

Fine-grained complexity of coloring geometric intersection graphs

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Paris

NP-hardness vs ETH-hardness

NP-hardness:

- ▶ your problem is not solvable in polynomial, unless 3-SAT is
- ▶ very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$.

NP-hardness vs ETH-hardness

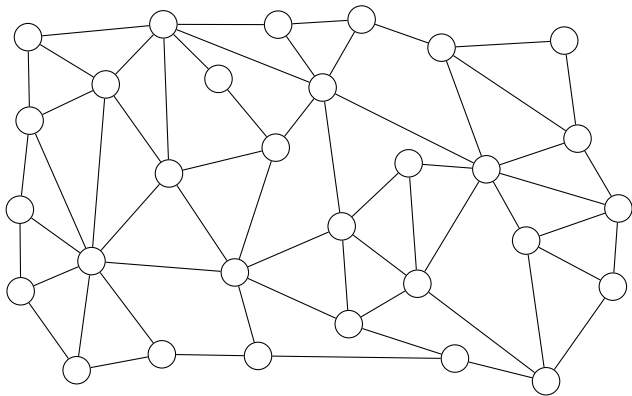
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ETH-hardness:

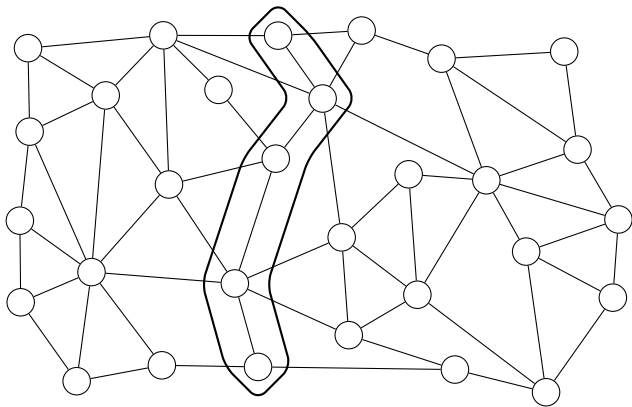
- ▶ stronger assumption than $P \neq NP$ is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT
- ▶ Allows to prove stronger conditional lower bounds
- ▶ linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.

Square root phenomenon on planar graphs



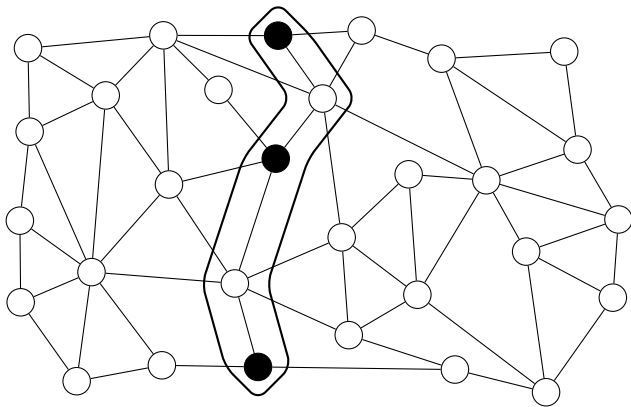
Many problems are solvable in $2^{O(\sqrt{n})}$ in **planar graphs**, and unlikely solvable in $2^{o(n)}$ in general graphs.

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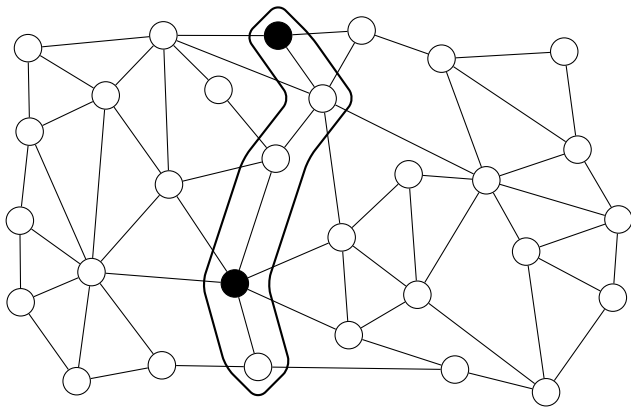
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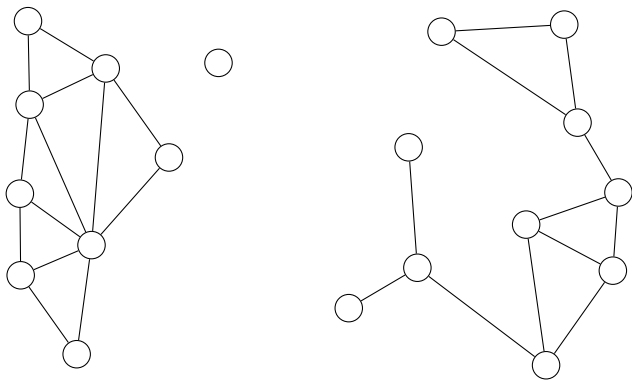
MAX INDEPENDENT SET, 3-COLORING, HAMILTONIAN PATH...

