COVER (corresponding to the family $\mathcal{S}' = \{S_i : i \in N'\}$, a solution N'' is a feasible 2-improvement of N' if there exist $j, k \in N'$ and $i \in \{1, \ldots, m\}$ such that $N'' = (N' \setminus \{j, k\}) \cup \{i\}$ is a feasible solution for MIN SET COVER in I. When $i \in N'$, the 2-improvement can be viewed as two (or one, if $i = j$ or $i = k$) consecutive 1-improvements. A *proper 2-improvement* is the one for which $i \in \{1, \ldots, m\} \setminus N'$. A set-cover N' is 2-optimal for MIN SET COVER if no 2-improvement is possible upon it. Remark that a 2-optimal set cover N' is 1-optimal (minimal for the inclusion).

The following algorithm, called `2_OPT` in the sequel, polynomially computes 2-local optima for MIN SET COVER:

1. set $N' = \{1, \ldots, m\}$;
2. while a 2-improvement of N' is possible do it;
3. output the 2-optimal set-cover N' computed in Step 2.

Theorem 2. *Algorithm `2_OPT` achieves differential approximation ratio bounded below by $2/(\Delta+1)$. This bound is tight.*

Proof. Fix an optimal solution N^* for MIN SET COVER. Note that, by construction, N' and N^* are minimal for the inclusion. Denote by C' the subset of C, the elements of which are not covered by $N' \cap N^*$. We can assume $C' \neq \emptyset$; otherwise, $|N'| = |N^*|$. We set: $\hat{N}' = \{i \in N' : S_i \cap C' \neq \emptyset\}$ and $\hat{N}^* = \{i \in N^* : S_i \cap C' \neq \emptyset\}$.

Fact 1. Both $\hat{N}^* \setminus \hat{N}'$ and $\hat{N}' \setminus \hat{N}^*$ cover C'. ▮

Fact 2. $N^* \setminus (N^* \cap N') \subseteq \hat{N}^*$ and $N' \setminus (N^* \cap N') \subseteq \hat{N}'$. ▮

Indeed, if $i \in N' \setminus (N^* \cap N')$ and $i \notin \hat{N}'$, then algorithm `2_OPT` should delete i to N', i.e., N' would not be minimal. Analogously, if $i \in N^* \setminus (N^* \cap N')$ and $i \notin \hat{N}^*$, then i should delete of N^*, i.e., N^* would not be minimal.

Immediate consequences of Fact 2 are the following equalities:

$$
\begin{array}{rcl}
|N'| & = & \left|\hat{N}'\right| + |N^* \cap N'| \qquad (22) \\
|N^*| & = & \left|\hat{N}^*\right| + |N^* \cap N'| \qquad (23)
\end{array}
$$

Transform arbitrarily the sets of the family associated to $\hat{N}' \setminus \hat{N}^*$ into a partition of C', and denote by N'_1 the resulting sets of size 1. Moreover, assume that $\hat{S}'_i = \{c_i\}$ for $i \in N'_1$. Then,

$$\left|N'_1\right| \leqslant \left|\hat{N}^* \setminus \hat{N}'\right| \tag{24}$$

Otherwise, there would exist $j, k \in N'_1$ and $i \in \hat{N}^* \setminus \hat{N}'$ such that S_i covers c_j and c_k. In this case, algorithm `2_OPT`, would have dropped j and k out of N' and would have introduced i into N'.

Using Fact 1 and (24), we get:

$$\left|\hat{N}' \setminus \hat{N}^*\right| \leqslant \left|N'_1\right| + \frac{|C'| - |N'_1|}{2} \leqslant \frac{\Delta + 1}{2}\left|\hat{N}^* \setminus \hat{N}'\right| \tag{25}$$

Moreover, we have:

$$\omega(I) = m \geqslant \left|N'\right| + |N^*| - \left|N' \cap N^*\right| \geqslant \left|\hat{N}'\right| + \left|\hat{N}^*\right| - \left|N' \cap N^*\right| \tag{26}$$

Then, using (22), (23), (25) and (26), we get:

$$\begin{aligned}
\delta_{\texttt{2_OPT}}\left(I, N'\right) &= \frac{m - |N'|}{m - |N^*|} \geqslant \frac{\left|\hat{N}^*\right|}{\left|\hat{N}'\right|} \\
&\geqslant \frac{\left|\hat{N}^* \setminus \hat{N}'\right|}{\left|\hat{N}' \setminus \hat{N}^*\right|} \geqslant \frac{2}{\Delta + 1}
\end{aligned}$$

