Fine-grained complexity of coloring geometric intersection graphs

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Joint works with Csaba Biró, Dániel Marx, Tillmann Miltzow, and Paweł Rzążewski and Stéphan Thomassé

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NP-hardness vs ETH-hardness

NP-hardness:

- your problem is not solvable in polynomial, unless 3-SAT is
- very widely believed but do not give evidence against algorithms running in say, $2^{n^{1/100}}$.

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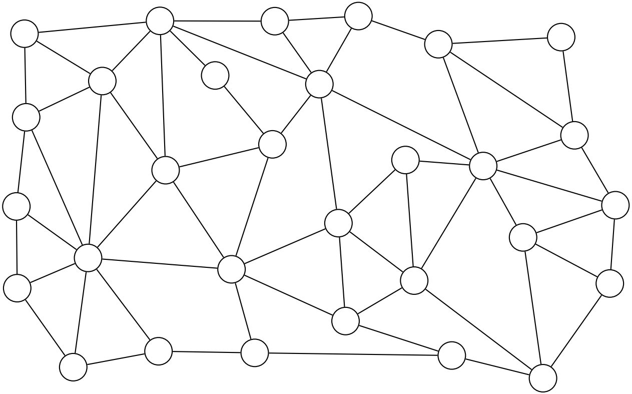
ETH-hardness:

- stronger assumption than P$\neq$NP is ETH asserting that no $2^{o(n)}$ algorithm exists for 3-SAT
- Allows to prove stronger conditional lower bounds
- linear reduction from 3-SAT: no $2^{o(n)}$ algorithm for your problem, quadratic reduction: no $2^{o(\sqrt{n})}$ algorithm, etc.

Square root phenomenon on planar graphs



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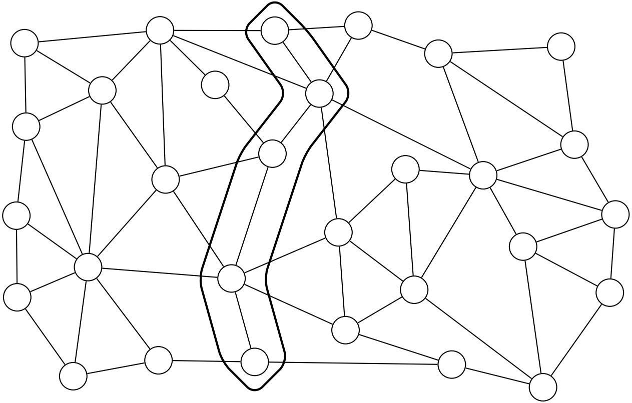

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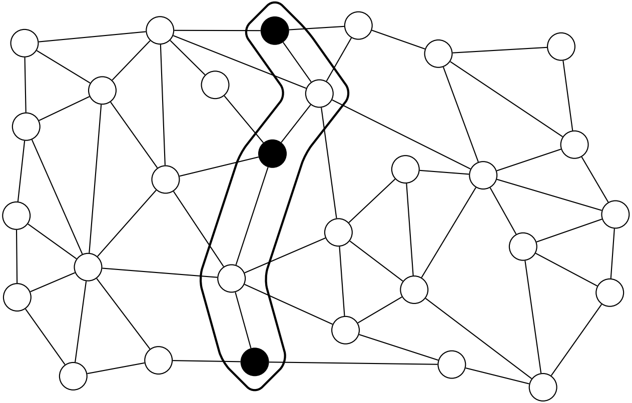

Max Independent Set, 3-Coloring, Hamiltonian Path...