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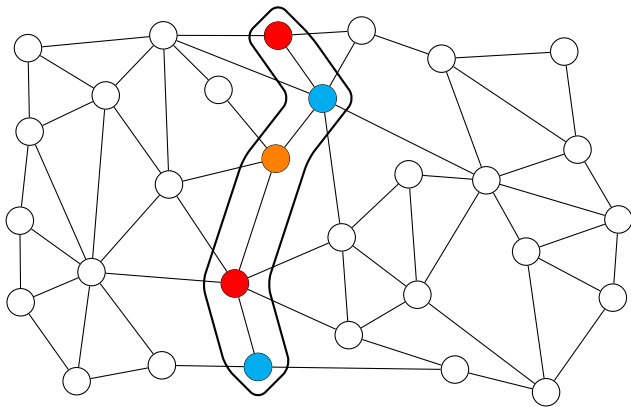
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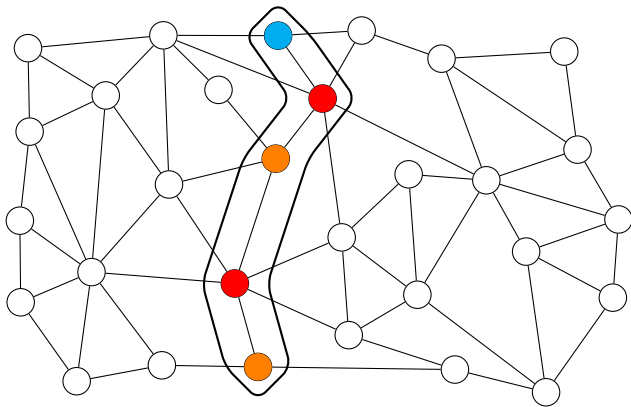
MAX INDEPENDENT SET, 3-COLORING, HAMILTONIAN PATH...
Dynamic programming would spare a $\log n$ in the exponent.

Square root phenomenon on planar graphs



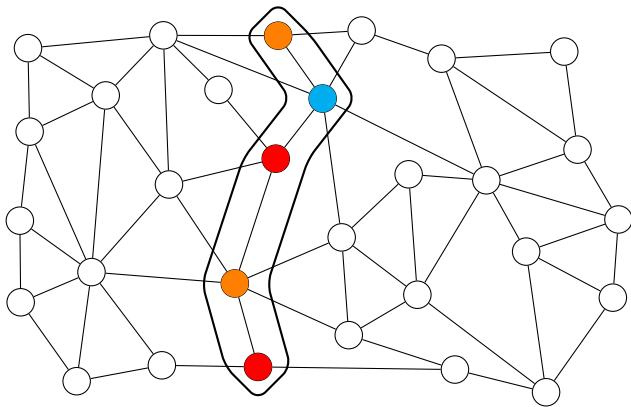
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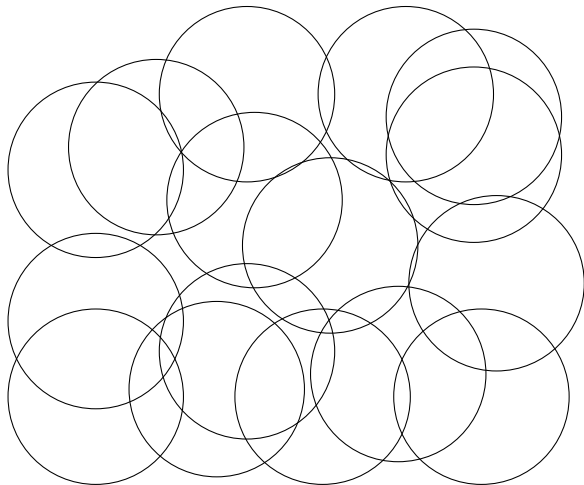
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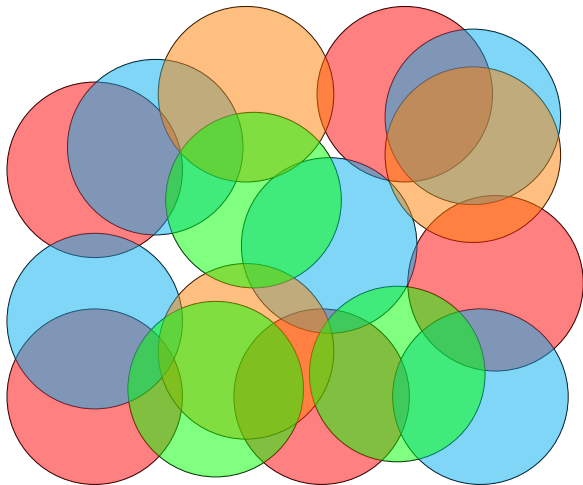
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Coloring Unit Disks



