Fine-grained complexity of coloring geometric intersection graphs

Édouard Bonnet joint work with Csaba Biró, Dániel Marx, Tillmann Miltzow, and Paweł Rzążewski

Middlesex University, London

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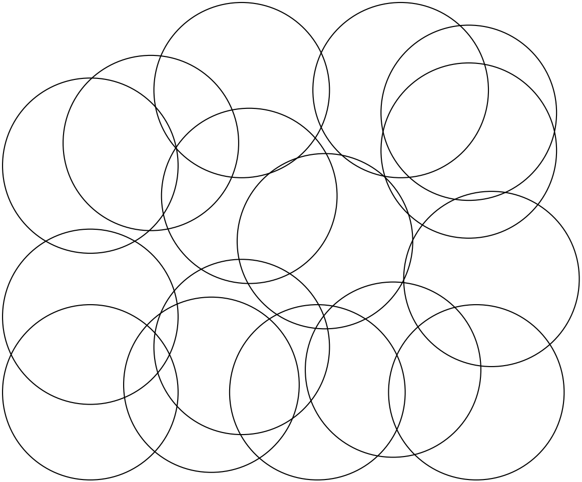
k-Coloring Unit Disks



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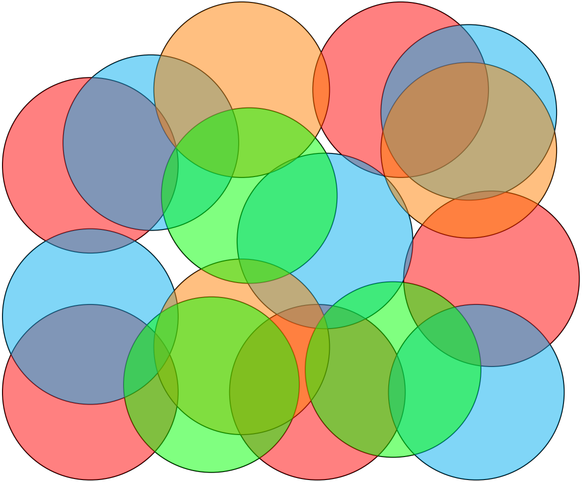

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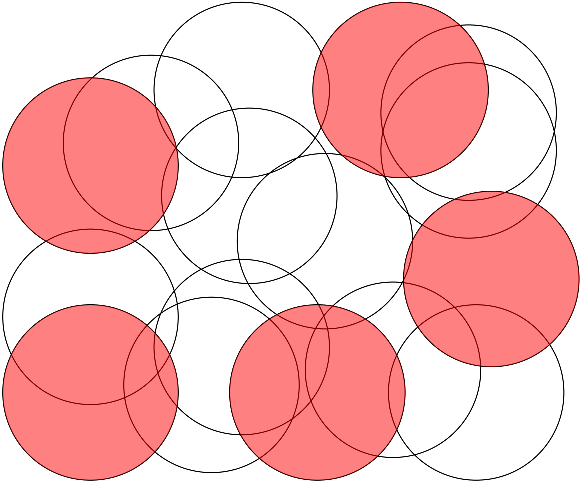

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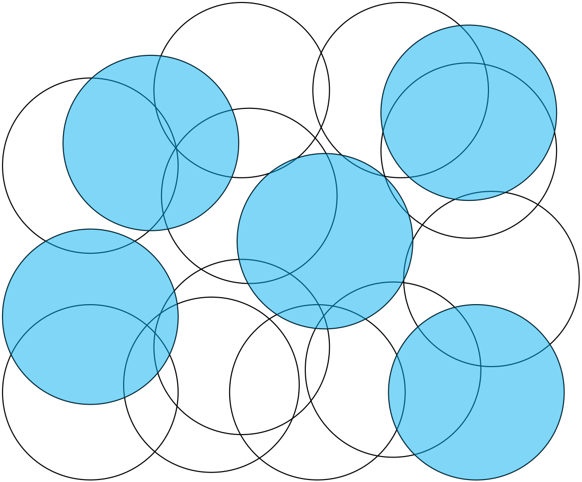