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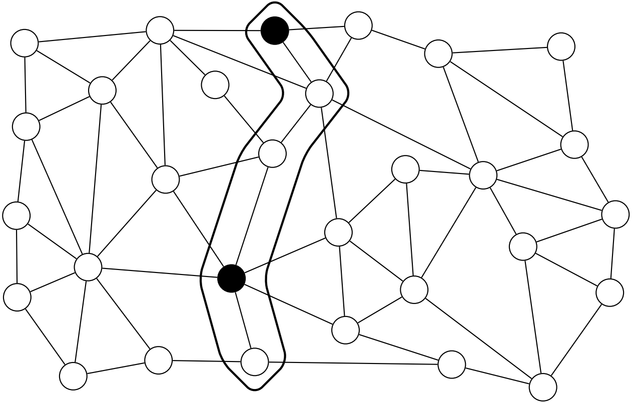

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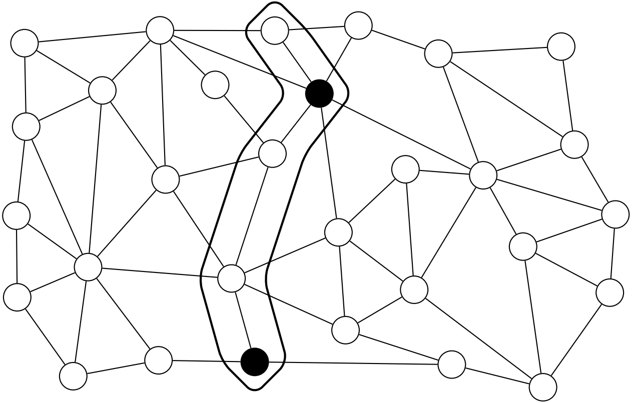

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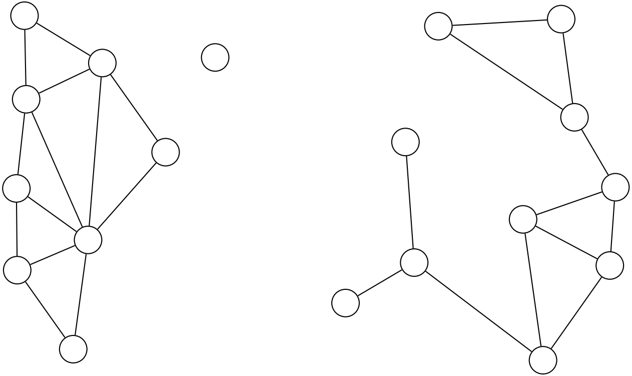

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Dynamic programming would spare a log n in the exponent.

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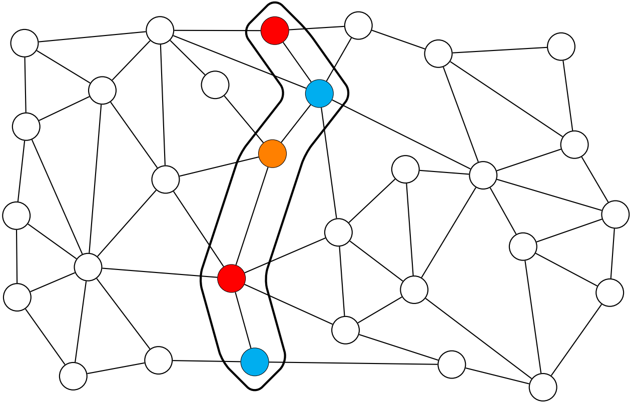

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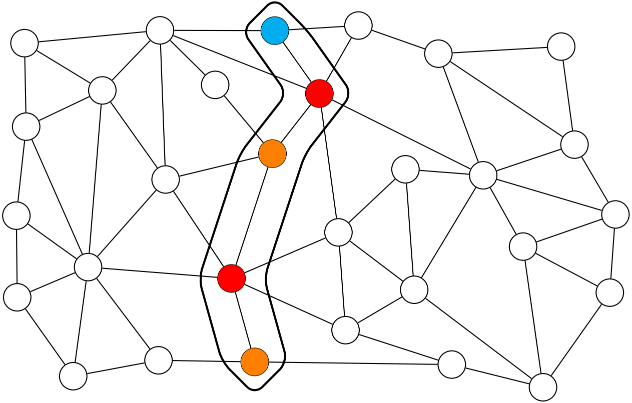

MAX INDEPENDENT SET, <u>3-COLORING</u>, HAMILTONIAN PATH...

