

Draws, Zugzwangs, and PSPACE-Completeness in the Slither Connection Game

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Abstract. Two features set SLITHER apart from other connection games. Previously played stones can be relocated and some stone configurations are forbidden. We show that the interplay of these peculiar mechanics with the standard goal of connecting opposite edges of a board results in a game with a few properties unexpected among connection games, for instance, the existence of mutual Zugzwangs. We also establish that, although there are positions where one player has no legal move, there is no position where both players lack a legal move and that the game cannot end in a draw. From the standpoint of computational complexity, we show that the game is PSPACE-complete, the relocation rule can indeed be tamed so as to simulate a HEX game on a SLITHER board.

Keywords: Complexity, Connection game, Slither, Hex, PSPACE

1 Introduction

Invented in 2010 by Corey Clark, SLITHER is relatively new connection game with an increasing popularity among online board game players. Unlike HEX and HAVANNAH which are played on a hexagonally-paved board, SLITHER is played on a grid and each player is trying to connect a pair of opposite edges corresponding to their color by constructing connected groups of stones. Whereas moves in most other connection games only involve putting down a new element on the board, moves in SLITHER also allow relocating previously played stones. Another important difference between usual connections games and SLITHER is that some stone configurations are forbidden in the latter. Namely, a player is not allowed to play a stone diagonally adjacent to a pre-existing stone of their color unless one of their already placed stones would be mutually adjacent.

The goal of this paper is to study the properties of SLITHER in order to understand better the impact of features such as forbidden configurations and stone relocation on a connection game.

Since its independent inventions in 1942 and 1948 by the poet and mathematician Piet Hein and the economist and mathematician John Nash, the game of HEX has acquired a special spot in the heart of abstract game aficionados. Its purity and depth has lead Jack van Rijswijck to conclude his PhD thesis with the following hyperbole [15].

HEX has a Platonic existence, independent of human thought. If ever we find an extraterrestrial civilization at all, they will know HEX, without any doubt.

HEX not only exerts a fascination on players, but it is the root of the field of connection games which is being actively explored by game designers and researchers alike [4]. The focuses of the latter include the design and programming of strong artificial players and solvers [1, 6], the computational complexity of determining the winner in arbitrary positions [5, 11, 2], and theoretical considerations on aspects specific to connection games such a virtual connections and inferior cells [15, 9, 13, 14]. The two-player game SLITHER that we study in this article should not be confused with the Japanese single-player puzzle SLITHER LINK which has been the object of independent papers [16, 17].

The paper is organized as follows. After describing the rules of the SLITHER game, we demonstrate via specific game positions that a few properties that typically hold in connection games may actually not hold for SLITHER. The last section establishes that determining the winner in a two-player game of SLITHER is a PSPACE-complete problem.

2 Rules

SLITHER is a two-player game starting on an empty n by n grid (or board). Let us call the players Black (or B) and White (or W). Black and White alternate *moves*. Before stating what a move consists of, and what the winning conditions are, we introduce some useful definitions.

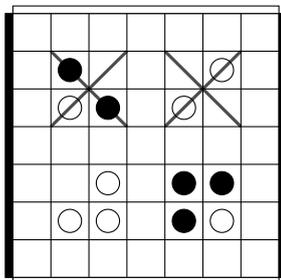
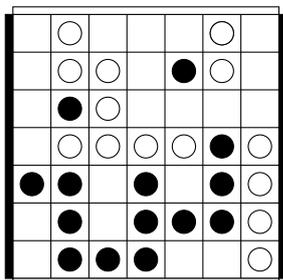
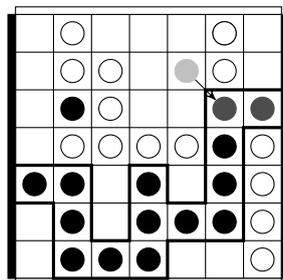


Fig. 1: Some examples of allowed and forbidden configurations. Forbidden configurations are crossed.

As the game proceeds, *squares* of the board can be empty, or contain a black stone, or contain a white stone. We refer to black (resp. white) stones as the stones of player B (resp. W). We say that two squares of the board are *adjacent* if they are in the same row and adjacent columns, or in the same column and adjacent rows. They are *king-adjacent* if a chess king can move from one square



(a) Black to play and win in one move.