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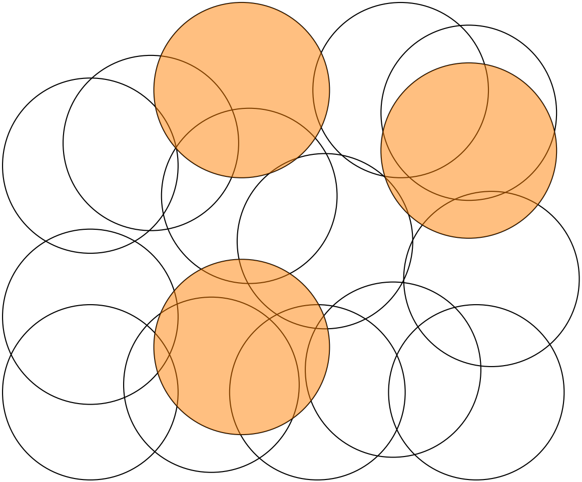

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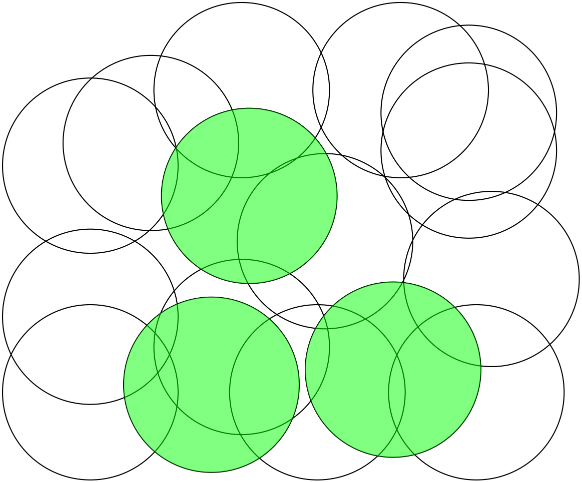

NP-hard for any integer $k \geqslant 3$

Subexponential algorithms?

NP-hardness:

- your problem is not solvable in polynomial, unless 3-SAT is
- very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$.

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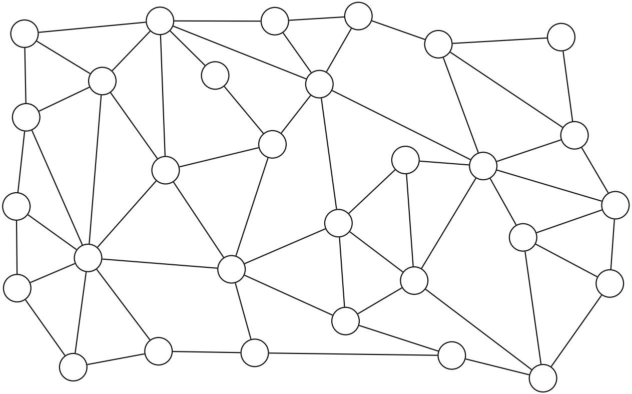
ETH-hardness:

- stronger assumption than P$\neq$NP is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT
- Allows to prove stronger conditional lower bounds
- linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.

Square root phenomenon on planar graphs



Many problems are solvable in $2^{O(\sqrt{n})}$ in **planar graphs**, and unlikely solvable in $2^{o(n)}$ in general graphs.