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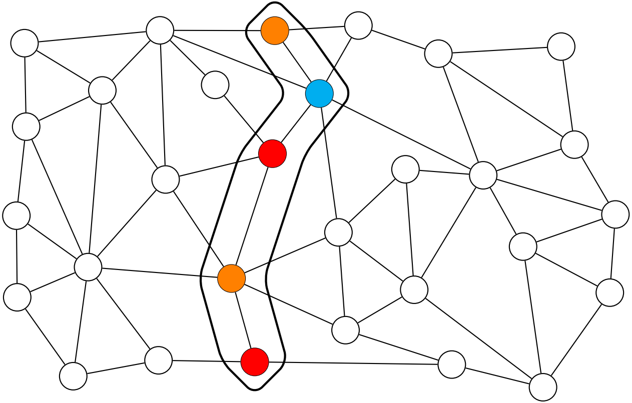
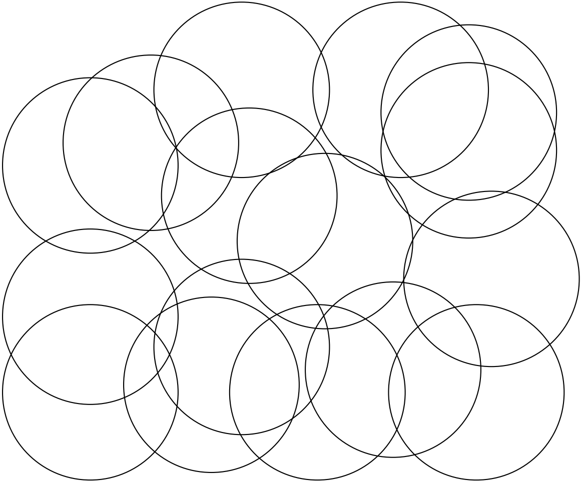


MAX INDEPENDENT SET, 3-COLORING, HAMILTONIAN PATH...

Coloring (Unit) Disks



It might also be that only the intersection graph is given and not a geometric representation.

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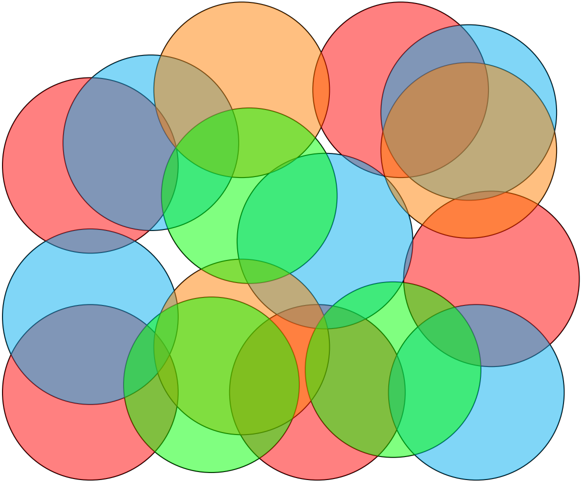

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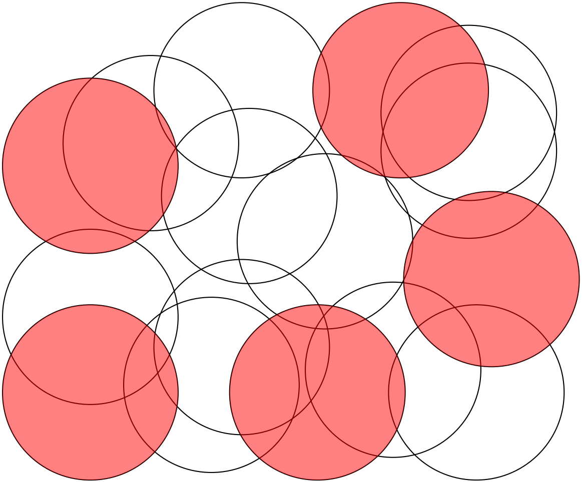