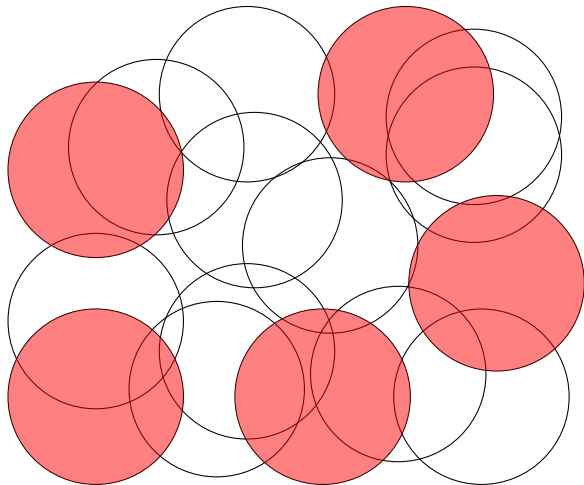
It might also be that only the intersection graph is given and not a geometric representation.

Coloring Unit Disks



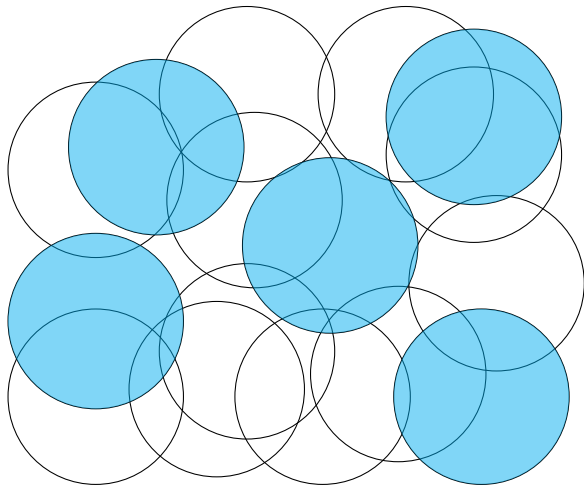
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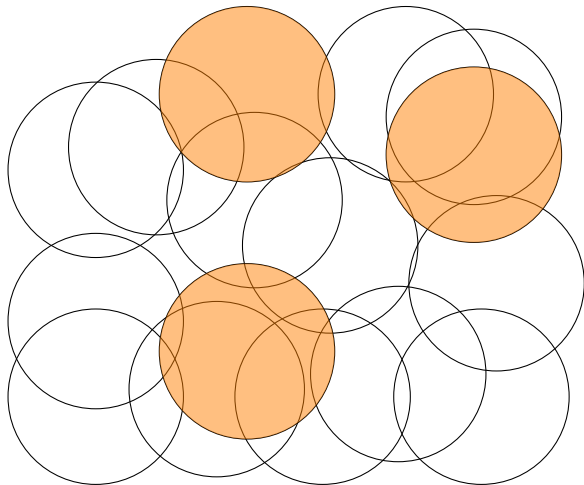
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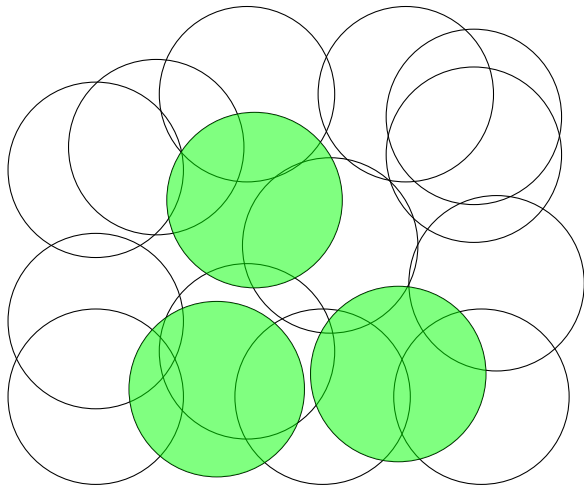
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Balanced separators