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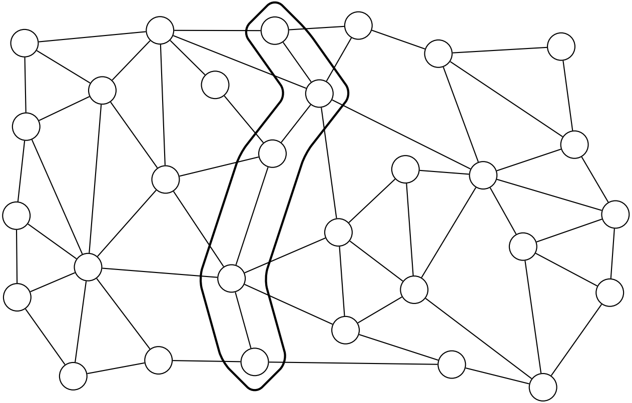

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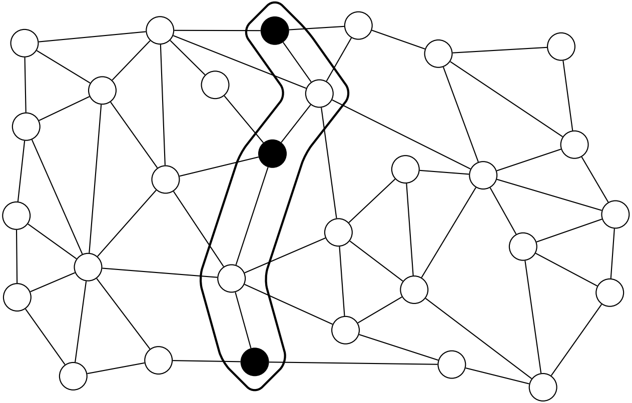
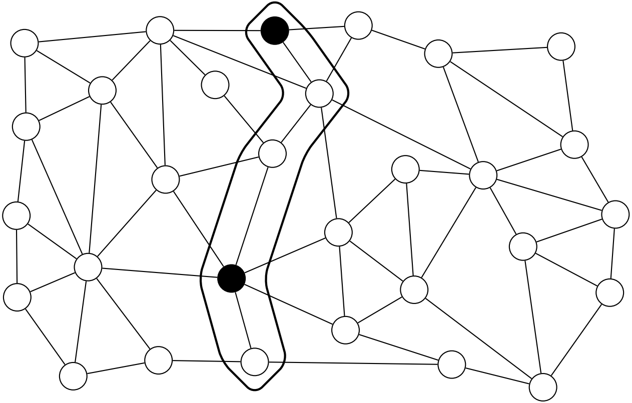


<u>Max Independent Set</u>, 3-Coloring, Hamiltonian Path...

Square root phenomenon on planar graphs



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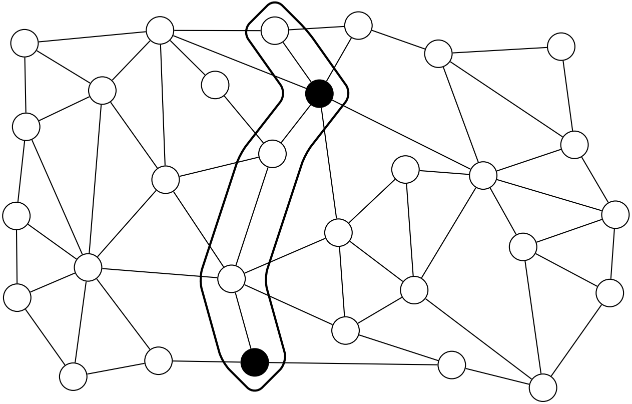

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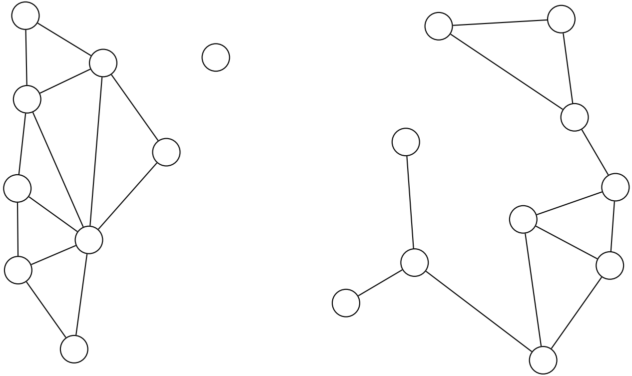

<u>MAX INDEPENDENT SET</u>, 3-COLORING, HAMILTONIAN PATH...