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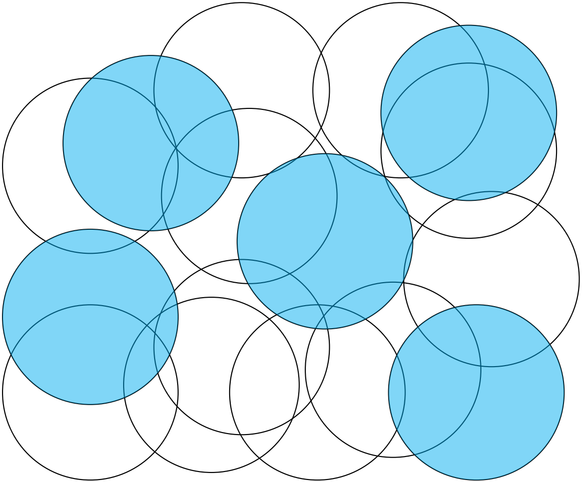

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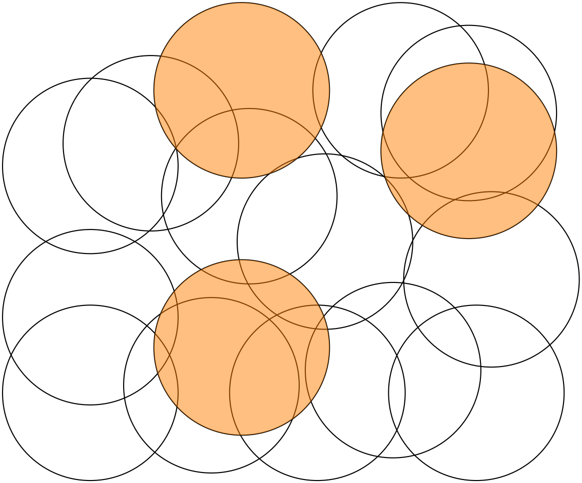

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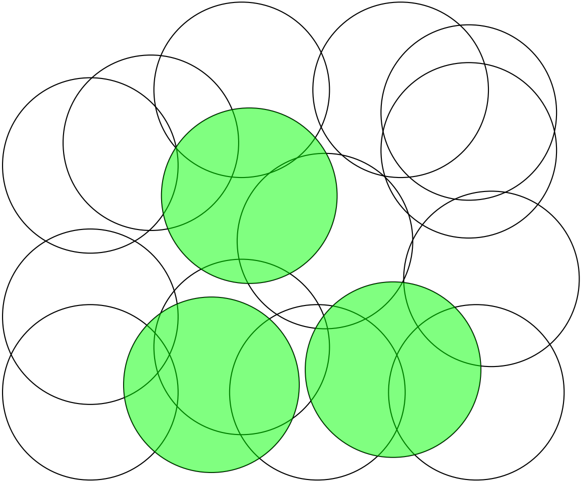

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Coloring Unit Disks

- Frequency assignment in broadcast networks
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For general graphs, the answer is yes: for any integer k,

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For planar graphs, only 3-Coloring is hard!

Balanced separators for unit disks

Theorem (Smith, Wormald '98, special case)

Given a collection $\mathcal{S}$ of n disks with ply at most ℓ, there exists a circle Q, such that:

- *at most $3n/4$ disks of $\mathcal{S}$ are entirely inside Q,*
- *at most $3n/4$ disks of $\mathcal{S}$ are entirely outside Q,*
- *at most $O(\sqrt{n\ell})$ disks of $\mathcal{S}$ intersect Q.*

Standard algorithm for ℓ-coloring (for unit disks)

If the ply is greater than ℓ, then more than ℓ colors are needed.

Otherwise, there is a balanced separator of size $O(\sqrt{n\ell})$ which can be exhaustively found in time $O(2^{\sqrt{n\ell}\log n})$.

