

On the Complexity of Various Parameterizations of Common Induced Subgraph Isomorphism[☆]

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Abstract

In the MAXIMUM COMMON INDUCED SUBGRAPH problem (henceforth MCIS), given two graphs G_1 and G_2 , one looks for a graph with the maximum number of vertices being both an induced subgraph of G_1 and G_2 . MCIS is among the most studied classical NP-hard problems. It remains NP-hard on many graph classes including forests. In this paper, we study the parameterized complexity of MCIS. As a generalization of CLIQUE, it is W[1]-hard parameterized by the size of the solution. Being NP-hard even on forests, most structural parameterizations are intractable. One has to go as far as parameterizing by the size of the minimum vertex cover to get some tractability. Indeed, when parameterized by $k := \text{vc}(G_1) + \text{vc}(G_2)$ the sum of the vertex cover number of the two input graphs, the problem was shown to be fixed-parameter tractable, with an algorithm running in time $2^{\mathcal{O}(k \log k)}$. We complement this result by showing that, unless the ETH fails, it cannot be solved in time $2^{o(k \log k)}$. This kind of tight lower bound has been shown for a few problems and parameters but, to the best of our knowledge, not for the vertex cover number. We also show that MCIS does not have a polynomial kernel when parameterized by k , unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. Finally, we study MCIS and its connected variant MCCIS on some special graph classes and with respect to other structural parameters.

Keywords: Induced Common Isomorphism, Parameterized Complexity, Kernelization, ETH based Lower Bounds

1. Introduction

A common induced subgraph of two graphs G_1 and G_2 is a graph that is isomorphic to an induced subgraph of both graphs. The problem of finding a common induced subgraph with the maximum number of vertices (or edges)

[☆]An extended abstract of this work appears in [1]

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has many applications in a number of domains including bioinformatics and chemistry [2, 3, 4, 5, 6]. In the decision version of the problem, we are given an integer k and the question is to decide whether there is a solution with at least k vertices. We say that the solution size k is the *natural parameter* of the problem.

Concerning its classical complexity, MAXIMUM COMMON INDUCED SUBGRAPH is NP-complete, and remains so on forests [10]. When the common subgraph is required to be connected, the problem is in P for trees [7]. Moreover, MAXIMUM COMMON INDUCED SUBGRAPH is also in P when the two input graphs are connected and (both) have bounded treewidth and bounded degree [8].

A particular case of MAXIMUM COMMON INDUCED SUBGRAPH is the well known INDUCED SUBGRAPH ISOMORPHISM (ISI) decision problem, where the question posed is whether G_1 is isomorphic to an induced subgraph of G_2 . In other words, it is equivalent to MAXIMUM COMMON INDUCED SUBGRAPH where $k = |G_1|$. In this case, G_1 is called the pattern graph and G_2 is called the host graph. ISI is known to be NP-hard, even when G_2 is an interval graph and G_1 is a proper interval graph, but it becomes polynomial-time solvable when G_1 is in addition connected [9]. Unlike SUBGRAPH ISOMORPHISM, INDUCED SUBGRAPH ISOMORPHISM remains NP-hard when both the host graph and the pattern graph consist of a disjoint union of paths [10]. From the parameterized complexity viewpoint, the problem is W[1]-hard in general for the natural parameter, by a straightforward reduction from k -CLIQUE. Therefore MCIS is also W[1]-hard. Moreover, ISI (and, therefore, MCIS) remains W[1]-hard even when both graphs are interval graphs [11]. On the other hand, ISI is FPT on nowhere dense graphs, being expressible by a first-order formula of length function of the natural parameter k [12]. This generalizes what was previously known about ISI on H -minor free graphs [13] and graphs of bounded degree [14]. We observe that whenever ISI is FPT then so is MCIS. To see this, note that an arbitrary instance (G_1, G_2, k) of MCIS can be reduced in fpt-time to instances of ISI by enumerating each graph H on k vertices and checking whether H is an induced subgraph of G_1 and G_2 . This implies that ISI and MCIS have the same parameterized complexity with respect to the natural parameter. It follows that MCIS is also FPT on H -minor free graphs and graphs of bounded degree.

Another way of dealing with the hardness of a problem is to study its complexity with respect to auxiliary (or structural) parameters, to better understand its algorithmic behavior (see for example [15]). Being NP-hard on disjoint union of chordless paths [10], MCIS is hard on graphs with bounded treewidth as well as graphs where the size of the minimum feedback vertex set is bounded. Thus the problem is paraNP-hard when parameterized by the treewidth of the input graphs, or by a bound on the sizes of their minimum feedback vertex sets. Therefore, we need to look for “bigger” parameters. And indeed, if the parameter k is the sum of the sizes of the minimum vertex covers of the input graphs, then the problem is in FPT, with a running time of $O((24k)^k) = 2^{O(k \log k)}$ [16]. In this paper, we show that this algorithm cannot be significantly improved: unless the Exponential Time Hypothesis (ETH) fails, there is no algorithm solving

MCIS in time $O^*(2^{o(k \log k)})$, where the O^* notation suppresses the polynomial factors. We also prove that MCIS does not have a polynomial-size kernel in this case unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. These two latter results answer open problems raised in [16]. Finally, we show that MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH (MCCIS), where the solution should be a connected graph, is also fixed-parameter tractable when parameterized by $k := vc(G_1) + vc(G_2)$.

2. Preliminaries

For any positive integer x , $[x]$ denotes the set $\{1, 2, \dots, x-1, x\}$. Two finite graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $\pi : V_1 \rightarrow V_2$ such that $\forall u, v \in V_1 : uv \in E_1 \Leftrightarrow \pi(u)\pi(v) \in E_2$. Given a graph $G = (V, E)$, a graph $G' = (V', E')$ is an *induced subgraph* of G if $V' \subseteq V$ and $E' = E(V') = \{uv \in E \mid u, v \in V'\}$, i.e. E' is the edge set with both extremities in V' . We also say that G' is the subgraph of G induced by V' .

