On Subexponential and FPT-time Inapproximability

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1. Inapproximability in subexponential time
   - APETH
   - designing a.p. sparsifiers
   - APETH and parameterised complexity

2. Inapproximability in FPT-time
   - LPC
   - results

ETH for approximations

Hypothesis (APETH(Π))

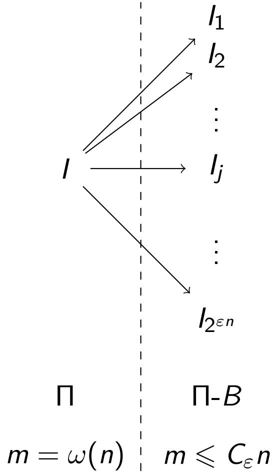
There exists r and ε such that Π cannot be r-approximated within time $O^*(2^{\epsilon n})$.

Definition (APETH-equivalent problems)

Π_1 and Π_2 are two APETH-equivalent problems denoted by $\Pi_1 \underset{ae}{\equiv} \Pi_2$ if APETH(Π_1) holds iff APETH(Π_2) holds

Standard Sparsification

Assumption: Π not solvable in $O(2^{o(n)})$



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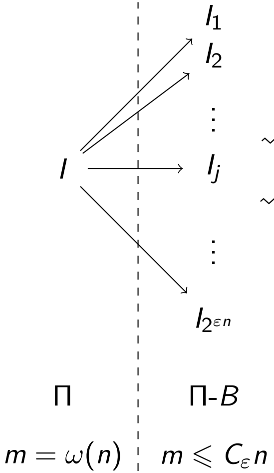

$\rightsquigarrow$ Π-B not solvable in $O(2^{o(n)})$

$\rightsquigarrow$ Π not solvable in $O(2^{o(m)})$

(Approximation preserving) Sparsification

Definition (approximation preserving sparsification)

Two functions (f, g) s.t. $\forall \varepsilon > 0$ and $\forall I$ instance of Π $\exists B_\varepsilon$ s.t.

- $f{:}I \mapsto I_1, I_2, \ldots, I_h$ in time $O^*(2^{\epsilon n})$, $h \leqslant 2^{\varepsilon n}$.
- $\forall i \in \{1, \ldots, h\}$, $I_i \leqslant n$ and $p(I_i) \leqslant B_\varepsilon$.
- $g{:}Sol(I_i) \mapsto Sol(I)$ in polynomial time.
- $\exists i, S_i$ r-approximation of $I_i \Rightarrow g(S_i)$ r-approximation of I.

Theorem (straightforward)

If Π admits an a.p. sparsification then $\Pi \underset{ae}{\equiv} \Pi\text{-}B$.

Aim

We want to give evidences that most inapproximable problems satisfy APETH:

- showing that many problems are APETH-equivalents.
- linking APETH to other complexity conjectures.

Recipe

- Design a.p. sparsifier for well-known problems $\rightsquigarrow$ $Pi_1 \underset{ae}{\equiv} Pi_1\text{-}B$, $Pi_2 \underset{ae}{\equiv} Pi_2\text{-}B$, $\ldots$, $Pi_l \underset{ae}{\equiv} Pi_l\text{-}B$.
- L-reduction in Max SNP [Papadimitriou, Yannakakis '91] $\rightsquigarrow$ $Pi_1\text{-}B \underset{ae}{\equiv} Pi_2\text{-}B \underset{ae}{\equiv} \ldots \underset{ae}{\equiv} Pi_l\text{-}B$.
- Conclude $Pi_1 \underset{ae}{\equiv} Pi_2 \underset{ae}{\equiv} \ldots \underset{ae}{\equiv} Pi_l$.

An a.p. sparsifier for Independent Set

Basic idea: to stop the branching tree at the right time.

B_ε: smallest integer such that the positive root of $X^{B_\varepsilon+1}$-X^{B_ε}-$1 = 0$ is smaller than 2^ε.

- $\Delta(G) \geqslant B_\varepsilon \rightsquigarrow n$-1, n-B_ε-1 branching.
- $\Delta(G) < B_\varepsilon \rightsquigarrow$ G B_ε-sparse.

- branching tree has size $(2^{\varepsilon})^n = 2^{\varepsilon n}$.
- f: building the tree.
- g: adding to S_j the vertices taken from I to I_j.
- approximation preserving: one branch takes only vertices of the optimal solution S^*. Let this number of vertices be k and the branch be the j-th:
$\frac{k+|S^* \cap G_j|}{k+|S_j|} \leqslant \frac{|S^* \cap G_j|}{|S_j|}$.

An a.p. sparsifier for Generalised Dominating Set

Generalised Dominating Set: $G = (V = V_1 \cup V_2 \cup V_3, E)$. Find a minimum subset of $V_1 \cup V_2$ which dominates $V_2 \cup V_3$.

- (i) While there exists $v \in V_1$ s.t. $d(v) \geqslant B'$, branch on v.
- (ii) While there exists $v \in V_2$ s.t. $d(v) \geqslant B'^2$, branch on v.
- (iii) While there exists $v \in V_3$ s.t. $d(v) \geqslant B'^3$, branch on a neighbor of v.