(b) The winning move for Black.

Fig. 2: Illustration of a move and a winning group in a 7 by 7 board.

to the other, and *diagonally-adjacent* if they are king-adjacent but not adjacent. Two stones are *adjacent* (resp. *king-adjacent*, *diagonally-adjacent*) if they are in adjacent (resp. king-adjacent, diagonally-adjacent) squares. For $P \in \{W, B\}$, let G_P be the graph whose vertices are the stones of player P placed in the board, and the edges encode the *adjacent* relation. That is, two vertices are linked by an edge if and only if they represent adjacent stones. Like in the game of GO, a *group* for player P is a maximal connected component in G_P .

A *move* for player P consists of an optional relocation of an existing stone of P on a king-adjacent empty grid square, followed by a mandatory placement a stone of P into an empty grid square (see Fig. 2). For a move to be legal, the resulting position may not have two diagonally-adjacent stones of P that do not also have an orthogonally-adjacent stone in common (see Fig. 1). In what follows, we refer to this restrictive rule as the *diagonal rule*.

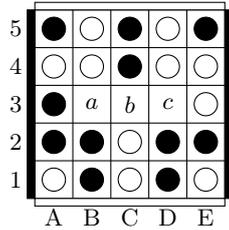
Black wins if they form a group with at least one stone in the first and in the last column. White wins if they form a group with at least one stone in the first and in the last row. Informally, Black wants to connect left-right and White wants to connect top-bottom (see Fig. 2b).

As in most connection games, a *swap* rule is usually implemented. That is, after the first move, the second player can decide either to play themselves a move and the game goes on normally, or to become first player with that very same move.

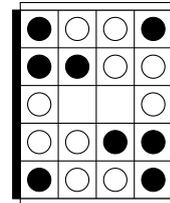
3 Elementary properties of the game

We now present some observations on and properties of SLITHER. Some of the observations made in this section have independently been pointed out earlier in the abstract game community, especially on BoardGameGeek.¹ However, we prove for the first time that the standard SLITHER variant cannot end in a draw, thereby settling an open-problem often raised in this community.

¹ <http://boardgamegeek.com/thread/692652/what-if-there-no-legal-move>



(a) Mutual zugzwang: no good move for Black and no good move for White.



(b) No move allowed for Black, only White has legal moves.

Fig. 3: SLITHER positions with a shortage of moves.

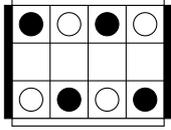
The concept of *zugzwang* appears in CHESS and denotes a position in which the current player has no desirable move and would rather pass and have the opponent act. A *mutual zugzwang* is a position in which both players would rather have the opponent play. Although zugzwangs are virtually unheard of in typical connection games, where additional moves can never hurt you, things are different in SLITHER.

Proposition 1. *Zugzwangs and mutual zugzwangs can occur in SLITHER.*

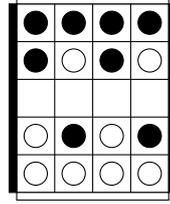
Proof. In Fig. 3a, if it is White (resp. Black) to play, only one move is available, moving stone on C2 (resp. C4) to *a* and placing a stone on *c* or equivalently moving to *c* and placing on *a*. Then Black (resp. White) wins by placing a stone on C2 (resp. moving stone B5 to C4 and placing a stone on *b*).

When John Nash discovered/invented the game HEX, one of his motivations was to find a non-trivial game with a non-constructive proof that the first player has a winning strategy in the initial position. To obtain this result, Nash developed a *strategy-stealing argument* which can be summed up as follows [15]. Suppose for a contradiction that the second player has a winning strategy σ in the initial position. Then a winning strategy for the first player can be obtained as: start with a random move, then apply σ pretending that the initial move did not occur. If σ ever recommends to play on the location of the initial move, then play another random move and carry on. Given that having an additional random stone on the board cannot hurt the first player, then we have developed a winning strategy for the first player too. Since the second and first player cannot both have a winning strategy at the same time, we can conclude that our hypothesis does not hold. Therefore the initial HEX position is not a second-player win when the swap rule is not used. Nash finishes his proof by showing that there can be no draw in HEX and concludes that the initial position is a first player win.