The *degeneracy* of a graph G is the minimum value d for which any induced subgraph of G has a vertex with degree at most d . The *girth* of a graph G is the length of the shortest cycle contained in G . Contracting an edge uv consists of deleting uv and replacing the vertices u and v by a single vertex w in the incidence relation (edges incident on u or v become incident on w). A graph H is a *minor* of graph G if H is obtained from a subgraph of G by applying zero or more edge contractions. Given a fixed graph H , a family \mathcal{F} of graphs is said to be *H -minor free* if H is not a minor of any element of \mathcal{F} .

The MAXIMUM COMMON INDUCED SUBGRAPH problem is defined formally as follows.

MAXIMUM COMMON INDUCED SUBGRAPH (MCIS):

- **Input:** Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, an integer k .
- **Output:** Does there exist an induced subgraph G'_1 of G_1 isomorphic to an induced subgraph G'_2 of G_2 with $|G'_1| \geq k$?

MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH (MCCIS) is defined as MCIS with the additional restriction that the solution must be *connected*.

For completeness, we also give the definition of INDUCED SUBGRAPH ISOMORPHISM, which is equivalent to MCIS when $k = |G_1|$:

INDUCED SUBGRAPH ISOMORPHISM (ISI):

- **Input:** Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.
- **Output:** Does there exist an induced subgraph G'_2 of G_2 that is isomorphic to G_1 ?

INDUCED CONNECTED SUBGRAPH ISOMORPHISM (ICSI) is defined as ISI but G_1 must be connected.

Parameterized complexity. A parameterized problem (I, k) is *fixed-parameter tractable* (or in the class FPT) with respect to parameter k if it can be solved in $f(k) \cdot |I|^c$ time (i.e. in fpt-time), where f is any computable function and c is a constant (see [17, 18] for more details about fixed-parameter tractability). The parameterized complexity hierarchy is composed of the classes $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots \subseteq \text{XP}$. The class XP contains problems solvable in time $f(k) \cdot |I|^{g(k)}$, where f and g are unrestricted functions. A problem is said to be *paraNP-hard* if it is NP-hard even for a constant value of the parameter (it hence cannot be in XP). A W[1]-hard problem is not fixed-parameter tractable, unless $\text{FPT} = \text{W}[1]$, and one can prove W[1]-hardness by means of a *parameterized reduction* from a W[1]-hard problem. This is a mapping of an instance (I, k) of a problem A_1 in $g(k) \cdot |I|^{O(1)}$ time (for any computable function g) into an instance (I', k') for A_2 such that $(I, k) \in A_1 \Leftrightarrow (I', k') \in A_2$ and $k' \leq h(k)$ for some function h .

A powerful technique to design parameterized algorithms is *kernelization*. In short, kernelization is a polynomial-time self-reduction algorithm that takes an instance (I, k) of a parameterized problem P as input and computes an equivalent instance (I', k') of P such that $|I'| \leq h(k)$ for some computable function h and $k' \leq k$. The instance (I', k') is called a *kernel* in this case. If the function h is polynomial, we say that (I', k') is a polynomial kernel. It is well known that a problem is in FPT iff it has a kernel, but this equivalence yields super-polynomial kernels (in general). To design efficient parameterized algorithms, a kernel of polynomial (or even linear) size in k is important. However, some lower bounds on the size of the kernel can be shown unless some polynomial hierarchy collapses. To show this result, we will use the cross composition technique developed by Bodlaender et al. [19].

Definition 1 (Polynomial equivalence relation [19]). *An equivalence relation \mathcal{R} on Σ^* is said to be polynomial if the following two conditions hold: (i) There is an algorithm that given two strings $x, y \in \Sigma^*$ decides whether x and y belong to the same equivalence class in time $(|x| + |y|)^{O(1)}$. (ii) For any finite set $S \subseteq \Sigma^*$ the equivalence relation \mathcal{R} partitions the elements of S into at most $(\max_{x \in S} |x|)^{O(1)}$ classes.*

Definition 2 (OR-cross-composition [19]). *Let $L \subseteq \Sigma^*$ be a set and let $Q \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem. We say that L cross-composes into Q if there is a polynomial equivalence relation \mathcal{R} and an algorithm which, given t strings x_1, x_2, \dots, x_t belonging to the same equivalence class of \mathcal{R} , computes an instance $(x^*, k^*) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $\sum_{i=1}^t |x_i|$ such that: (i) $(x^*, k^*) \in Q \Leftrightarrow x_i \in L$ for some $1 \leq i \leq t$. (ii) k^* is bounded by a polynomial in $\max_{i=1}^t |x_i| + \log t$.*

Proposition 3 ([19]). *Let $L \subseteq \Sigma^*$ be a set which is NP-hard under Karp reductions. If L cross-composes into the parameterized problem Q , then Q has no polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.*

The *Exponential Time Hypothesis* (ETH) is a conjecture by Impagliazzo et al. asserting that there is no $2^{o(n)}$ -time algorithm for 3-SAT on instances

with n variables [20]. The ETH, together with the sparsification lemma [20], even implies that there is no $2^{o(n+m)}$ -time algorithm solving 3-SAT. Many algorithmic lower bounds have been proved under the ETH, see for example [21].

We say that a parameterized problem is *fixed-parameter enumerable* if all feasible solutions can be enumerated in $O(f(k)|I|^c)$ time, where f is a computable function of the parameter k only, and c is a constant.

3. Parameterized Complexity with respect to the natural parameter

We study the parameterized complexity of INDUCED SUBGRAPH ISOMORPHISM, MAXIMUM COMMON INDUCED SUBGRAPH, INDUCED CONNECTED SUBGRAPH ISOMORPHISM, and MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH with respect to the natural parameter. We will in particular study these problems in graphs of bounded degeneracy, chordal graphs, and graphs of large girth.