Dynamic programming would spare a log n in the exponent.

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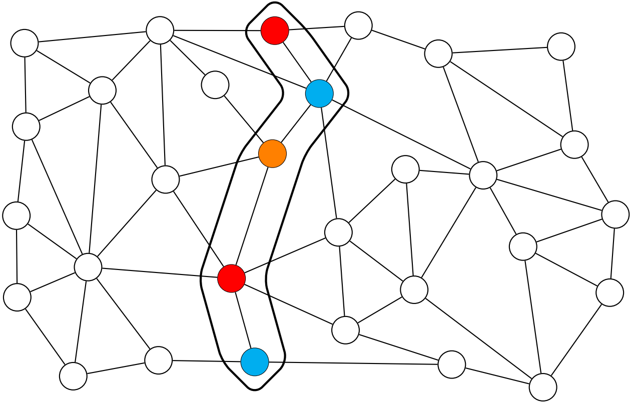

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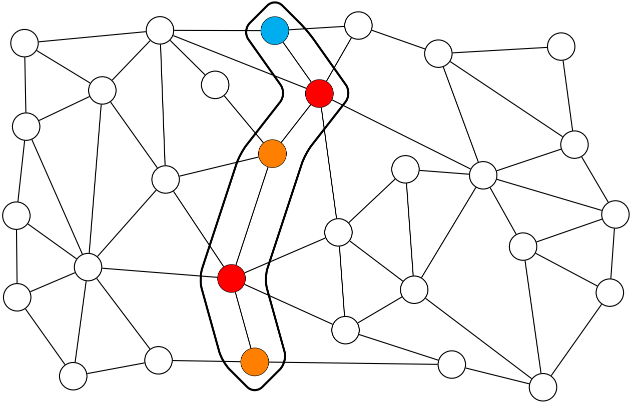

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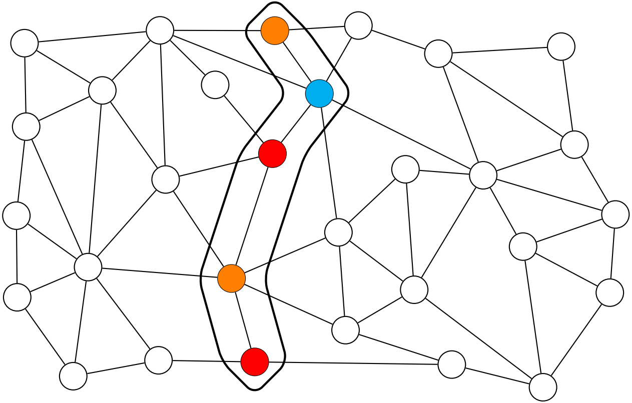
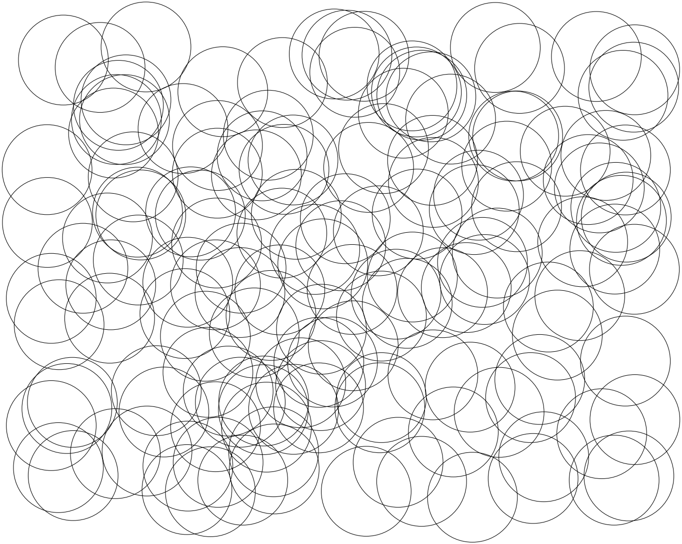
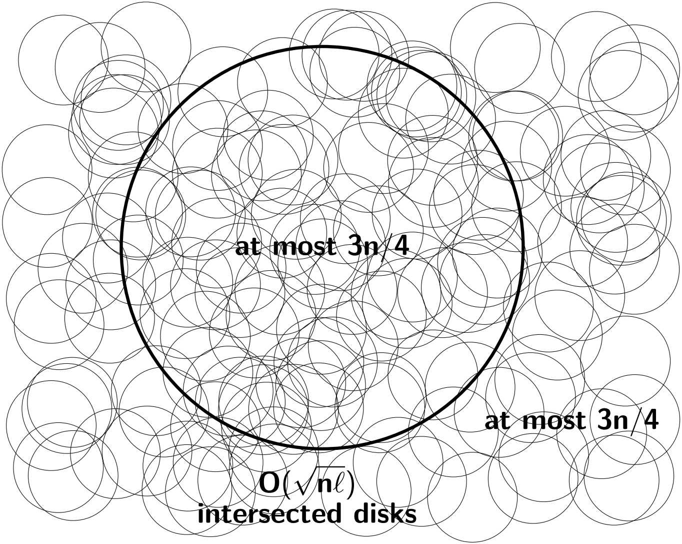