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Theorem

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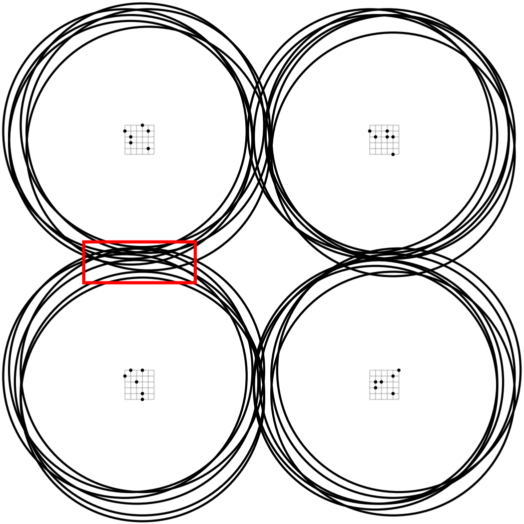
And everything in between (hard part).
For instance, $\sqrt{n}$-coloring cannot be done in $2^{o(n^{3/4})}$.

Roadmap

3-SAT $\rightarrow$ 2-grid 3-SAT $\rightarrow$ Partial 2-grid Coloring $\rightarrow$ coloring unit disks

Partial 2-grid Coloring → coloring unit disks



Partial 2-Grid Coloring

Input: An induced subgraph G of the $g \times g$-grid, a positive integer ℓ. Each cell of this grid is mapped to a set of ℓ points (in a smaller grid $[\ell]^2$).

Question: Is there an ℓ-coloring of all the points such that:

- two points in the same cell get different colors;
- if v and w are adjacent in G, say, $w = v + (1, 0)$, p, resp. q, are points in the smaller grid of v resp. w, receiving the same color, then q has at a second coordinate which is at least the second coordinate of p?

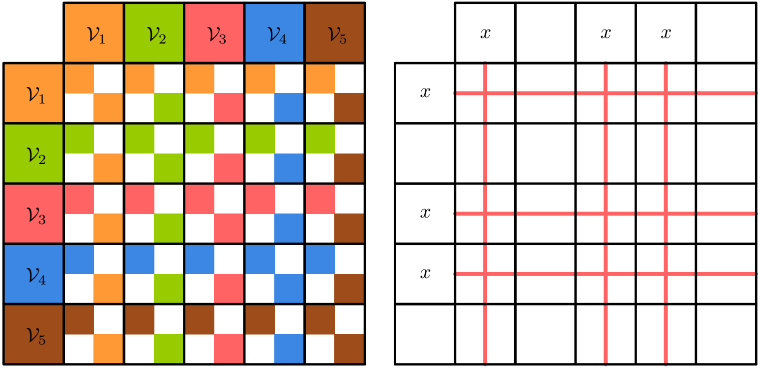
2-Grid 3-SAT

Input: A $g \times g$ grid, a positive integer k, each vertex (or cell) of the grid is associated to k variables, and a set $\mathcal{C}$ of constraints of two kinds:

- **clause constraints**: for each cell of the grid, a set of pairwise variable-disjoint 3-clauses on its variables;
- **equality constraints**: for two adjacent cells of the grid, a set of pairwise variable-disjoint equality constraints.

Question: Is there an assignment of the variables such that all constraints are satisfied?

3-SAT → 2-Grid 3-SAT

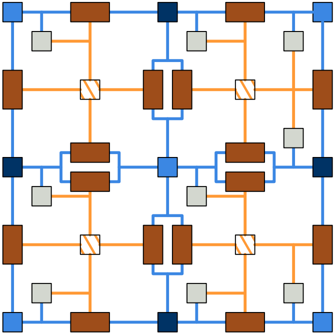


3-SAT on N variables with bounded number of occurrences (Sparsification Lemma) $\rightsquigarrow$ split the variables into $\approx k$ blocks $\rightsquigarrow$ split the clauses on one block into a constant number of sub-blocks (clauses vertex-disjoint)

The size of the created instance is $n = g^2 k$.