This strategy-stealing argument can be applied to many other games including TWIXT, HAVANNAH, and games of the CONNECT(m, n, k) family [10]. Since draws are not ruled-out in these games, the argument only shows that the ini-



(a) Cylinder board: left and right edges are connected.



(b) Torus board: left and right edges are connected and top and bottom edges are connected.

Fig. 4: Drawn SLITHER positions on non-planar boards.

tial situation is not a second-player win. Unfortunately, the strategy-stealing argument cannot be applied to SLITHER.

Proposition 2. *Nash’s strategy-stealing argument does not apply in SLITHER.*

Proof. Consider the White zugzwang position in Fig. 3a. Had the A4 square not been occupied with a White stone, White would have a winning move: move B4 to b and place a stone on c . Since there are positions where having one too many stones on the board can make a player lose the game, Nash’s strategy-stealing argument does not apply.

Therefore, there is no theoretical indication yet that SLITHER is *not* a second-player win on an empty board. However, in practice it is a huge advantage to play first, so much that if the swap rule is used, it is recommended to swap no matter where the first move is played, including corner locations. The SLITHER-specific intuition behind this practical advice is that the game is dynamic and a player can bring back a stone from a corner towards the center, moving it closer every turn.

Proposition 3. *There exist positions in which a player has no legal move.*

Proof. For instance, Black has no legal move in the position Fig. 3b.

Proposition 4. *Draws are possible when SLITHER is played on a cylinder or on a torus.*

Proof. In Fig. 4a (resp. Fig. 4b), if black (resp. black and white) sides are connected, then both players have no legal moves.

That draws are possible for some exotic boards should enhance the accidental aspect of our result on rectangular boards. There is probably no fundamental reasons why the following result is true, and the proof, which we defer to the appendix, consists of a large case analysis on the consequences of forbidding diagonal configurations and the possibility of moving stones.

Theorem 1. *Draws are not possible in SLITHER on rectangular boards.*

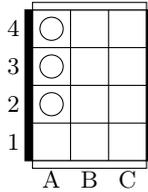


Fig. 5: Ladders.

We end this section, remarking that the connection tactics (for instance ladders) are very different in SLITHER than in other connection games. For example, in Fig. 5, White is connected to the bottom. Black has to defend on A1, but White can play on C1 and then Black has to play on B1, White plays on C2, Black answers on B2 and White ends the game moving stone A2 to B3 and placing a stone on C3. This last winning move for White is typical in SLITHER and would obviously not be allowed in a game like HEX.

4 Computational complexity

Here, we show that deciding if one player has a winning strategy from a given position is intractable.

A staple in proofs of PSPACE-hardness for two-player games seems to be GENERALIZED GEOGRAPHY [12]. This problem has been used to show the intractability of games as different as HEX [11], AMAZONS [7], BRIDGE [3], and many more [8]. Our result for SLITHER also relies on GENERALIZED GEOGRAPHY, albeit indirectly since we reduce from HEX.

Theorem 2. *It is PSPACE-complete to decide which player has a winning strategy from a given SLITHER position.*

Proof. The membership of this problem to PSPACE boils down to noticing that the length of a game is bounded from above by the number of empty squares. Indeed, at each move, one stone is added to the board. Thus, a minimax depth-first search uses a polynomial amount of space.

We now present a reduction from HEX which is known to be PSPACE-complete (see [11]). A hexagonal cell of HEX is encoded by the gadget depicted in Fig. 6. More precisely, an empty cell (resp. a cell containing a black stone, resp. containing a white stone) is transformed into the portion of position of Fig. 6a (resp. Fig. 6b, resp. Fig. 6c). The cell gadgets are glued together and attached to the edges of the board as described in Fig. 7. The empty squares which do not correspond to one of the eight squares designated by letter *a* to *h* in Fig. 6a can be filled with black stones. For convenience, we do not represent those stones.