MAX INDEPENDENT SET, <u>3-COLORING</u>, HAMILTONIAN PATH...

Smith and Wormald '98: $\forall n$ disks with ply ℓ,



Smith and Wormald '98: $\forall n$ disks with ply ℓ, $\exists$ **O**



Standard algorithm for ℓ-coloring

If the ply is greater than ℓ, then more than ℓ colors are needed.

Otherwise, there is a balanced separator S of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell}\log n})$.

Trying all the ℓ-colorings on S takes time $O(2^{\sqrt{n\ell}\log \ell})$.

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Overall running time: $O(2^{\sqrt{n\ell}\log n})$.

We will see that this running time is optimal up to logarithmic factors in the exponent.

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Theorem

For any $\alpha \in [0,1]$, coloring n unit disks with $\ell = \Theta(n^{\alpha})$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

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Linear number of colors $\rightsquigarrow$ no subexponential-time algorithm.

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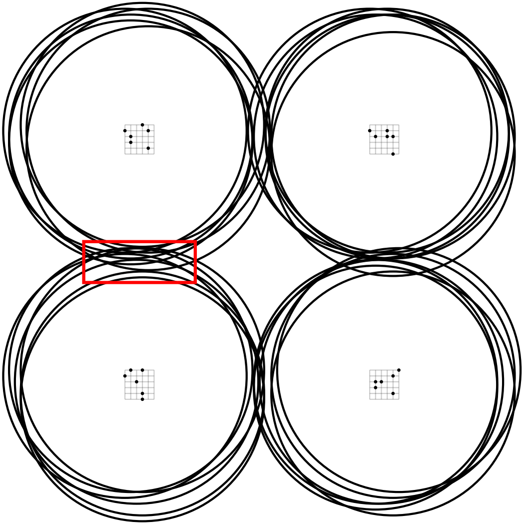
And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$.

Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks

Partial 2-grid Coloring → coloring unit disks



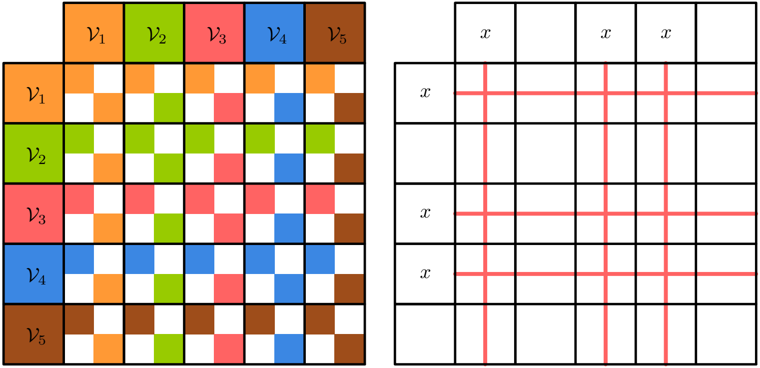
2-Grid 3-SAT

Input: A $g \times g$ grid, a positive integer k, each vertex (or cell) of the grid is associated to k variables, and a set $\mathcal{C}$ of constraints of two kinds:

- **clause constraints**: for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints**: for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

Question: Is there an assignment of the variables such that all constraints are satisfied?

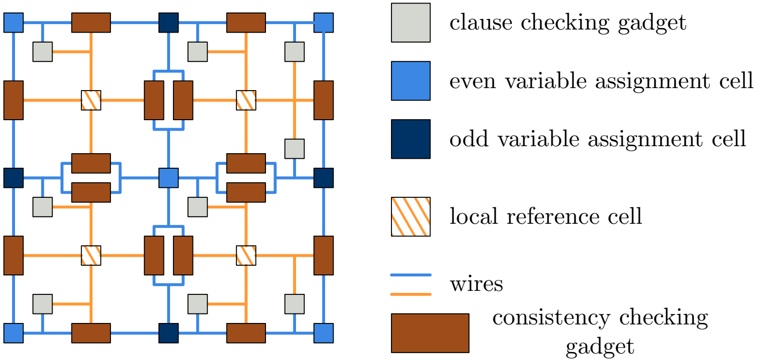
3-SAT $\to$ 2-Grid 3-SAT



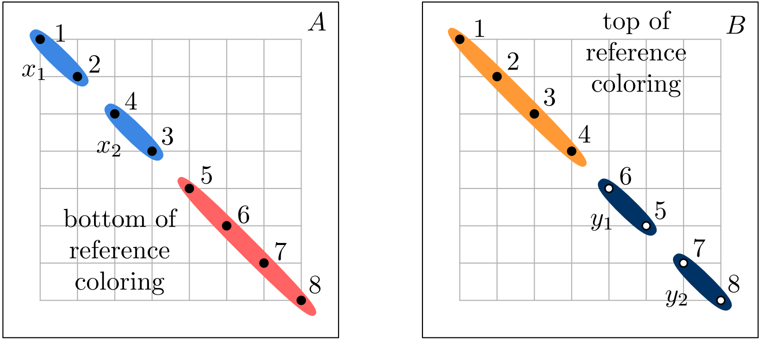
3-SAT on N variables with bounded number of occurrences (Sparsification Lemma) $\rightsquigarrow$ split the clauses into $\approx g$ blocks $\rightsquigarrow$ split again the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2 k$.
$N = \Theta(gk) = \Theta(\sqrt{nk})$