For tightness, fix a $\Delta \in \mathbb{N}$ and consider the following instance of MIN SET COVER:

$$\begin{aligned}
C &= \{x_i, y_i : i = 1, \ldots, \Delta\} \\
S_1 &= \{x_i : i = 1, \ldots, \Delta\} \\
S_2 &= \{y_i : i = 1, \ldots, \Delta\} \\
S_{i+2} &= \{x_i, y_i\} \quad i = 1, \ldots, \Delta - 1 \\
S_{\Delta+2} &= \{x_\Delta\} \\
S_{\Delta+3} &= \{y_\Delta\} \\
N &= \{S_i : i = 1, \ldots, \Delta + 3\}
\end{aligned}$$

Then, $N' = \{3, \ldots, \Delta + 1\}$ and $N^* = \{1, 2\}$. The approximation ratio achieved by `2_OPT` on this instance is equal to $2/(\Delta + 1)$, and the proof of the theorem is complete. ∎

Remark that the result of Theorem 2, identically applies to MAX SET SAVING defined as follows: given a family $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of a ground set $C = \{c_1, c_2, \ldots, c_n\}$ such that $\mathcal{S}$ is a cover for C, determine the maximum-size sub-family $\mathcal{S}^*$ such that $\mathcal{S} \setminus \mathcal{S}^*$ is a cover for C.

MAX SET SAVING is clearly equivalent to MIN SET COVER for the differential approximation, since the objective function of the one is an affine transformation of the objective function of the other and differential ratio is stable under such transformations ([8]). Moreover, for MAX SET SAVING, standard and differential approximation ratios coincide since empty set is a feasible solution for it. Equivalence of MAX SET SAVING and MIN SET COVER for the differential approximation implies that if one runs `2_OPT`, in order to return N', the set $\{1, \ldots, m\} \setminus N'$ is a solution for MAX SET SAVING that achieves (standard and differential) approximation ratio $2/(\Delta + 1)$. Hence, the following corollary immediately holds from Theorem 2.

Corollary 1. *Algorithm `2_OPT` with complementation of its output (with respect to $\mathcal{S}$) approximately solves* MAX SET SAVING *within both standard and differential approximation ratio bounded below by $2/(\Delta + 1)$. This ratio is tight.*

4 Differential approximation for MIN WEIGHTED SET COVER

Consider an instance $I = (\mathcal{S}, C, \vec{w})$ of MIN WEIGHTED SET COVER, where $\vec{w}$ denotes the vector of the weights on the subsets of $\mathcal{S}$ and the following algorithm, denoted by WSC in what follows:

- order sets in $\mathcal{S}$ in decreasing weight-order (i.e., $w_1 \geqslant \ldots \geqslant w_m$); let $N = \{1, \ldots, m\}$ be the set of indices in the (so-ordered) $\mathcal{S}$;
- set $N' = N$;
- for $i = 1$ to m: if $N' \setminus \{i\}$ covers C, then set $N' = N' \setminus \{i\}$;
- output N'.

Proposition 3. *Algorithm* `WSC` *achieves differential approximation ratio bounded below by* $1/\Delta$. *This bound is asymptotically tight.*

Proof. We use in what follows notations introduced in Section 2. Observe that $N \setminus N' = \bar{N} \cup (N^* \setminus N')$ and $N \setminus N^* = \bar{N} \cup (N' \setminus N^*)$ where we recall that $\bar{N} = N \setminus (N^* \cup N')$. Denoting, for any $i \in N$, by w_i the weight of S_i, and, for any subset $\mathcal{X} \subseteq N$, by $w_{\mathcal{X}}$ the total weight of the sets with indices in $\mathcal{X}$, i.e., the quantity $\sum_{i \in \mathcal{X}} w_i$, the differential approximation ratio of WSC becomes

$$\delta\left(I, N'\right) = \frac{w_{N \setminus N'}}{w_{N \setminus N^*}} \tag{27}$$

Let $C_c = \{c_j : \exists i \in N' \cap N^*, c_j \in S_i\}$ be the set of elements covered by $N' \cap N^*$ and let $\bar{C}_c = C \setminus C_c$ be the complement of C_c with respect to C. It is easy to see that both $N' \setminus N^*$ and $N^* \setminus N'$ cover $\bar{C}_c$. Obviously, $C'_1 \subseteq \bar{C}_c$ (recall that $C'_1 = \{c_i : i \in N' \setminus N^*\}$ is a set of non-redundant elements with respect to sets of $N' \setminus N^*$ and that any element of C'_1 is covered by sets in $N^* \setminus N'$).