Observation 1 *When a player places a stone in an empty cell gadget (that is, on a square marked with letter *a* to *h*), they create a configuration which is*

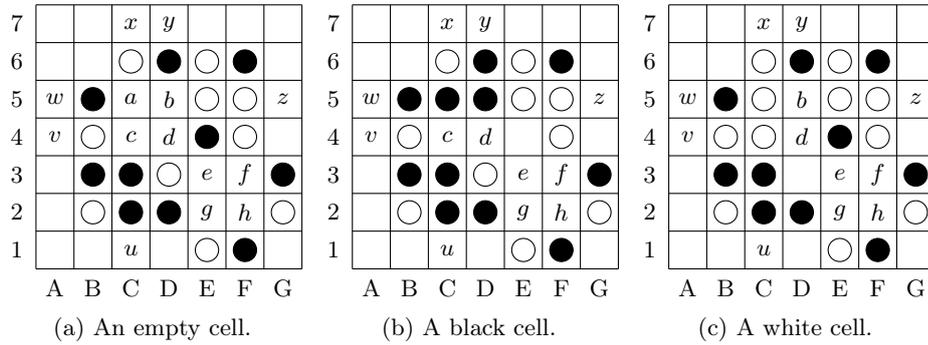


Fig. 6: The cell gadget.

forbidden by the diagonal rule. Thus, they should also move one of their stone in the same cell gadget.

Lemma 1. *In a black cell C (see Fig. 6b), White cannot prevent Black from having a group containing stones in w , y , and u .*

Proof. Black stones on w and y are already in the same group. Because of the diagonal rule, White cannot place a stone in cell C and move a stone in another cell gadget. By Observation 1 and the previous remark, if White moves a stone in cell C , but decides to place a stone in another cell gadget, they can only do so in a white cell, which turns out to be useless. Thus, White might as well place a stone *and* move in cell C . After their move, White should occupy square c ; otherwise, Black places a stone on c and thereby connects their group containing u to their group containing w and y .

There are three ways for W to occupy square c : (1) move stone on B4 to c , or (2) move stone on D3 to c , or (3) place a new stone on c . The first option cannot be extended into a legal move. Indeed, the diagonal rule would impose that a stone is placed on d , to connect the two diagonally-adjacent white stones on c and D3. But then white stones on d and E5 would form a forbidden configuration. In the second option, White cannot place a stone on D3 nor on d , because of the diagonal rule. And Black's next move would consist of moving the stone on D2 to D3, and placing a stone on d , which connects u to w and y . Finally, in the third option, White is forced to move their stone in D3 to a square other than d . And Black connects in the same manner.

Since the cell gadget is symmetric, the following holds similarly.

Lemma 2. *In a white cell (see Fig. 6c), Black cannot prevent White from having a group containing stones on v , x , and z .*

The following observation is outlined by Fig. 7.