Theorem 4. *MCIS, MCCIS, ISI, and ICSI are $W[1]$ -complete.*

Proof. Since those problems are $W[1]$ -hard by a straightforward reduction from k -CLIQUE, it suffices to show membership in $W[1]$. In [22], it is shown that if a problem can be reduced in FPT time to simulating a non-deterministic single-taped Turing machine halting in at most $f(k)$ steps, for some function f , then it is in $W[1]$. The Turing machine can have an alphabet and a set of states of size depending on the size of the input of the initial problem. In our case, we can design a Turing machine that guesses in $2k$ steps the right k vertices in G_1 (for I(C)SI this part is not necessary) and the right k vertices in G_2 (our alphabet being isomorphic to an indexing of $V(G_1) \cup V(G_2)$ and the correspondence between the k vertices of G_1 and the k vertices of G_2 is implicitly given by the order the vertices are guessed). Then it checks in time $O(k^2)$ whether the two induced subgraphs are isomorphic (and that they are connected for ICSI and MCCIS). \square

In [23] it was shown that MAXIMUM INDUCED MATCHING¹ is $W[1]$ -hard on bipartite graphs. This implies that MCIS is $W[1]$ -hard on bipartite graphs. In fact, we show that MCIS remains $W[1]$ -hard on more restricted graph classes, namely C_4 -free bipartite graphs with degeneracy 2. In particular, those graphs have girth at least 6. This result tells us two things about MC(C)IS. The first is that the fixed-parameter algorithm of Cai et al. [14, Theorem 1] cannot be extended from bounded degree to bounded degeneracy (note that some W -hard problems on general graphs become FPT on graphs with bounded degeneracy, such as the $W[2]$ -complete DOMINATING SET problem [24]). The second is that short cycles are not making MC(C)IS $W[1]$ -hard; they are $W[1]$ -hard even without them. In [25], the authors present fixed-parameter algorithms on graphs

¹where one looks for a largest subset of vertices that induce a disjoint union of edges

of girth 5, for some problems which are W-hard on general graphs. MCIS and MCCIS are also resistant to this approach.

Theorem 5. INDUCED SUBGRAPH ISOMORPHISM *and* INDUCED CONNECTED SUBGRAPH ISOMORPHISM are W[1]-complete even when both graphs are C_4 -free bipartite graphs with degeneracy at most 2.

Proof. The incidence graph $I(G)$ of any graph $G = (V, E)$, obtained by subdividing each edge of G once, has degeneracy 2. Indeed, graph $I(G)$ is the bipartite graph $(V \uplus E, F)$ where the edges of F are all the ue for which $u \in V$, $e \in E$, and u is an endpoint of e . All the vertices $e \in E$ of $I(G)$ have degree 2. Therefore, they can be removed first. Then, what is left in $I(G)$ is the independent set V .

We transform any input $G = (V, E)$, $k > 3$ of k -CLIQUE, into the instance $I(K_k), I(G)$ of I(C)SI, where both graphs have degeneracy 2. The problem consists of finding the incidence graph of a k -clique within the incidence graph of G . We show that it is equivalent to finding a k -clique in G . Obviously, if there is a k -clique S in G , then the graph $I(G)[S \cup E(S)]$ is isomorphic to $I(K_k)$. Now, let us assume that $I(K_k)$ is isomorphic to an induced subgraph of $I(G)$. We denote by a_1, \dots, a_k the vertices of $I(K_k)$ with degree $k - 1$, and by $b_1, \dots, b_{\binom{k}{2}}$ the vertices of $I(K_k)$ with degree 2. We denote by $\psi : V(I(K_k)) \rightarrow V(I(G))$ the isomorphism from graph $I(K_k)$ to an induced subgraph of $I(G)$. Let $u_i = \psi(a_i)$ for each $i \in [k]$, and $v_j = \psi(b_j)$ for each $j \in [\binom{k}{2}]$. We set $S = \{u_1, \dots, u_k, v_1, \dots, v_{\binom{k}{2}}\}$. For every $i \in [k]$, $u_i \in V$ since the degree of a_i in $I(K_k)$ is $k - 1 > 2$ (hence, the degree of u_i in S is also $k - 1 > 2$). Now, for every $j \in [\binom{k}{2}]$, $v_j \in E$ since v_j has two neighbors in V (recall that $I(G)$ is bipartite). Therefore, u_1, \dots, u_k are k vertices in V inducing precisely $\binom{k}{2}$ edges. Hence, $\{u_1, \dots, u_k\}$ is a k -clique in G .

Membership in W[1] comes from [Theorem 4](#). □

Corollary 6. MAXIMUM COMMON INDUCED SUBGRAPH *and* MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH remain W[1]-complete on bipartite graphs of girth 6 and degeneracy 2.

The absence of triangles and cycles of length four in the input graphs does not make the problems tractable. We show that the absence of a long induced cycle does not help either (in [\[26\]](#), the authors show that the W[2]-hard problem DOMINATOR COLORING is in FPT when the input graph is chordal). More specifically, all four problems are W[1]-hard on chordal graphs. In fact, we can even show that these problems remain W[1]-hard on a proper subclass of chordal graphs called split graphs. A split graph is a graph whose vertex set can be partitioned into a set inducing a clique and an independent set.

Theorem 7. ISI (hence MCIS) and ICSI (hence MCCIS) remain W[1]-complete on split graphs.

Proof. Similarly to the previous construction, we define $I'(G)$ as the graph $(V \uplus E, F)$ where the edges of F are the edges ue for which $u \in V$, $e \in E$, and u

is an endpoint of e , plus all the edges uv with $u, v \in V$. The graph $I'(G)$ is split: V induces a clique in $I'(G)$ and E induces an independent set. From an instance G of k -CLIQUE with $k > 3$, we build the equivalent instance $I'(K_k), I'(G)$ of MC(C)IS and I(C)SI. The soundness can be obtained in the same way as in the previous proof.

Membership in $W[1]$ comes from [Theorem 4](#). \square

Let us now say some words about the complexity of the connected version. First we note that MCIS is NP-hard on forests while MCCIS is solvable in polynomial-time in this case: given two forests G_1 and G_2 , run the polynomial-time MCIS algorithm of Akutsu on every pair of trees from G_1 and G_2 [\[27\]](#). From the parameterized complexity standpoint, MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH is FPT whenever INDUCED SUBGRAPH ISOMORPHISM is FPT since the enumeration of all $O(2^{k^2})$ possible induced *connected* subgraphs can be used as described in the introduction. The converse is true on classes of graphs which are closed by adding a universal vertex (i.e., a vertex linked to all the other vertices). An instance (G_1, G_2, k) of ISI can be reduced to an equivalent instance $(G'_1, G'_2, k+1)$ of MCCIS by letting G'_i be the graph obtained by adding a vertex to G_i that is made adjacent to all other vertices of G_i .