$$N = \Theta(gk) = \Theta(\sqrt{nk})$$

2-Grid 3-SAT → Partial 2-Grid Coloring



clause checking gadget

even variable assignment cell

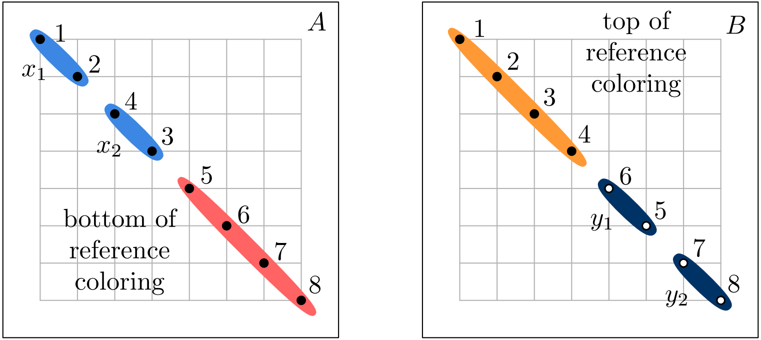
odd variable assignment cell

local reference cell

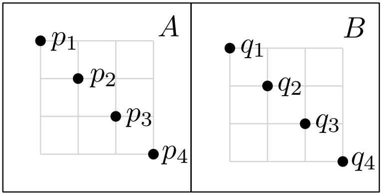
wires

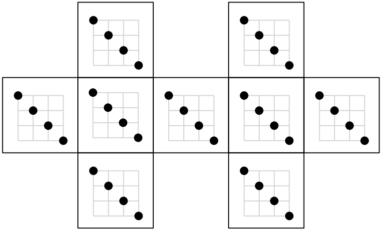
consistency checking gadget

Encoding information and reference coloring

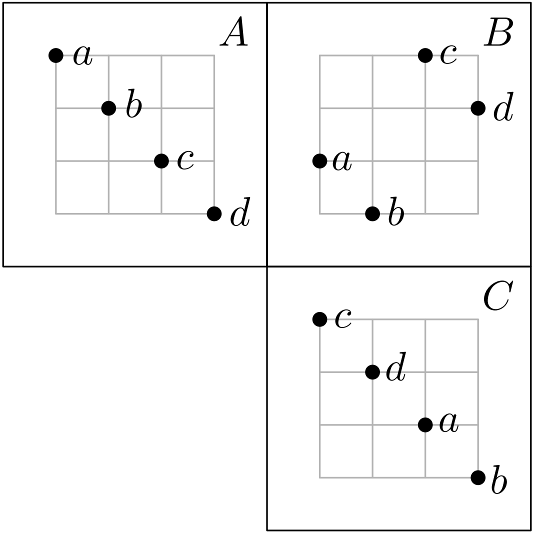


Wires

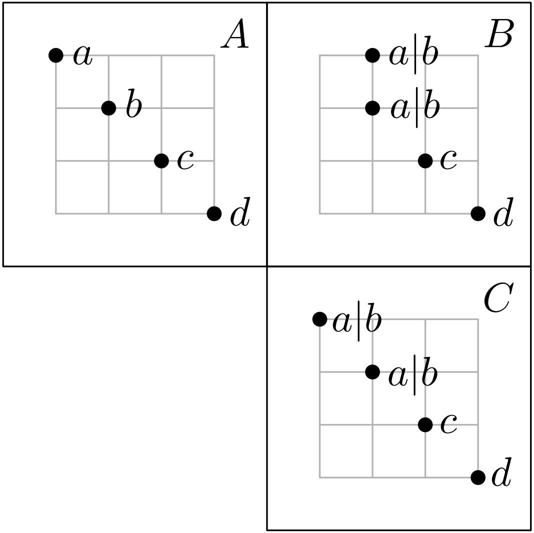




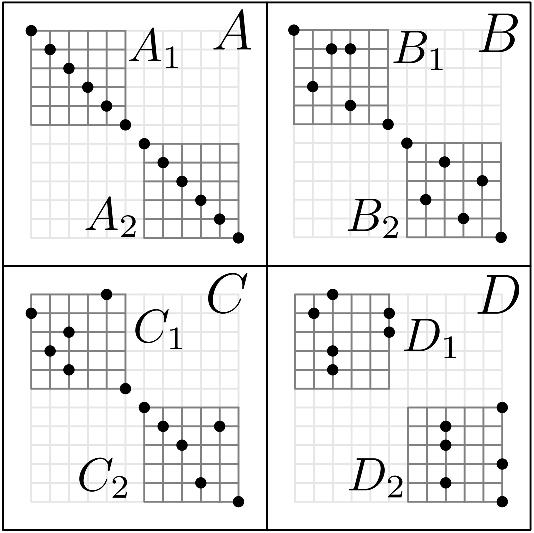
Permutation



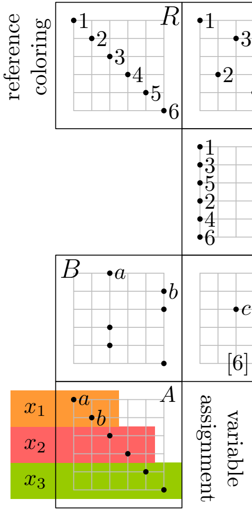
Forget



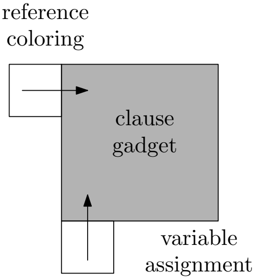
Independence



Clauses