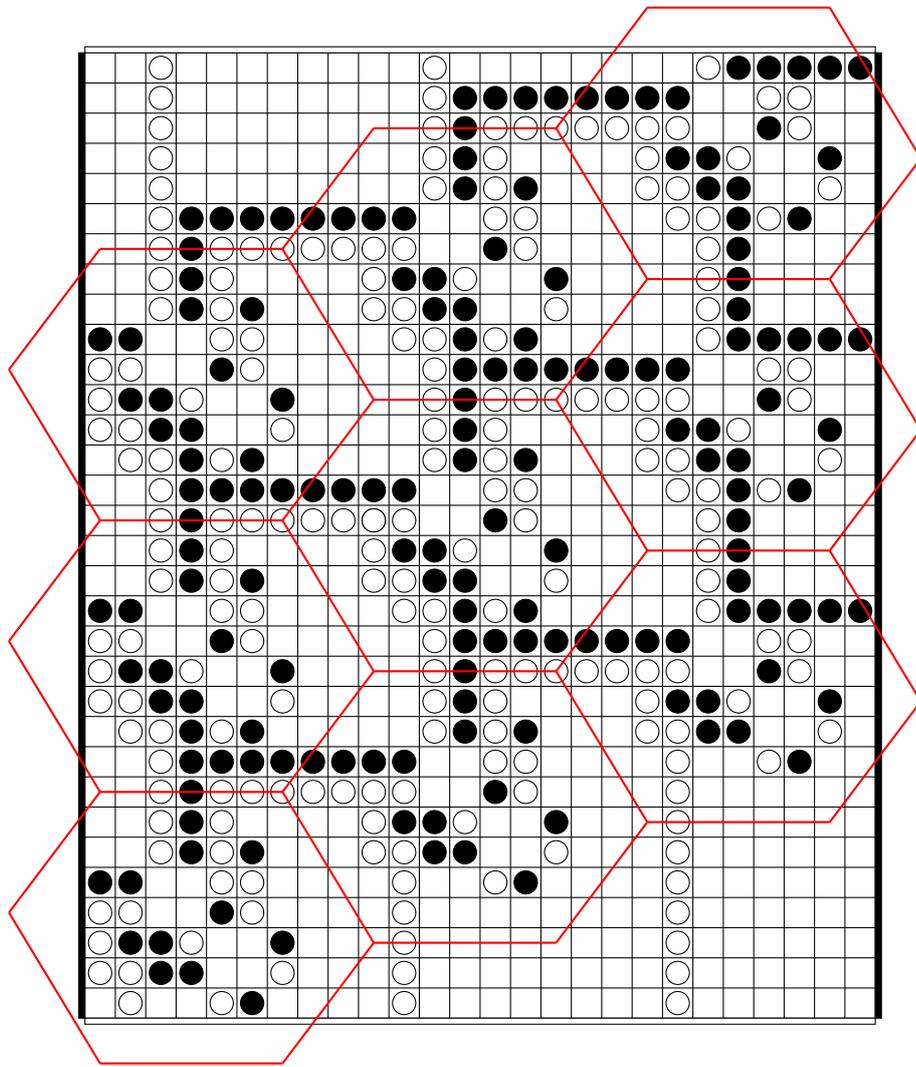


Fig. 7: An empty 3 by 3 HEX board reduced to a SLITHER position.

Observation 2 *When playing in a empty cell gadget, Black cannot do more than connecting u , w , and y , and White cannot do more than connecting v , x , and z .*

From the empty cell gadget, Black can move stone E4 to b and place a stone on a , resulting in the black cell configuration. By Lemma 1 and Observation 2, it is the optimal play within this cell. Similarly, the optimal play for White in a given empty cell, is to move stone D3 to c and place a stone on a , yielding the white cell. Thus, having chosen the cell gadget where to play, the optimal move is to connect six paths going from this cell to the six adjacent cells in a hexagonal paving (see Fig. 7). Hence, the built SLITHER position simulates a game of HEX, and so, SLITHER is as hard as HEX.

5 Conclusion

This paper establishes the PSPACE-completeness of a connection game, SLITHER. Although the diagonal rule and the relocation part of a move may seem, at first sight, somewhat artificial, they provide a mechanism to avoid draw possibilities. As such, they can be seen as a direct consequence of bringing HEX to a more natural grid board. Both rules are necessary: the diagonal rule prevents a straightforward drawing strategy, and draws would arise quite often if it was not for the right to relocate stones (see, for instance, Fig. 3a, Fig. 3b, and Fig. 7).

SLITHER is not the first connection game played on a grid, but unlike in TWIXT where nodes can have direct links to up to 8 neighbors, the largest number of neighbors of a node is 4 in SLITHER. Going from an hexagonal paving to a grid lowers the number of neighbors of each node, the degree. This change rules out a direct reduction from HEX as was possible for TWIXT [2]. Our main technical contribution is a hybrid connection mechanism between 6 groups on a degree 4 grid: in one move, a player connects *physically* two pairs of paths together and connects *virtually* the third pair to the other two. Our main practical contribution is to have ruled out the possibility of a draw on rectangular boards.

A seemingly reasonable heuristic consists of playing the first move of a shortest sequence leading to victory, supposing the opponent passes. We leave as an open question if such a shortest sequence can be efficiently computed. This problem might already be NP-hard. In fact, deciding if such a sequence exists at all, is not necessarily easy to do.

Acknowledgments

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A Draws are impossible in SLITHER

Stating that SLITHER does not feature any draw actually corresponds to the following three more elementary statements: each filled SLITHER board has winning groups for at least one player, each filled SLITHER board has winning groups for no more than one player, and each non-filled SLITHER board has at least one legal move for one of the two players. The first two statements can be obtained rather directly from the equivalent ones in HEX, thanks to the diagonal rule. The third statement is much more involved and requires a careful case analysis on non-filled boards.

Lemma 3. *If a SLITHER board is filled, then exactly one of the two players wins.*

Proof. Recall that HEX can be played on a rectangular board provided we add a link between each pair of king-adjacent squares along one specified diagonal direction, as in Fig. 8. The forbidden configuration rule ensures that this king-adjacent diagonal connection is respected in SLITHER, although it is indirect. Therefore, any filled $m \times n$ SLITHER board can be mapped onto an equivalent $m \times n$ HEX board such that any pair of SLITHER squares is connected if and only if the corresponding pair of HEX cells is connected. Since any filled HEX board is won by exactly one player, we have the desired SLITHER result.