4. Structural parameterization

Let us first recall that $\text{tw}(G) \leq \text{fvs}(G) + 1 \leq \text{vc}(G) + 1$, where $\text{tw}(G)$ (resp. $\text{fvs}(G)$, $\text{vc}(G)$) represents the treewidth (resp. the feedback vertex set number, the vertex cover number) of G [\[28\]](#). As noted before, if the parameter is the combination of $\text{tw}(G_1)$ and $\text{tw}(G_2)$ then MCIS is known to be $W[1]$ -hard. Even more, if the parameter is the combination of $\text{fvs}(G_1)$ and $\text{fvs}(G_2)$ (which is bigger than the combination of the treewidth), then the problem is not even in XP since MAXIMUM COMMON INDUCED SUBGRAPH and INDUCED SUBGRAPH ISOMORPHISM are NP-hard on disjoint union of chordless paths, a case where the parameter is equal to 0 [\[10, 7\]](#).

Theorem 8 ([\[10, 7\]](#)). MAXIMUM COMMON INDUCED SUBGRAPH is *paraNP-hard* when parameterized by $\text{fvs}(G_1) + \text{fvs}(G_2)$ (and hence by $\text{tw}(G_1) + \text{tw}(G_2)$).

One can extend this result to make it valid for the connected version.

Theorem 9. INDUCED CONNECTED SUBGRAPH ISOMORPHISM, and as a corollary MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH, are *paraNP-hard* when parameterized by $\text{fvs}(G_1) + \text{fvs}(G_2)$.

Proof. Given an instance of INDUCED SUBGRAPH ISOMORPHISM on forests G_1 and G_2 (each with at least 2 trees), we build an instance of INDUCED CONNECTED SUBGRAPH ISOMORPHISM by adding a universal vertex (connected to every node) in G_1 and in G_2 . Both graph have thus a feedback vertex set of value one. One can see that these two universal vertices must be matched together since they are the only ones with sufficiently high degree. Then, there

is a solution for INDUCED SUBGRAPH ISOMORPHISM iff there is a solution for INDUCED CONNECTED SUBGRAPH ISOMORPHISM. The result of course holds for MCCIS, too. \square

It was shown in [16] that MCIS is in FPT if the parameter is the combination of $\text{vc}(G_1)$ and $\text{vc}(G_2)$. Accordingly, the problem has a kernel, but no polynomial bound is known on its size. We show that, in this case, the kernel cannot be polynomial unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.

Theorem 10. *Unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, MAXIMUM COMMON INDUCED SUBGRAPH has no polynomial kernel when parameterized by the sum of the sizes of vertex covers in the two input graphs.*

Proof. We will define an OR-cross-composition from the NP-complete CLIQUE, problem, where the given instance is a tuple (G^c, l) and the question is whether the graph G^c contains a clique on l vertices.

Given t instances, $(G_1^c, l_1), (G_2^c, l_2), \dots, (G_t^c, l_t)$, of CLIQUE, where G_i^c is a graph and $l_i \in \mathbb{N}$, for all $i \in [t]$, we define our equivalence relation \mathcal{R} such that any strings that are not encoding valid instances are equivalent, and $(G_i^c, l_i), (G_j^c, l_j)$ are equivalent iff $|V(G_i^c)| = |V(G_j^c)|$, and $l_i = l_j$. Hereafter, we assume that $V(G_i^c) = [n]$ and $l_i = l$, for any $i \in [t]$. We will build an instance of MAXIMUM COMMON INDUCED SUBGRAPH parameterized by the vertex cover (G_1, G_2, l', Z) where G_1 and G_2 are two graphs, $l' \in \mathbb{N}$ and $Z \subseteq V(G_2)$ is a vertex cover of G_2 computed in fpt-time, such that there is a solution of size l' for MAXIMUM COMMON INDUCED SUBGRAPH iff there is an $i \in [t]$ such that there is a solution of size l in G_i^c . We now describe how to build G_1 and G_2 .

The graph G_2 is defined by (see also Figure 1):

- $V(G_2) = \{p, q, r\} \cup \{a_i \mid i \in [t]\} \cup \{e_{uv} \mid u < v \in [n]\} \cup \{x_u \mid u \in [n]\}$,
- $E(G_2)_1 = \{pq, pr, qr\}$,
- $E(G_2)_2 = \{ra_i \mid i \in [t]\}$,
- $E(G_2)_3 = \{a_i e_{uv} \mid uv \in E(G_i^c)\}$,
- $E(G_2)_4 = \{e_{uv} x_u, e_{uv} x_v \mid u < v \in [n]\}$,
- $E(G_2) = E(G_2)_1 \cup E(G_2)_2 \cup E(G_2)_3 \cup E(G_2)_4$.

To build G_1 (see also Figure 1):

- $V(G_1) = \{p, q, r, a\} \cup \{e_i \mid i \in \binom{[t]}{2}\} \cup \{x_u \mid u \in [l]\}$,
- $E(G_1)_1 = \{pq, pr, qr, ra\}$,
- $E(G_1)_2 = \{ae_i \mid i \in \binom{[t]}{2}\}$,
- $E(G_1)_3 = \{e_i x_u, e_i x_v \mid i \in \binom{[t]}{2} \text{ and } e_i = uv\}$,