Theorem (Smith, Wormald '98)

For every $d \geq 1$ and $B \geq 0$, there exists a constant $c = c(d, B)$, such that for every B -fat collection S of n d -dimensional convex sets with ply at most ℓ , there exists a d -dimensional sphere Q , such that:

- ▶ *at most $\frac{d+1}{d+2}n$ elements of S are entirely inside Q ,*
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ply: maximum number of objects covering a point.

B -fat objects: aspect ratios diameter/width are bounded by B .



Balanced separators for unit disks

Theorem (Smith, Wormald '98, special case)

Given a collection S of n disks with ply at most ℓ , there exists a circle Q , such that:

- ▶ *at most $3n/4$ disks of S are entirely inside Q ,*
- ▶ *at most $3n/4$ disks of S are entirely outside Q ,*
- ▶ *at most $O(\sqrt{n\ell})$ disks of S intersect Q .*

Standard algorithm for ℓ -coloring (for unit disks)

If the ply is greater than ℓ , then more than ℓ colors are needed.

Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell} \log n})$.

Trying all the ℓ -colorings on S takes time $O(2^{\sqrt{n\ell} \log \ell})$.

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Overall running time: $O(2^{\sqrt{n\ell} \log n})$.

We will see that this running time is optimal up to logarithmic factors in the exponent.

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Theorem

For any $\alpha \in [0, 1]$, coloring n unit disks with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{o(n^{\frac{1+\alpha}{2}})} = 2^{o(\sqrt{n\ell})}$, under the ETH.

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Linear number of colors \rightsquigarrow no subexponential-time algorithm.

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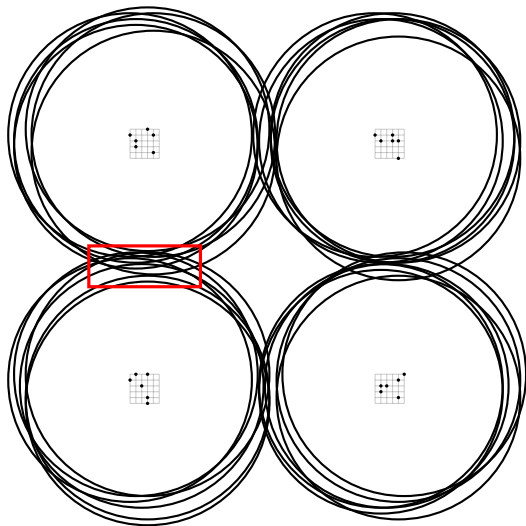
And everything in between (hard part).

For instance, \sqrt{n} -coloring cannot be done in $2^{o(n^{3/4})}$.

Roadmap

3-SAT \rightarrow 2-grid 3-SAT \rightarrow Partial 2-grid Coloring \rightarrow coloring unit disks

Partial 2-grid Coloring \rightarrow coloring unit disks



Partial 2-Grid Coloring

Input: An induced subgraph G of the $g \times g$ -grid, a positive integer ℓ . Each cell of this grid is mapped to a set of ℓ points (in a smaller grid $[\ell]^2$).

Question: Is there an ℓ -coloring of all the points such that:

- ▶ two points in the same cell get different colors;
- ▶ if v and w are adjacent in G , say, $w = v + (1, 0)$, p , resp. q , are points in the smaller grid of v resp. w , receiving the same color, then q has at a second coordinate which is at least the second coordinate of p ?