Consider the sub-instance of I induced by $(N' \setminus N^* \cup N^* \setminus N', C'_1)$. Fix an index $i \in N^* \setminus N'$ and denote by $S^*_i = \{c_{i_1}, \ldots, c_{i_k}\}$ the restriction of S_i to C'_1, i.e., $S^*_i = S_i \cap C'_1$. Assume that $S^*_i \neq \emptyset$; as it will be understood just below, if this is not the case, then the approximation ratio of WSC will be even better. Obviously, since sets $i_1, \ldots, i_k$ have been chosen by WSC (i.e., $\{i_1, \ldots, i_k\} \subseteq N'$), $w_{i_j} \leqslant w_i$ and, $k \leqslant \Delta$, we get:

$$\sum_{j=1}^{k} w_{i_j} \leqslant \Delta w_i \tag{28}$$

Summing (28) for all $i \in N^* \setminus N'$, we obtain $w_{N' \setminus N^*} \leqslant \Delta w_{N^* \setminus N'}$ and then, $w_{N \setminus N^*} \leqslant \Delta w_{N^* \setminus N}$. Expression (27) suffices now to conclude the proof of the ratio.

For tightness, fix $\Delta \in \mathbb{N}$, $w \in \mathbb{R}^+$ and consider the following instance $(\mathcal{S}, C, \vec{w})$ for MIN WEIGHTED SET COVER:

$$\begin{aligned}
C &= \{1, \ldots, \Delta\} \\
S_0 &= C \\
S_i &= \{i\} \quad i = 1, \ldots, \Delta \\
\mathcal{S} &= \{S_0, S_1, \ldots, S_\Delta\} \\
w_0 &= w + 1 \\
w_i &= w \quad i = 1, \ldots, \Delta
\end{aligned}$$

Application of WSC in the instance above gives: $w_{\bar{\mathcal{S}}'} = w + 1$, while $w_{\bar{\mathcal{S}}^*} = \Delta w$. Hence the differential approximation ratio achieved is $(w+1)/w\Delta \rightarrow 1/\Delta$ with $w \rightarrow \infty$. ∎

5 Concluding remarks and open problems

As we have already mentioned, Proposition 1 implies that, unless $\mathbf{P} = \mathbf{NP}$, MIN SET COVER is not polynomially approximable within differential ratios of $O(\log^{-1} n)$. Consequently, differential approximation bounds, even trivial, cannot be directly derived from the standard ones. The results of Theorem 1 and of Proposition 2 exhibit an important gap between lower and upper bounds for `SCGREEDY`. It seems very interesting to reduce this gap by improving both of them.

Other interesting questions about differential approximability of MIN SET COVER are, for instance:

- can MIN SET COVER be as "well differentially approximable" as MAX INDEPENDENT SET, i.e, is MIN SET COVER differentially approximable within $O(\log^2 n/n)$ ([10]) or (asymptotically) within k/Δ, $\forall k \in \mathbb{N}$ ([9])? (recall that standard and differential approximation ratios coincide for MAX INDEPENDENT SET);
- does there exist for MIN SET COVER an upper bound tighter than the one for MAX INDEPENDENT SET?

The result of Proposition 3, even if it slightly improves the approximation ratio of [6], is, to our opinion, far from being optimal for MIN WEIGHTED SET COVER. It would be interesting to improve it. Also, questions mentioned above hold for the unweighted case too. Another question that seem to us very interesting to handle, is the investigation of structural links between MIN SET COVER and MIN WEIGHTED SET COVER with respect to the differential approximation. Are these problems approximate equivalent or not?

An instance $(\mathcal{S}, C)$ of MIN SET COVER can also be seen as a hypergraph H where C is the set of its vertices and $\mathcal{S}$ is the set of its hyper-edges. Then MIN SET COVER consists of determining the smallest set of hyper-edges covering C. The "dual" of this problem is the well-known MIN HITTING SET problem, where, on $(\mathcal{S}, C)$, one wishes to determine the smallest subset of C hitting any set in $\mathcal{S}$. MIN HITTING SET and MIN SET COVER are approximate equivalent in both standard and differential paradigms (see, for example, [2]; the former is the same as the latter modulo the inter-change of the roles of $\mathcal{S}$ and C). On the other hand another well-known combinatorial problem is MAX HYPERGRAPH INDEPENDENT SET where given $(\mathcal{S}, C)$, one wishes to determine the largest subset C' of C such that no $S_i \in \mathcal{S}$ is a proper subset of C'. It is easy to see that for MAX HYPERGRAPH INDEPENDENT SET and MIN HITTING SET, the objective function of the one is an affine transformation of the objective function of the other, since a hitting set is the complement with respect to C of a hypergraph independent set. Consequently, the differential approximation ratios of these two problems coincide, and coincide also (as we have seen just above) with the differential approximation ratio of MIN SET COVER. Hence, our results identically apply for MAX HYPERGRAPH INDEPENDENT SET and hold in both the standard and the differential approximation paradigms.

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