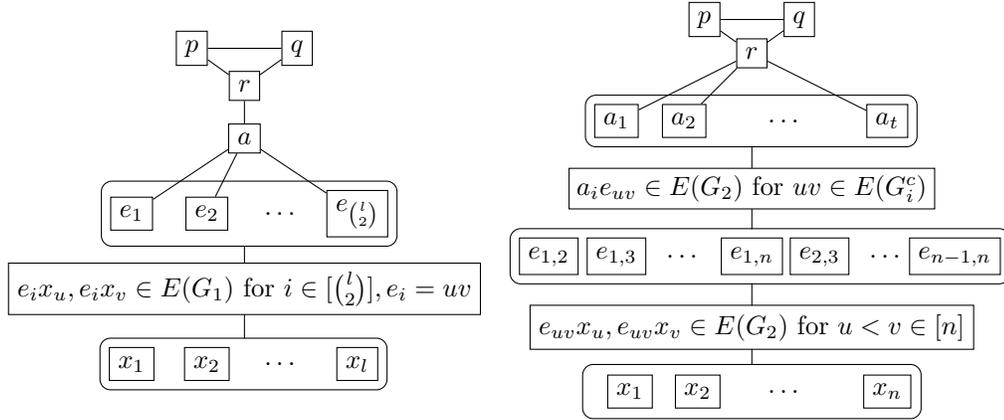


Figure 1: Illustration of the construction of G_1 (left) and G_2 (right).

- $E(G_1) = E(G_1)_1 \cup E(G_1)_2 \cup E(G_1)_3$.

We set $l' = |V(G_1)|$, and $Z = \{p, r\} \cup \{e_{uv} \mid u < v \in [n]\}$. It is easy to see that Z is indeed a vertex cover for G_2 and that its size is equal to $\frac{n(n-1)}{2} + 2$, which is polynomial in n and hence in the size of the largest instance. Note that the size of the graph G_1 does not depend on t and is polynomial in n , so the size of its vertex cover is also polynomial in n and independent of t .

Let us show that G_1 is an induced subgraph of G_2 iff at least one of the G_i^c 's has a clique of size l .

(\Leftarrow) Suppose that G_i^c has a clique of size l . We denote by $S \subseteq V(G_i^c)$ a clique of size exactly l in G_i^c . We show that there is an induced subgraph S' of G_2 of size l' , isomorphic to G_1 . We set $V(S') = \{p, q, r\} \cup \{a_i\} \cup \{e_{uv} \mid uv \in E(S)\} \cup \{x_u \mid u \in S\}$. One can easily check that this subgraph is isomorphic to G_1 .

(\Rightarrow) Assume now that G_1 is an induced subgraph of G_2 . Denote by S' the subgraph of G_2 isomorphic to G_1 . Note that the only triangle in G_2 is pqr . Indeed, $G_2 - p$ is bipartite. The triangle pqr in G_1 has therefore to match pqr in G_2 . Moreover, r in G_1 has to match r in G_2 since p and q have no edges besides the clique pqr . The vertex a in G_1 can only match a vertex a_i for some $i \in [t]$. Then, e_1 up to $e_{\binom{l}{2}}$ in G_1 has to match $\binom{l}{2}$ vertices in $\{e_{uv} \mid u < v \in [n]\}$ of G_2 which correspond to actual edges in G_i^c . Finally, x_1 up to x_l in G_1 has to match l vertices among the x_j 's in G_2 . Note that the number of edges in $E(G_1)_3$ between the e_j 's and the x_j 's is exactly $2\binom{l}{2} = l(l-1)$. More precisely, each e_j touches 2 edges in $E(G_1)_3$ and each x_j touches $l-1$ edges in $E(G_1)_3$. In order to get a match in G_2 , one should find a set of $\binom{l}{2}$ edges inducing exactly l vertices. So, this set of l vertices is a clique in G_i^c . \square

Note that the parameter of MCIS in the previous reduction is exactly the size of G_1 and the graphs used in the proof are connected. Therefore, we have the following:

Corollary 11. INDUCED SUBGRAPH ISOMORPHISM and MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH, parameterized by a bound on the minimum vertex covers of input graphs, do not have a polynomial-size kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.

The algorithm of [16] is not single-exponential for parameter sum of the vertex cover numbers. In fact, we show that a single-exponential algorithm is very unlikely. This is, to the best of our knowledge, the first result of this type for parameter vertex cover.

Theorem 12. Under the ETH, IS(C)I cannot be solved in time $2^{o(k \log k)}$ when parameter k is the sum of the vertex cover number of both graphs.

Proof. We give a reduction from $k \times k$ PERMUTATION CLIQUE which linearly preserves the parameter k . It is known that this problem does not admit an algorithm with running time $2^{o(k \log k)}$ unless the ETH fails [29]. In the $k \times k$ PERMUTATION CLIQUE problem, one is given a graph over the set of vertices $[k] \times [k]$ and the goal is to find a clique of size k such that in each row and in each column exactly one vertex is part of the clique, where a row is the set of vertices $\{(i, 1), (i, 2), \dots, (i, k)\}$ for some $i \in [k]$, and a column is the set of vertices $\{(1, j), (2, j), \dots, (k, j)\}$ for some $j \in [k]$.

We first describe how the graph G_2 is built from any instance $G = ([k] \times [k], E)$ of $k \times k$ PERMUTATION CLIQUE. For each row (resp. column) index $i \in [k]$, we add two vertices r_i^1 and r_i^2 (resp. c_i^1 and c_i^2) that we link by an edge. For $j \in [2]$, we set $R_j = \{r_1^j, r_2^j, \dots, r_k^j\}$ (resp. $C_j = \{c_1^j, c_2^j, \dots, c_k^j\}$) and $R = R_1 \cup R_2$ (resp. $C = C_1 \cup C_2$). Then, to distinguish row indices from column indices, we add a clique D_r of size 6, and we link one designated vertex r of D_r to all the vertices in R . We also add a clique D of size 5 with a special vertex v in D such that v is linked to all the vertices in $R_1 \cup C_1$.

Finally, for each edge $e = (i, j)(i', j')$ of G with $i \neq i'$ and $j \neq j'$ ², we add a vertex $v(e, 1)$ that we link to the four vertices $r_i^1, c_j^1, r_{i'}^2$, and $c_{j'}^2$, and a vertex $v(e, 2)$ that we link to the four vertices $r_i^2, c_j^2, r_{i'}^1$, and $c_{j'}^1$. This ends the construction of G_2 (see Figure 2). The pattern G_1 depends only on k and is defined as the graph one obtains following the above construction when G_1 have all the edges of the form $(i, i)(i', i')$ and no other edges (in other words, G_1 has a k -clique on the diagonal and nothing else).