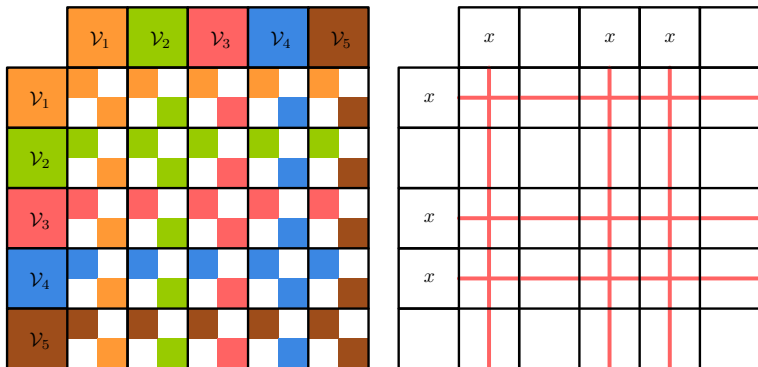
2-Grid 3-SAT

Input: A $g \times g$ grid, a positive integer k , each vertex (or cell) of the grid is associated to k variables, and a set \mathcal{C} of constraints of two kinds:

- ▶ **clause constraints:** for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- ▶ **equality constraints:** for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

Question: Is there an assignment of the variables such that all constraints are satisfied?

3-SAT \rightarrow 2-Grid 3-SAT

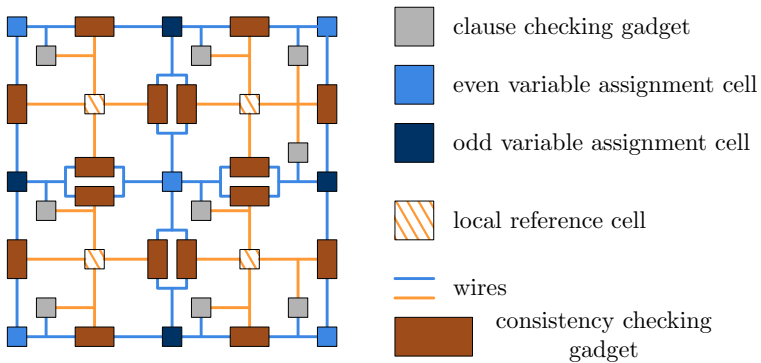


3-SAT on N variables with bounded number of occurrences (Sparsification Lemma) \rightsquigarrow split the clauses into $\approx g$ blocks \rightsquigarrow split again the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

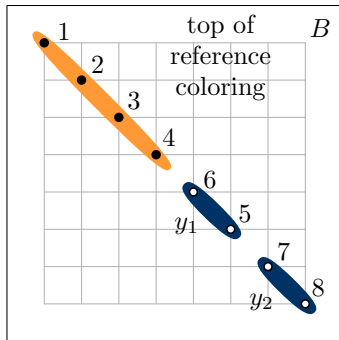
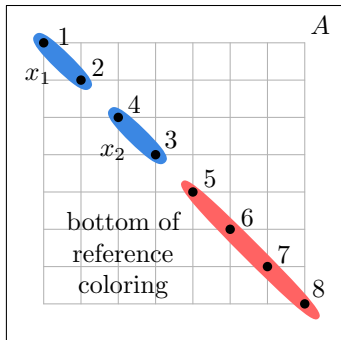
The size of the created instance is $n = g^2 k$.

$$N = \Theta(gk) = \Theta(\sqrt{nk})$$

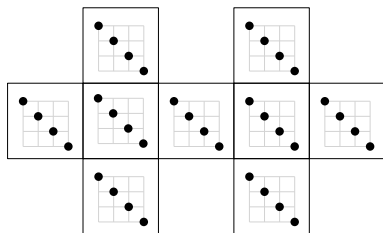
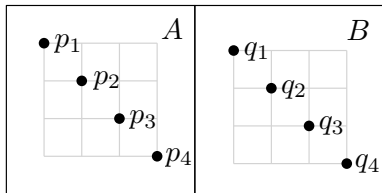
2-Grid 3-SAT \rightarrow Partial 2-Grid Coloring



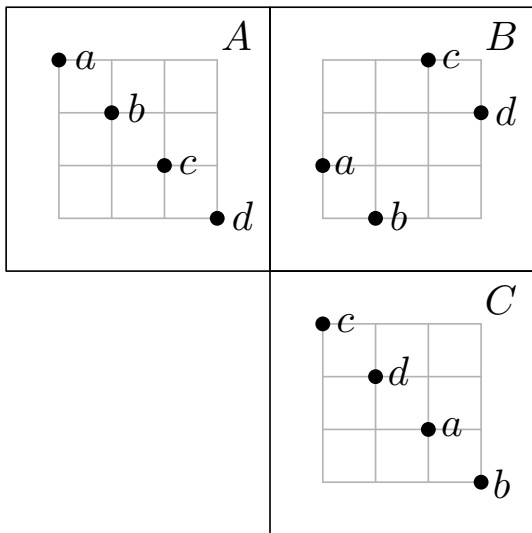
Encoding information and reference coloring