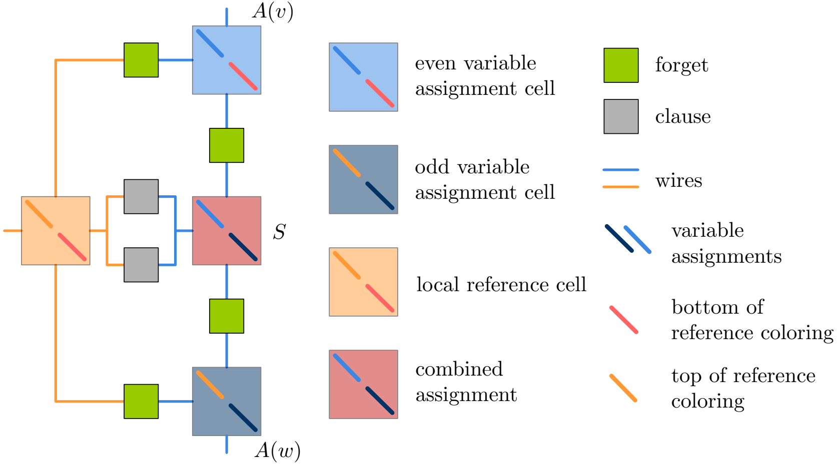


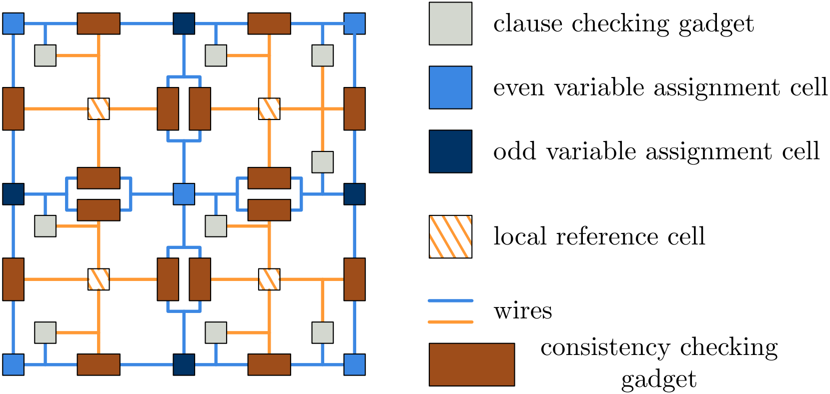
reference coloring
R
1
2
3
4
5
6
S
1
3
5
2
4
6
T
1
3
5
2
4
6
B
a
b
U
c
[6] \ c
A
a
b
x_1
x_2
x_3
variable assignment



reference coloring
clause gadget
variable assignment

Consistency gadget (also crossing)




clause checking gadget
even variable assignment cell
odd variable assignment cell
local reference cell
wires
consistency checking gadget

Higher dimension

Theorem

For $\alpha \in [0,1]$ and dimension $d \geqslant 2$, coloring n unit d-balls with $\ell = \Theta(n^\alpha)$ colors cannot be solved in time $2^{n^{\frac{d-1+\alpha}{d}-\epsilon}}$ for any $\epsilon > 0$, under the ETH.