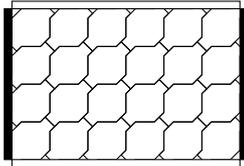


Fig. 8: 6×4 HEX board represented on a rectangular SLITHER board.

Theorem 3. *On a rectangular board, as long as there is at least one empty square, at least one of the two players has a legal move.*

Proof. We adopt the following proof technique.² We assume for a contradiction that we have a non-filled position with no legal moves for any players. We start from an empty square and make deduction concerning its surrounding so as to constrain the occupancy of the nearby squares. Each constraint is deduced based on the established occupancies and from the no legal moves assumption or from the diagonal rule. We may perform a split case analysis on squares that are not constrained enough to have a definite status. In each case, however, we finally arrive at a position with a legal move that cannot be prevented by adding any further constraints.

² No part of the argument will rely upon the color of the board edge.

As we add constraints to forbid legal moves from either player, we liberally extend the size of the pattern around the empty square. If any such extension was not possible because we would have reached the limit of the board, then it would not be possible to forbid the desired legal move and our case would be proved. We can therefore disregard the possibility of inadvertently meeting an edge of the board as we extend our patterns, at least for the sake of this argument.

In addition to the regular three types of squares, \square white stone, \blacksquare black stone, and \square empty, we add the following ones: \square *no constraints yet*, \square *cannot hold a white stone*, and \square *cannot hold a black stone*.

Consider a rectangular board. If there is at least an empty square on the board, then there is at least an empty square s such that one of the following 4 conditions on the bottom and left neighbors of s is satisfied. Either s is in the bottom-left corner (Fig. 9a), or s is on the bottom edge and its left neighbor is occupied (Fig. 9b), or s has three neighboring stones of different colors (Fig. 9c), or these stones are all of the same color (Fig. 9d). We can assume w.l.o.g. that a majority of the bottom-left neighboring stones are white.

The first two cases are treated in Section B, the third case is treated in Section C, and the last case in Section D. In each case, we arrive to the conclusion that at least one player can move.

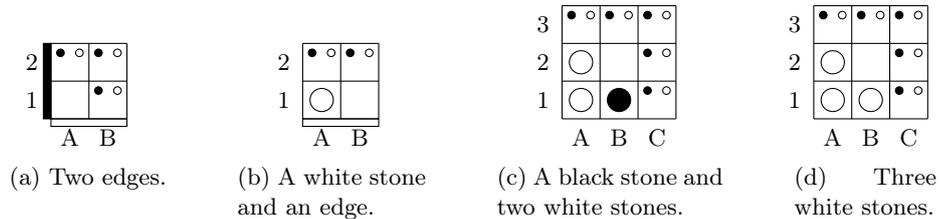


Fig. 9: Case analysis for the bottom left surroundings of the empty square.

B A square in a corner or on the edge of the board

If there is an empty square in a corner, as depicted in Fig. 9a, then placing a stone on that very square is a legal move for at least one player.

If there is an empty square on an edge, we start from the situation in Fig. 9b and use the following reasoning to constrain the surrounding and obtain Fig. 10. C2 needs a white stone to forbid White's move B1, and C1 cannot be white. A2 needs a black stone to forbid Black's move B1. B2 needs to be empty to forbid White's and Black's move B1. C1 cannot be empty to forbid White's move C1. A3 needs a white stone to forbid White's move A1B1-B2, and B3 cannot be

white. C3 needs a black stone to forbid Black's move C1B1-B2, and B3 cannot be black. Similarly, A4 needs a black stone, C4 needs a white stone, and B4 needs to be empty, so as to forbid White's move A1B2-B3 and Black's move C1B2-B3.

But then, C3B2-B1 is a legal move for Black.

4	●		○
3	○		●
2	●		○
1	○		●
	A	B	C

Fig. 10: The case in Fig. 9b with a few deducible constraints filled in.