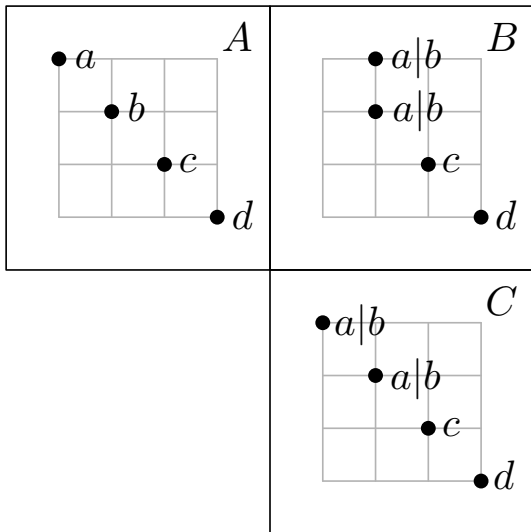
Wires



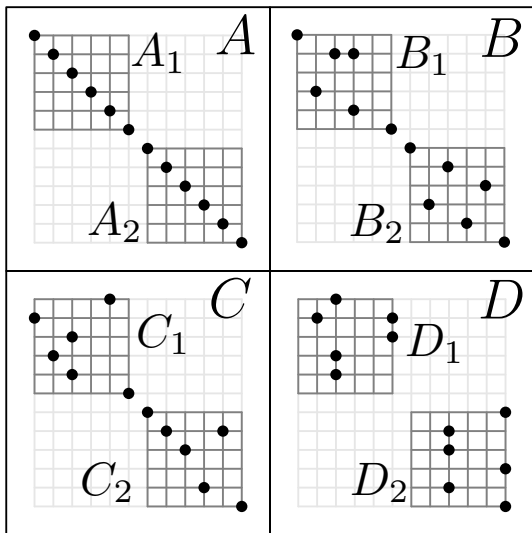
Permutation



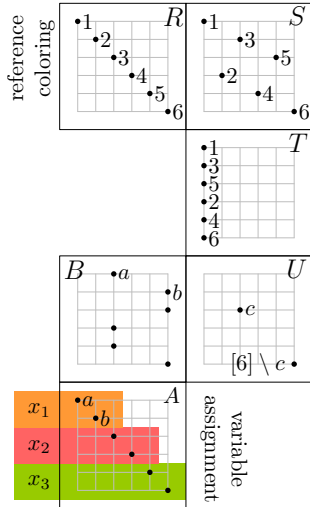
Forget



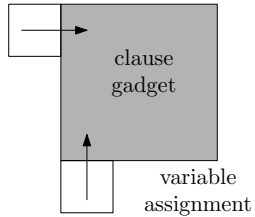
Independence



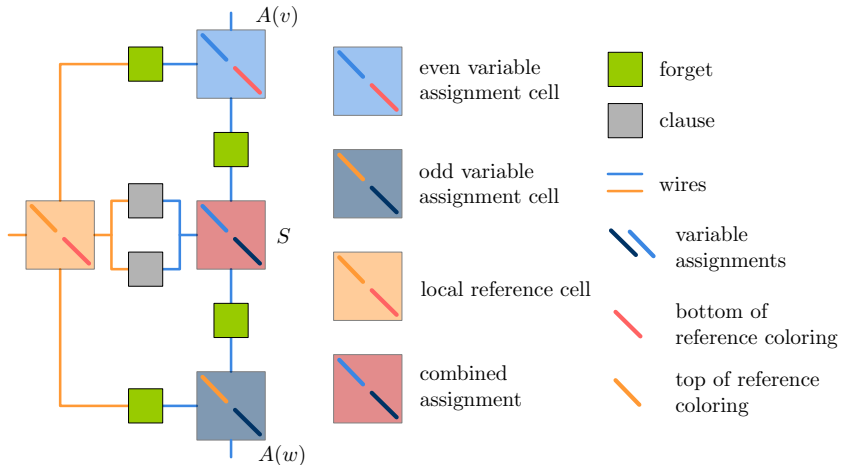
Clauses

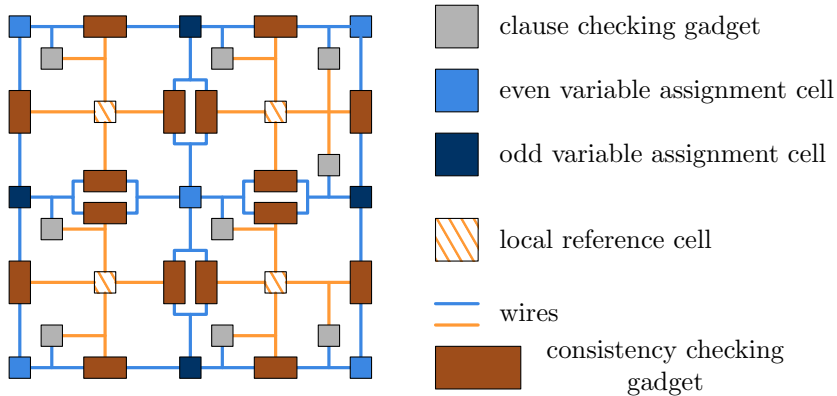


reference coloring



Consistency gadget (also crossing)





Higher dimension

Theorem

For $\alpha \in [0, 1]$ and dimension $d \geq 2$, coloring n unit d -balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{n^{\frac{d-1+\alpha}{d}-\epsilon}}$ for any $\epsilon > 0$, under the ETH.

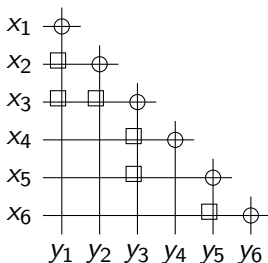
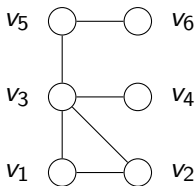
The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

The second and third steps work similarly.

(Longer and longer) Segments

Theorem