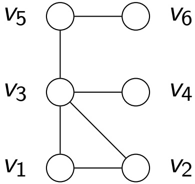
The first step in the chain is trickier: the higher dimensional grid should embed the SAT instance in a more compact way.

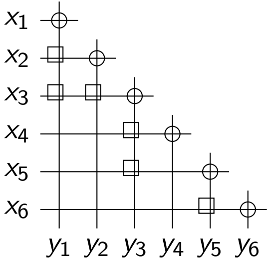
The second and third steps work similarly.

(Longer and longer) Segments

Theorem

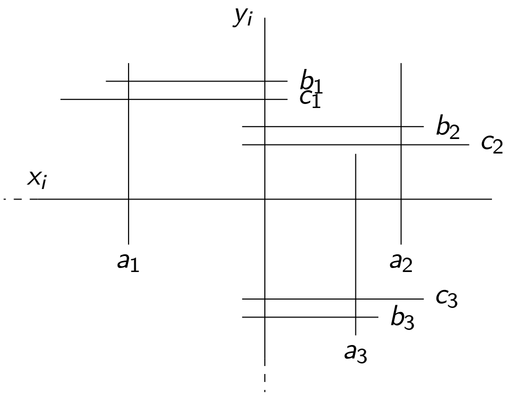
6-coloring 2-Dir is not solvable in $2^{o(n)}$, under the ETH.





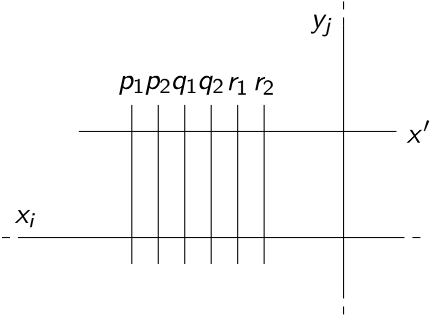
Reduction from 3-coloring on degree-4 graphs to list 6-coloring of segment intersection graphs.
The x_i's lists are $[1, 2, 3]$, the y_j's lists are $[4, 5, 6]$.
Circles are equality gadgets ($1 \equiv 4, 2 \equiv 5, 3 \equiv 6$), squares are inequality gadgets.

Equality



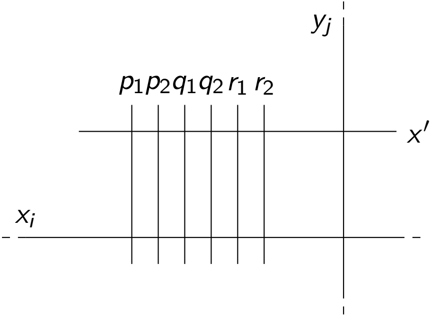
vertex	list
x_i	1,2,3
y_i	4,5,6
a_1	1,4
b_1	4,5
c_1	4,6
a_2	2,5
b_2	4,5
c_2	5,6
a_3	3,6
b_3	4,6
c_3	5,6

Inequality



vertex	list
x_i	1,2,3
y_j	4,5,6
x'	4,5,6
p_1	1,5
p_2	1,6
q_1	2,4
q_2	2,6
r_1	3,4
r_2	3,5

Inequality



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y_j	4,5,6
x'	4,5,6
p_1	1,5
p_2	1,6
q_1	2,4
q_2	2,6
r_1	3,4
r_2	3,5

Some extra gadgets permit to remove the lists.

Same lower bound for 4 colors.
What happens with 3-colors? (whiteboard)

Thanks for your attention!