Both G_1 and G_2 have $R \cup C \cup D_r \cup D$ as a vertex cover of size $4k + 11$. G_2 has $|E| + 4k + 11 = O(k^4)$ vertices and G_1 has $2\binom{k}{2} + 4k + 11 = O(k^2)$ vertices. To avoid confusion about vertices in G_1 and G_2 we will denote the vertices and sets of vertices of G_1 with a tilde. We now show that there is a solution of size k to the instance of $k \times k$ PERMUTATION CLIQUE if and only if there is a solution to the built IS(C)I instance.

Suppose there is a solution $\{(a_1, b_1), \dots, (a_k, b_k)\}$ to the instance of $k \times k$ PERMUTATION CLIQUE. Then, G_1 is an induced subgraph of G_2 with the

²We ignore the other edges since they are not relevant in finding a permutation clique.

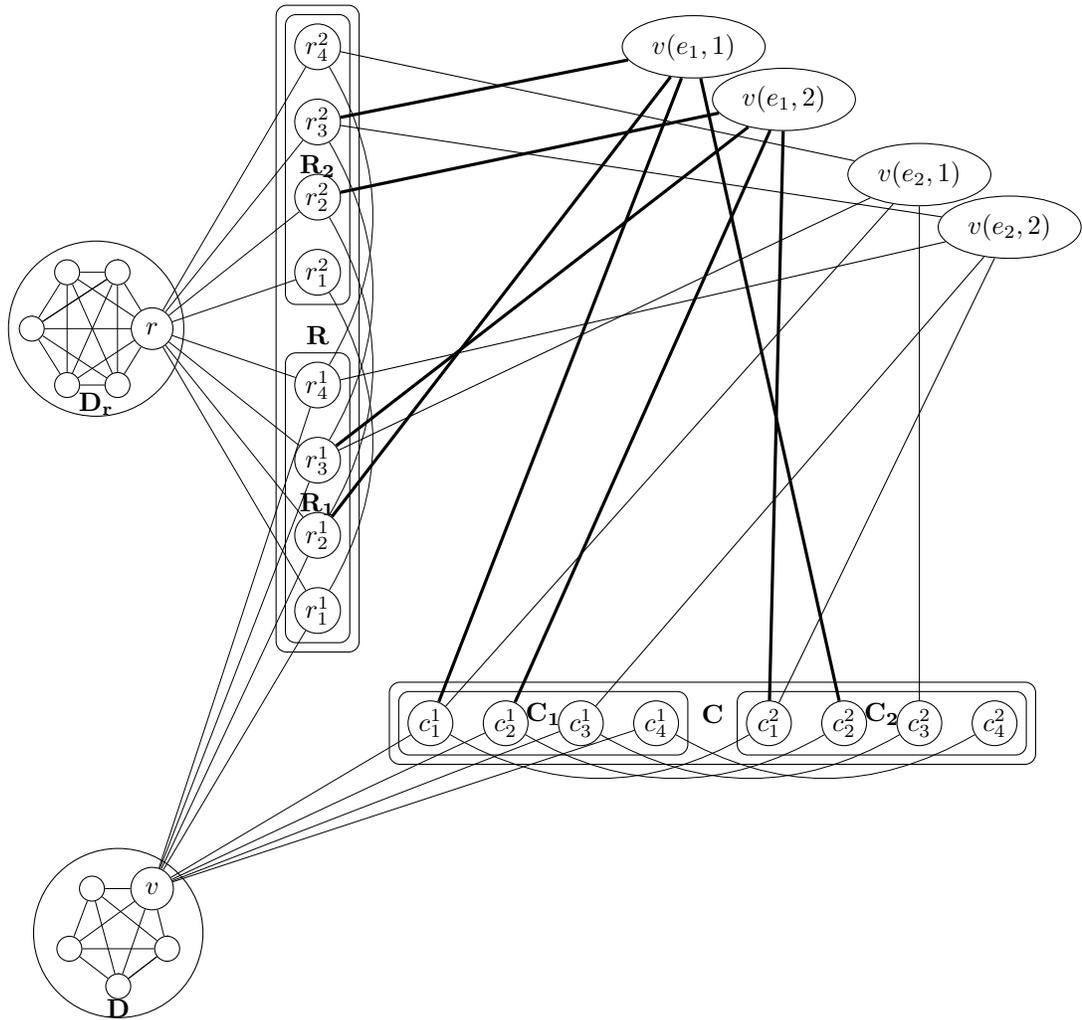


Figure 2: The overall construction of G_2 in the proof of Theorem 12. We represented only two edges of G : $e_1 = (2, 1)(3, 2)$ and $e_2 = (3, 1)(4, 3)$. For the sake of readability, the edges encoding e_1 are enhanced to distinguish them easily from the edges encoding e_2 .

following mapping. We map \tilde{r} to r and \tilde{v} to v . We map $\tilde{D}_r \setminus \{\tilde{r}\}$ to $D_r \setminus \{r\}$ and $\tilde{D} \setminus \{\tilde{v}\}$ to $D \setminus \{v\}$ in an arbitrary way. Then, for each $i \in [k]$ and $j \in [2]$, we map \tilde{r}_i^j to $r_{a_i}^j$ and \tilde{c}_i^j to $r_{b_i}^j$. We observe that this mapping is one-to-one since $(a_1, b_1), \dots, (a_k, b_k)$ is a *permutation* clique, i.e., $\{a_1, a_2, \dots, a_k\} = [k] = \{b_1, b_2, \dots, b_k\}$. Finally, for any $j \in [2]$, and any $i \neq i' \in [k]$ we map $\tilde{v}(e, j)$ to $v((a_i, b_i)(a_{i'}, b_{i'}), j)$. Note that vertex $v((a_i, b_i)(a_{i'}, b_{i'}), j)$ always exists because $\{(a_1, b_1), \dots, (a_k, b_k)\}$ is a clique.