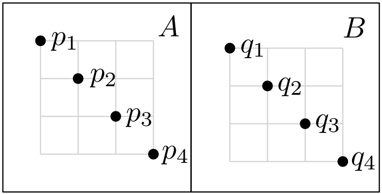
2-Grid 3-SAT → Partial 2-Grid Coloring

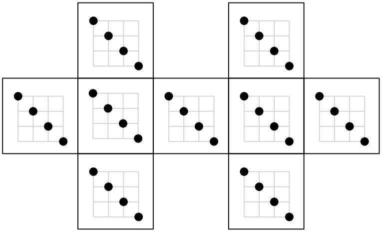


Encoding information and reference coloring

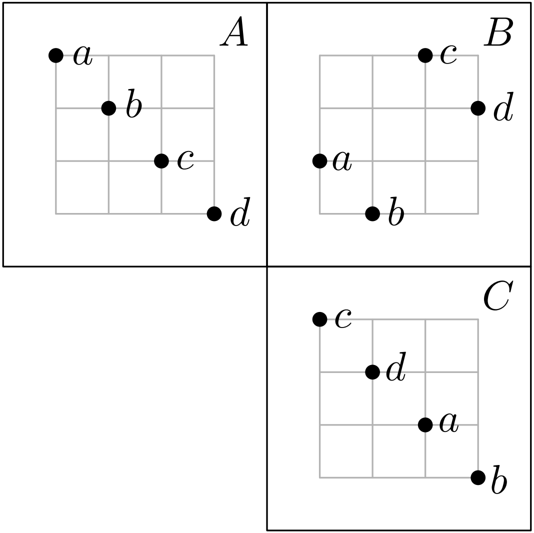


Wires

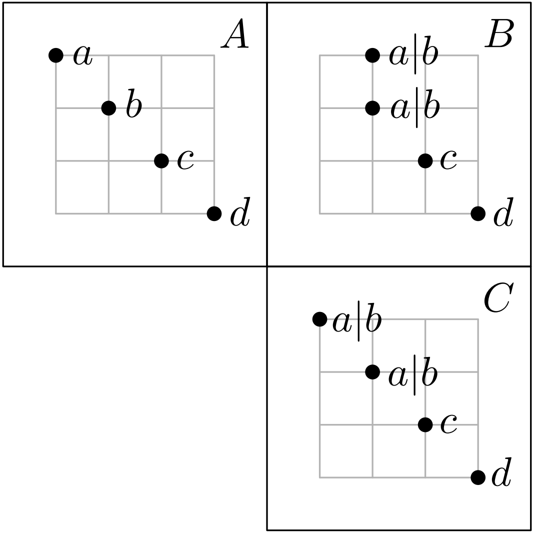




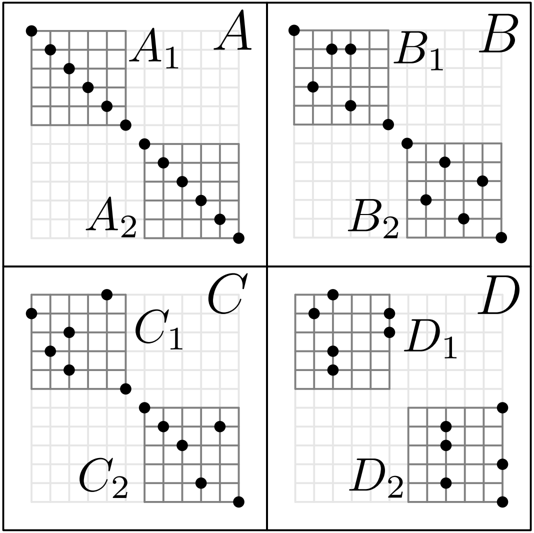
Permutation



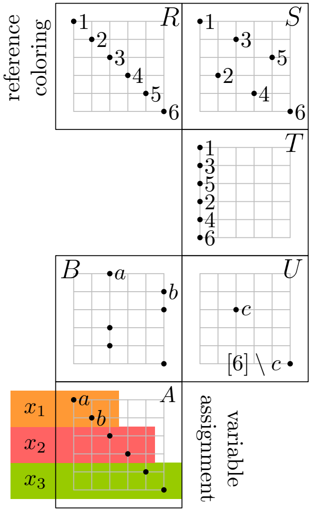
Forget

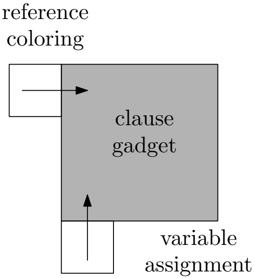


Independence

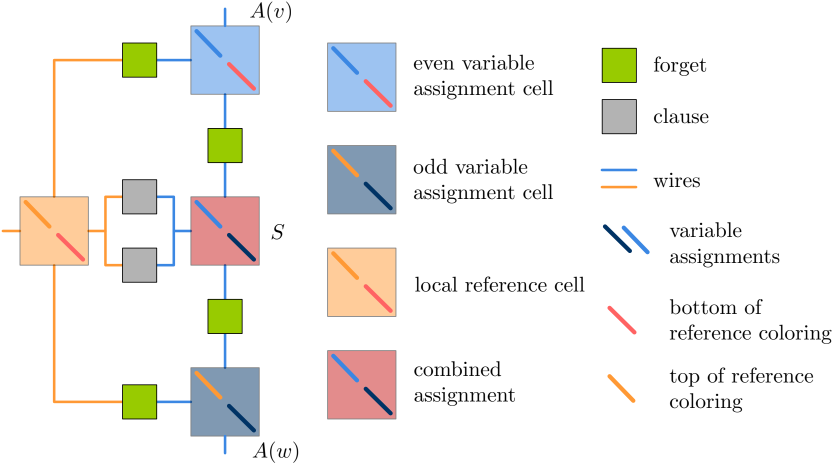


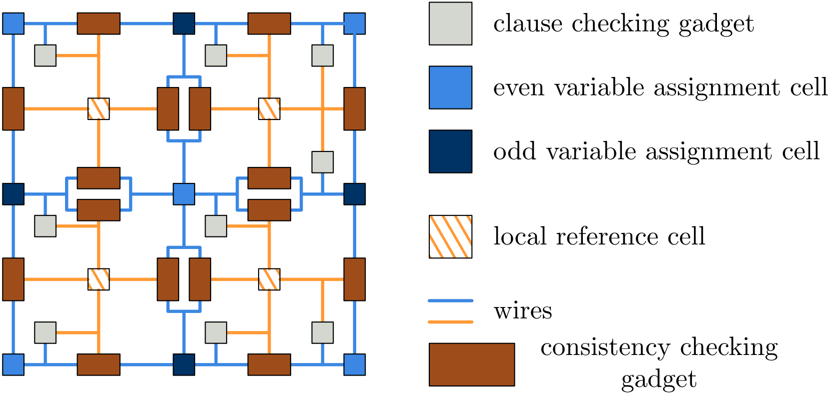
Clauses


reference coloring
R
1
2
3
4
5
6
S
1
3
5
2
4
6
T
1
3
5
2
4
6
B
a
b
U
c
[6] \ c
A
a
b
x_1
x_2
x_3
variable assignment


reference coloring
clause gadget
variable assignment

Consistency gadget (also crossing)




clause checking gadget
even variable assignment cell
odd variable assignment cell
local reference cell
wires
consistency checking gadget

Generalization to higher dimension

Theorem

For $\alpha \in [0,1]$ and dimension $d \geqslant 2$, coloring n unit d-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{n^{\frac{d-1+\alpha}{d}-\epsilon}} \approx 2^{o(n^{1-1/d}\ell^{1/d})}$ for any $\epsilon > 0$, under the ETH.

The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.

Generalization to other shapes

Smith and Wormald's result is more general.
The almost tight lower bound also generalizes.

Fatness of the family is crucial:

Theorem

4-coloring axis-parallel segment intersection graphs (2-Dir) is not solvable in $2^{o(n)}$, under the ETH.

Generalization to other shapes

Smith and Wormald's result is more general.
The almost tight lower bound also generalizes.

Fatness of the family is crucial:

Theorem

4-coloring axis-parallel segment intersection graphs (2-Dir) is not solvable in $2^{o(n)}$*, under the ETH.*

Thanks for your attention!