6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.

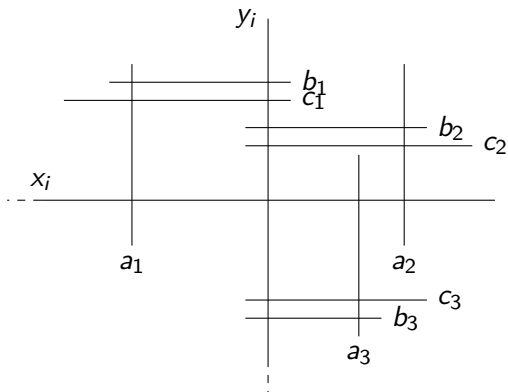


Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.

The x_i 's lists are $[1, 2, 3]$, the y_j 's lists are $[4, 5, 6]$.

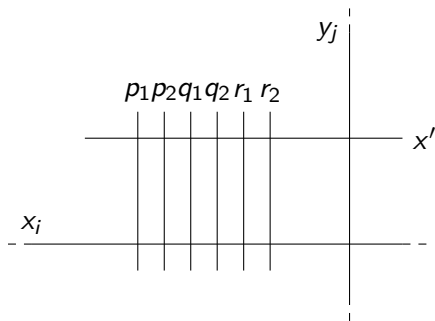
Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.

Equality



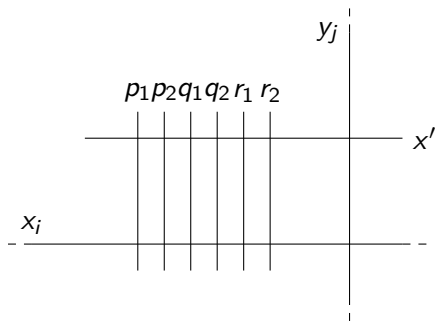
vertex	list
x_i	1,2,3
y_i	4,5,6
a_1	1,4
b_1	4,5
c_1	4,6
a_2	2,5
b_2	4,5
c_2	5,6
a_3	3,6
b_3	4,6
c_3	5,6

Inequality



vertex	list
x_j	1,2,3
y_j	4,5,6
x'	4,5,6
p_1	1,5
p_2	1,6
q_1	2,4
q_2	2,6
r_1	3,4
r_2	3,5

Inequality



vertex	list
x_j	1,2,3
y_j	4,5,6
x'	4,5,6
p_1	1,5
p_2	1,6
q_1	2,4
q_2	2,6
r_1	3,4
r_2	3,5

Some extra gadgets permit to remove the lists.

Same lower bound for 4 colors.

What happens with 3-colors? (whiteboard)

Thanks for your attention!