C A square with two white stones and a black stone

We start from the situation in Fig. 9c and use the following reasoning to constrain the surrounding and obtain Fig. 11. The C2 square cannot contain a white stone, otherwise B2 is a legal move for White. Similarly, B3 cannot contain a black stone, otherwise B2 is a legal move for Black. To forbid White's move B2, there should be a white stone on C1 or on C3 (Fig. 12).

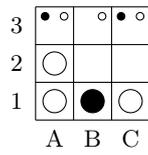
3	● ○	○ ●	○ ○
2	○		●
1	○	●	○ ○
	A	B	C

Fig. 11: The case in Fig. 9c with a few deducible constraints filled in.

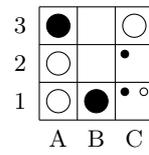
C.1 Fig. 12a

C2 cannot contain a black stone due to the diagonal rule (since B2 is empty by assumption), and has to be empty. Now, we distinguish two subcases: either B3 is white, or it is empty (Fig. 13).

C.1.1 Fig. 13a A3 contains a white stone, by the diagonal rule. C3 needs a black stone to forbid Black's move B2. D1 needs a black stone to forbid Black's move B1B2-C2. D3 needs a white stone to forbid White's move C1B2-C2. D2

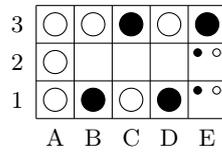


(a) Assume C1 is white.

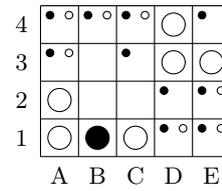


(b) Assume C3 is white.

Fig. 12: Case analysis for Fig. 11, either C1 is white or C3 is white.



(a) Assume B3 is white.



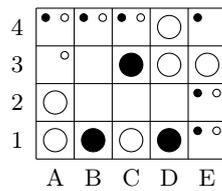
(b) Assume B3 is empty.

Fig. 13: Case analysis for Fig. 12a: B3 is either white or empty.

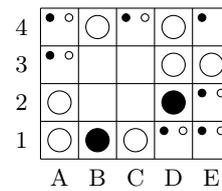
needs to be empty, by the diagonal rule on stones C1 and C3. E3 needs a black stone to forbid Black's move B1C2-D2.

But, in that situation, D3C2-B2 is a legal move for White.

C.1.2 Fig. 13b To forbid White's move C2, D3 needs a white stone and C3 and D2 cannot contain a white stone. To forbid White's move D3C2-B2, there should be white stones on E3 and D4 and E4 should not contain a white stone. Now, we distinguish two subcases: either C3 is black, or it is empty (Fig. 14).



(a) Assume C3 is black.



(b) Assume C3 is empty.

Fig. 14: Case analysis for Fig. 13b: C3 is either black or empty.

C.1.2.1 Fig. 14a A3 cannot contain a black stone to forbid Black's move B3. D1 needs a black stone to forbid Black's move C3B2-C2. D2 cannot contain a black stone by the diagonal rule, and has to be empty.

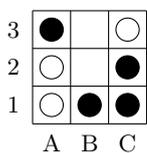
But then, move B1C2-D2 is legal for Black.

C.1.2.2 Fig. 14b B4 needs a white stone to forbid White's move C3. D2 needs a black stone to forbid Black's move C3.

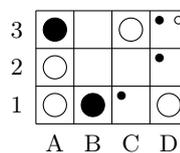
But then, B1C2-C3 is legal for Black.

C.2 Fig. 12b

A3 needs a black stone to forbid Black's move B2. B3 cannot contain a white stone by the diagonal rule (with A2) and has to be empty. Now, we distinguish two subcases: either C2 is black, or it is empty (Fig. 15).



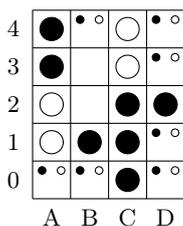
(a) Assume C2 is black.



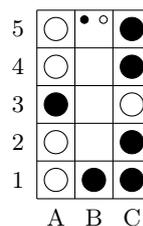
(b) Assume C2 is empty.

Fig. 15: Case analysis for Fig. 12b: C2 is either black or empty.

C.2.1 Fig. 15a To forbid White's move C3B2-B3 and Black's move A3B2-B3, there should be at least one white stone and one black stone in {A4, C4}. So, we distinguish further between {A4 black, C4 white} and {A4 white, C4 black} (Fig. 16).