Conversely, if there is a solution to the IS(C)I instance, we will show that there is a permutation k -clique in G . There is only one clique of size 6 in G_2 , so the clique \tilde{D}_r of size 6 has to be mapped to D_r . Then, \tilde{r} , as the unique vertex of \tilde{D}_r of degree larger than 5, should be mapped to r . Now, for the same reasons, \tilde{D} should be mapped to D and \tilde{v} to v . Vertices of $\tilde{R}_1 \cup \tilde{C}_1$ are the only $2k$ unmatched vertices having \tilde{v} as a neighbor, so those vertices should be matched to the only $2k$ unmatched vertices having v as a neighbor, namely $R_1 \cup C_1$. For similar reasons, \tilde{R} should be mapped to R . Now, $\tilde{R}_2 \cup \tilde{C}_2$ can only be mapped to $R_2 \cup C_2$ as the only unmatched vertices having exactly one neighbor in $\tilde{R}_1 \cup \tilde{C}_1$ ($R_1 \cup C_1$).

Thus, the $4k$ vertices of $\tilde{R} \cup \tilde{C}$ can only be mapped to $R \cup C$, such that for $j \in [2]$, \tilde{R}_j is mapped to R_j and \tilde{C}_j is mapped to C_j . The edges $\tilde{r}_i^1 \tilde{r}_i^2$ and $r_i^1 r_i^2$ (resp. $\tilde{c}_i^1 \tilde{c}_i^2$ and $c_i^1 c_i^2$) further constrain the mapping: if \tilde{r}_i^1 is mapped to $r_{i'}^1$ then \tilde{r}_i^2 has to be mapped to $r_{i'}^2$ (resp. if \tilde{c}_i^1 is mapped to $c_{i'}^1$ then \tilde{c}_i^2 has to be mapped to $c_{i'}^2$). Hence, we can see the mapping from $\tilde{R} \cup \tilde{C}$ to $R \cup C$ as two permutations σ_r and σ_c on k elements, such that for $j \in [2]$, for $i \in [k]$, \tilde{r}_i^j is mapped to $r_{\sigma_r(i)}^j$ and \tilde{c}_i^j is mapped to $c_{\sigma_c(i)}^j$. Then, the current partial mapping can be extended to a solution only if $\{(\sigma_r(1), \sigma_c(1)), \dots, (\sigma_r(k), \sigma_c(k))\}$ is a clique in G . Indeed, $\forall j \in [2], \forall i \neq i' \in [k], \tilde{v}((i, i')(i', i'), j)$ can only be mapped to a potential $v((\sigma_r(i), \sigma_c(i))(\sigma_r(i'), \sigma_c(i')), j)$ so that vertex has to exist, meaning that there should be an edge in G between $(\sigma_r(i), \sigma_c(i))$ and $(\sigma_r(i'), \sigma_c(i'))$.

An algorithm solving IS(C)I in time $poly(|G_1|, |G_2|)2^{o(k \log k)}$ with $k := \text{vc}(G_1) + \text{vc}(G_2)$ would therefore translate into an algorithm running in time $2^{o(k \log k)}$ for $k \times k$ PERMUTATION CLIQUE and contradict the ETH. \square

As ISI is a particular case of MCIS, we have the following direct corollary:

Corollary 13. *Under the ETH, MCIS and MCCIS cannot be solved in time $2^{o(k \log k)}$ when parameter k is the sum of the vertex cover number of both graphs.*

Despite the fact that ISI and MCIS have the same parameterized complexity with respect to the natural parameter, they exhibit different behaviors when considering structural parameters. In fact, the latter is paraNP-hard when parameterized by the vertex cover of only one of the two graphs, whereas ISI is FPT when parameterized by the vertex cover of the second (host) graph. To see this, note that when the host graph has a vertex cover of size k , the minimum size of a vertex cover in the pattern graph must be bounded by the parameter k ; otherwise we have a NO-instance. The claim follows from the fixed-parameter tractability of MCIS in this case [16].

Given the negative result of Theorem 9, the next question we pose is whether MCCIS is in FPT with respect to the size of a minimum vertex cover. In [16], a parameterized algorithm is presented for MCIS when the parameter is a bound on the minimum vertex cover number of the input graphs. However, that algorithm cannot help us much for solving MCCIS since it relies on the existence of a feasible solution of size at least $\approx n - k$ which consists of mapping the two *big* independent sets of the two graphs onto each other. Of course, this is not a feasible solution for MCCIS. In the following we prove that MCCIS is fixed-parameter tractable w.r.t. $k := \text{vc}(G_1) + \text{vc}(G_2)$. In fact, we will first prove a more general result, namely that MCIS is fixed-parameter enumerable and use this algorithm to discard non-connected solutions. Note that similar result can be derived for any condition on the solution graph which can be checked in polynomial (actually FPT) time.

Theorem 14. MAXIMUM COMMON INDUCED SUBGRAPH *parameterized by $k := \text{vc}(G_1) + \text{vc}(G_2)$ is fixed-parameter enumerable.*

Proof. In time $O^*(2^k)$ (even $O^*(1.2738^k)$ [30]), we can find minimum vertex covers C_1 and C_2 in G_1 and G_2 respectively. Let $I^{(j)}$ be the independent set $V(G_j) \setminus C_j$ for $j \in \{1, 2\}$. By assumption, our parameter k is $\max(C_1, C_2)$, so we can enumerate all tripartitions of C_1 and C_2 in time $O^*(9^k)$. We denote by $C_{1,m}$, $C_{1,u}$ and $C_{1,i}$ (respectively $C_{2,m}$, $C_{2,u}$ and $C_{2,i}$) the three sets of a tripartition of C_1 (respectively C_2). For $j \in \{1, 2\}$, $C_{j,u}$ corresponds to the vertices of C_j that are not matched, so they may be deleted. $C_{j,m}$ comprises the vertices matched to $C_{3-j,m}$ (that is, to the vertex cover of the other graph), and $C_{j,i}$ are the vertices matched to $I^{(3-j)}$, the independent set of the other graph. See Figure 3.