Weights

$$w(v) = \begin{cases} \min(\frac{1}{2}, \frac{1}{4} + \frac{d(v)}{4B'}) & \text{if } v \in V_1. \\ \min(1, \frac{3}{4} + \frac{d(v)}{4B'}) & \text{if } v \in V_2. \\ \frac{1}{2} & \text{if } v \in V_3. \end{cases}$$

- (i) n-$\frac{1}{2}$,n-$\frac{B'}{2}$-$\frac{1}{2}$ branching, neighbors in V_3 removed, neigbors in V_2 transferred to V_1.
- (ii) n-$\frac{1}{2}$,n-$\frac{B'^2}{B'}$ branching.
- (iii)n-$\frac{1}{2}$,n-$\frac{B'^3}{B'^2}$ branching.

In any case, roughly a n-c,n-B' branching.

Theorem (Th1)

Set Cover, Independent Set, Independent Set-B, Vertex Cover, Vertex Cover-B, Dominating Set, Dominating Set-B, Max Cut-B, Max kSAT-B ($k \geqslant 2$) are APETH-equivalent.

Theorem (Th2)

The followings are equivalent:

- *(i) APETH holds for one problem of Th1*
- *(ii)* $\exists\Pi$ *Max SNP-complete,* $\exists r, \varepsilon$ *s.t.* Π *cannot be* r*-approximated in* $O^*(2^{\varepsilon k})$*.*
- *(iii)* $\forall\Pi$ *Max SNP-complete,* $\exists r, \varepsilon$ *s.t.* Π *cannot be* r*-approximated in* $O^*(2^{\varepsilon k})$*.*

Theorem (Th2)

The followings are equivalent:

- *(i) APETH holds for one problem of Th1*
- *(ii) $\exists \Pi$ Max SNP-complete, $\exists r, \varepsilon$ s.t. Π cannot be r-approximated in $O^*(2^{\varepsilon k})$.*
- *(iii) $\forall \Pi$ Max SNP-complete, $\exists r, \varepsilon$ s.t. Π cannot be r-approximated in $O^*(2^{\varepsilon k})$.*

(i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i): Contrapositives are straightforward.
(ii) $\Rightarrow$ (iii): Suppose there is a Max SNP-complete problem Π' r-approximable in $O^*(2^{\varepsilon k})$ for all r and ε. For any Max SNP-complete problem Π, consider an L-reduction from Π to Π' to show that so does Π.

Linear PCP Conjecture

Conjecture (LPC)

$3SAT \in PCP_{\beta,1}[\log|\phi| + D, E]$.

It is more an open question than a conjecture but:

Theorem (Dinur '07)

$\forall \varepsilon > 0$, $3SAT \in PCP_{\varepsilon,1}[(1 + o(1)) \log n + O(log(\frac{1}{\varepsilon})), O(log(\frac{1}{\varepsilon}))]$.

Theorem (Moshkovitz, Raz '08)

Under ETH, $\forall \varepsilon, \delta > 0$, you cannot tell apart instances of Max 3SAT where:

- *at least $(1 - \varepsilon)m$ clauses are satisfiable.*
- *at most $(\frac{7}{8} + \varepsilon)m$ clauses are satisfiable.*

in time $O(2^{m^{1-\delta}})$.

Lemma (Lem1)

Under LPC+ETH, $\exists r < 1$, $\forall \varepsilon > 0$, you cannot tell apart instances of Max 3SAT where:

- *at least $(1 - \varepsilon)m$ clauses are satisfiable.*
- *at most $(r + \varepsilon)m$ clauses are satisfiable.*

in time $O(2^{o(m)})$.

Sparsification
Reduction: 3SAT formula $\phi \to$ 3SAT formula ψ simulating the prover of ϕ implied by LPC.
Solving the gap for ψ in subexponential time $\to$ solving ϕ in subexponential time
Contradiction of ETH.

Lemma (Lem2, self-improvement property)

If there exists an FPT-time r-approximation for Independent Set for some r, then there is one for all $r \in (0, 1)$.

Theorem (Chen, Huang, Kanj, Xia '06)

Under ETH, Independent Set cannot be solved in time $f(k)n^{o(k)}$.

Theorem (Th3)

Under LPC+ETH, there exists r s.t. Independent Set cannot be r-approximated in time $f(k)n^{o(k)}$.

Combination of previous theorem and gap-preserving reduction.

Corollary

Under LPC+ETH, for any r there is no r-approximation for Independent Set in FPT-time.

Th3+Lem2

Open Questions

- Inapproximability results upon ETH only, or a more standard conjecture than LPC?
  - $\forall \varepsilon$, $\exists r_0 = h(n, \varepsilon)$, $\forall r \geqslant r_0$ Independent Set cannot be r-approximated in $O(2^{\frac{n^{1-\varepsilon}}{r^{1+\varepsilon}}})$ [Chalermsook, Laekhanukit, Nanongkai, FOCS '13].
  - See also [Chitnis, Hajiaghayi, Kortsarz, IPEC '13].
- Approximation preserving sparsifiers for Max Cut, Max 3SAT?