(a) Assume A4 is black and C4 is white.



(b) Assume A4 is white and C4 is black.

Fig. 16: Case analysis for Fig. 15a: the contents of A4 and C4 is white.

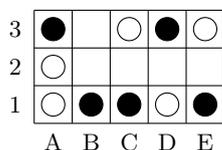
C.2.1.1 Fig. 16a. D2 needs a black stone to forbid Black's move C2B2-B3, and C0 needs a black stone to forbid Black's move C1B2-B3.

But then, B1B2-B3 is a legal move for Black.

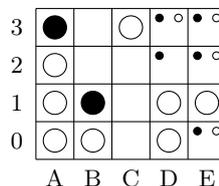
C.2.1.2 Fig. 16b. By the diagonal rule, square B4 has to be empty. C5 (as well as D4) needs a black stone to forbid Black's move C4B3-B2. Symmetrically, A5 needs a white stone to forbid White's move A4B3-B2.

But then, C3B2-B4 is a legal move for White.

C.2.2 Fig. 15b. To forbid White's move C2, there should be a white stone on D1 but no white stones on C1 nor D2. We distinguish two subcases: C1 contains a black stone, or it is empty (Fig. 17).



(a) Assume C1 is black.



(b) Assume C1 is empty.

Fig. 17: Case analysis for Fig. 15b: C1 is either black or empty.

C.2.2.1 Fig. 17a. D3 needs a black stone to forbid Black's move C2 (since B3 is empty). Then, by the diagonal rule, square D2 can only be empty. E3 needs a white stone to forbid White's move C3D2-C2. E1 needs a black stone to forbid Black's move D3C2-D2.

But then, D1C2-B2 is a legal move for White.

C.2.2.2 Fig. 17b. The only way to forbid White's move D1C2-B2 is to add two white stones on D0 and on E1. To forbid White's move C1, there should be a white stone on B0 (and a white stone on A0, by the diagonal rule), and no white stones on C0. C0 cannot contain a black stone because of the diagonal rule

But then, C0 is a legal move for White.

D A square with three white stones

We start from the situation in Fig. 9d. To forbid White's move B2, there should be a white stone on C3, but no white stones on B3 nor C2. To forbid Black's move B2, there should be a black stone on C1 or A3, say C1 w.l.o.g. (see Fig. 18).

Therefore, B3 and C2 are empty or contain a black stone. They cannot both contain a black stone since B2 is empty. We thus distinguish three cases: B3 and C2 are empty, B3 contains a black stone, and C2 contains a black stone (Fig. 19).

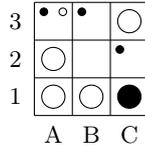
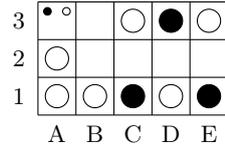
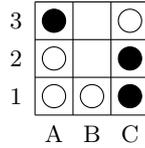


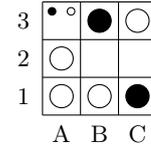
Fig. 18: The case in Fig. 9d with a few deducible constraints filled in.



(a) Assume B3 and C2 are empty.



(b) Assume B3 is empty and C2 is black.



(c) Assume B3 is black and C2 is empty.

Fig. 19: Case analysis for Fig 18: the contents of B3 and C2.

D.1 Fig. 19a

D3 needs a black stone to forbid Black's move C2. D1 needs a white stone to forbid White's move C3B2-C2. E3 needs a white stone to forbid White's move C3B2-D2. E1 needs a black stone to forbid Black's move D3C2-D2.

But then, D1C2-B2 is legal for White.

D.2 Fig. 19b

A3 needs a black stone to forbid Black's move B2. This case is equivalent to the case of Fig. 15a under color and spatial symmetry.

D.3 Fig. 19c

Let us consider cases for the contents of A3. If A3 contains a black stone, then we obtain a position equivalent to Fig. 15a under color and spatial symmetry.

If A3 is empty or white, then a similar proof to Fig. 19a still holds. Indeed, D3 needs a black stone to forbid Black's move B3B2-C2 and D1 needs a white stone to forbid White's move C3B2-C2. The same way, E3 needs a white stone to forbid White's move C3B2-D2, and then E1 needs a black stone to forbid Black's move C1B2-D2.

But then, D1C2-B2 is legal for White.