We observe that for $j \in \{1, 2\}$, $I^{(j)}$ can be partitioned into at most 2^k classes of twins $I_1^{(j)}, I_2^{(j)}, \dots, I_{2^k}^{(j)}$ in polynomial time. A class of twins in this context is a set of vertices with an identical neighborhood in the vertex cover and there are at most 2^k subsets of C_j . Potentially, some classes can be empty: they correspond to a subset of the vertex cover C_j that is not the (exact) neighborhood of any vertex in $I^{(j)}$.

At this point, we can enumerate the mappings between $C_{1,m}$ and $C_{2,m}$ in time $O^*(k^k)$ and the mappings between $C_{j,i}$ and $I^{(3-j)}$ in time $O^*((2^k)^k) = O^*(2^{k^2})$. Indeed, to match a vertex u with a vertex v or a twin of v is equivalent. Thus, in time $O^*((9k)^k 2^{k^2})$ we can enumerate all the solutions of MCIS where only vertices of $I^{(1)}$ could still be matched to vertices of $I^{(2)}$. The optimal map of the independent sets can be done in polynomial time by matching the greatest number of vertices in each *equivalent* twin class (which is the size of the smaller of the two equivalent twin classes), where a twin class $I_r^{(j)}$ in $I^{(j)}$ is equivalent to a twin class $I_s^{(3-j)}$ in $I^{(3-j)}$ if the vertices of $N(I_r^{(j)}) \setminus C_{j,u}$ and $N(I_s^{(3-j)}) \setminus C_{3-j,u}$ are in one-to-one correspondence. \square

To find a solution for MCCIS, the algorithm described in the above proof enumerates all possible maximal common induced subgraphs in time $O^*((9k)^k 2^{k^2}) =$

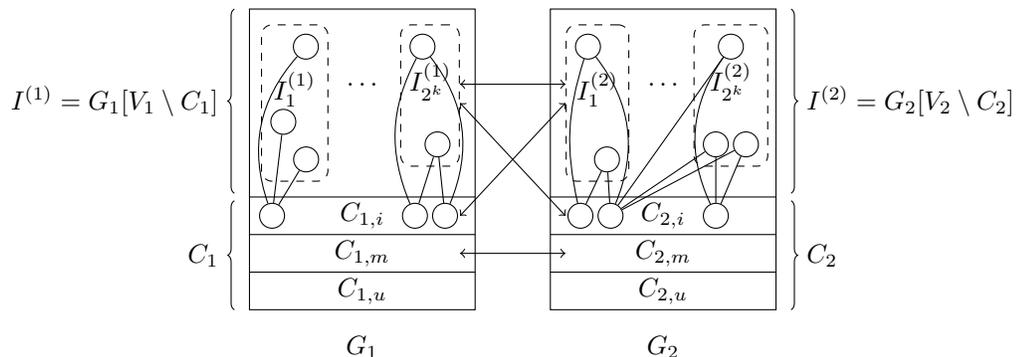


Figure 3: Illustration of the proof of Theorem 14. Dashed boxes represent the classes inside the independent set. Arrows represent the matching between sets of vertices. Sets C_1 (resp. C_2) represents a vertex cover for G_1 (resp. G_2).

$O^*(2^{O(k^2)})$. The current bottleneck to improve it is when we try to match vertices of the vertex cover with vertices of the independent set. For the not connected version of the problem, a trivial argument can bound the size of the independent set (if this one is big, there is a trivial solution), which cannot be used for the connected version.

Corollary 15. *MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH parameterized by $k := vc(G_1) + vc(G_2)$ is fixed-parameter tractable.*

Let us finish this section with some general considerations. Note that for ISI, the parameter $vc(G_1) + fvs(G_2)$ is not the same as $fvs(G_1) + vc(G_2)$. In the latter, the parameter is a bound on the vertex cover of G_2 (as well as the feedback vertex set of G_1) which makes ISI in FPT, while it remains open for $vc(G_1) + fvs(G_2)$. We also note that ISI is not in XP w.r.t. $vc(G_1)$ by a simple reduction from INDEPENDENT SET: let G_2 be an edgeless graph on k vertices, then its vertex cover number is 0.

We now turn our attention to the case where MCIS is parameterized by a combination of the natural parameter and some structural parameter. We note that, in general, such parameterization reduces the problem's complexity. This is most often due to the fixed-parameter tractability of MCIS in H -minor free graphs (again, since ISI is FPT in this case [13]). For example, consider the case where the parameter is the sum of some bound t on the feedback vertex set of the input graphs and the natural parameter k . The problem is FPT in this case since graphs of t -feedback vertex set are H -minor free where H is the “fixed” graph consisting of a disjoint union of $t + 1$ triangles. The same applies to parameterization by treewidth and the natural parameter by considering H to be the complete graph on $t + 2$ vertices, for example.

5. Conclusion

We studied the MAXIMUM COMMON INDUCED SUBGRAPH and MAXIMUM COMMON CONNECTED INDUCED SUBGRAPH problems with respect to the solution size on special graph classes such as forests, bipartite graphs, bounded degree graphs, bounded degeneracy graphs, graphs without small (length 3 or 4) cycles. The two problems are fixed-parameter tractable on H -minor free graphs, which include forests, and bounded degree graphs, but they are W[1]-complete on bipartite graphs of girth 6 and degeneracy 2. This ruled out at the same time two approaches to get fixed-parameter algorithms on subclasses of graphs for W-hard problems.

We then considered the use of structural parameters, such as a bound on the minimum vertex covers of the input graphs. Although both MCIS and MCCIS are in FPT in this case, we proved that no kernel of polynomial bound can be obtained unless $\text{NP} \subseteq \text{coNP}/\text{poly}$ and that they cannot be solved in time $2^{o(k \log k)}$ under the ETH. We noted that MCIS is not even in XP with respect to other (smaller) auxiliary parameters, such as treewidth and feedback vertex set. A few open problems remain to be addressed. For example, is MCIS/MCCIS in FPT when parameterized by the combination of the vertex cover number and the feedback vertex set number, or by the vertex cover number and the treewidth? Moreover, it would be interesting to know whether the algorithm for MCCIS (Corollary 15) can be improved to match the lower bound of Theorem 12.

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