A note on Candeal and Induráin's semiorder separability condition [1]

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Abstract

We show that the semiorder separability condition used by Candeal and Induráin in their characterization of semiorders having a strict representation with positive threshold can be factorized into two conditions. The first says that the trace of the semiorder must have a numerical representation. The second asserts that the number of "noses" in the semiorder must be finite or countably infinite. We discuss the interest of such a factorization.

Keywords: Utility theory, Semiorders, Separability, Noses.

1 Introduction

Although the idea of introducing a threshold into preference or perception models has distant origins (see Pirlot and Vincke, 1997 and Fishburn and Monjardet, 1992, for historical accounts of the idea) the formal definition of semiorders is due to Luce (1956). Shortly after, Scott and Suppes (1958) showed that a semiorder defined on a *finite* set always has a strict representation with positive threshold (see also Scott, 1964).

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Because the threshold used in the representation is constant and positive, it is clear that the result does not carry over to countably infinite sets (see, e.g., Fishburn, 1985, p. 30). This makes semiorders at variance with what happens with many other preference structures (e.g., weak orders, biorders, interval orders, suborders, see Aleskerov, Bouyssou, and Monjardet, 2007 and Bridges and Mehta, 1995) for which the finite and the countably infinite cases are identical.

The characterization of semiorders on countably infinite sets having a strict representation with positive threshold was achieved by Manders (1981) and Beja and Gilboa (1992). Basically, what is required is to prevent the existence of infinite (ascending or descending) chains of strict preference that are bounded.

Finally, building upon these results [1], Candeal and Induráin (2010) have achieved a general characterization of semiorders having a strict positive threshold representation by the addition of a condition called *semiorder separability* (see Candeal, Estevan, Gutiérrez García, and Induráin, 2012, Estevan, Gutiérrez García, and Induráin, 2013, for further results).

This note is part of a long-term project aiming at giving new proofs for the existence of numerical representations of semiorders that would unify the finite, countably infinite and general cases (Bouyssou and Pirlot, 2019a,b). Its purpose is to show that the semiorder separability condition proposed in Candeal and Induráin (2010) can be factorized into two conditions. The first says that the trace of the semiorder must have a numerical representation. This condition holds for strict as well as for nonstrict representations, as defined below. The second condition asserts that the number of "noses" in the semiorder must be finite or countably infinite. It is highly specific to strict representations. We discuss the interest of such a factorization.

The paper is organized as follows. Section 2 introduces our notation and framework. Section 3 presents our results. They are discussed in Section 4.

[1] Earlier results in the general case include Abrísqueta, Candeal, Induráin, and Zudaire (2009), Candeal, Induráin, and Zudaire (2002), Gensemer (1987a,b, 1988), Narens (1994).

2 Notation and framework

2.1 Notation and definitions

In the rest of this text, we say that a set is *denumerable* if it is finite or countably infinite.

Let S be a binary relation on a set X. We often write $x\ S\ y$ instead of $(x,y) \in S$. The relation S is a *semiorder* if it is complete ($x\ S\ y$ or $y\ S\ x$, for all $x,y \in X$), Ferrers ($x\ S\ y$ and $z\ S\ w$ imply $x\ S\ w$ or $z\ S\ y$, for all $x,y,z,w \in X$) and semi-transitive ($x\ S\ y$ and $y\ S\ z$ imply $x\ S\ w$ or $w\ S\ z$, for all $x,y,z,w \in X$). In the sequel, we shall often write the semiorder S as a pair (P,I) of relations, where P (resp. I) denotes the asymmetric (resp. symmetric) part of S. The asymmetric part of S is the relation P, often called the "strict preference" relation. It is a partial order on X, i.e., an asymmetric and transitive relation, which is also Ferrers and semitransitive. The symmetric part of S is the relation I, often called the "indifference" relation. It is reflexive and symmetric but not necessarily transitive. Because S is complete, notice that we could have alternatively defined a semiorder, giving its asymmetric part P, while letting I be the symmetric complement of P (i.e., $x\ I\ y$ iff $[Not[x\ P\ y]$ and $Not[y\ P\ x]])$ and $S = P \cup I$. We refer to Aleskerov et al. (2007), Fishburn (1985), Monjardet (1978), Pirlot and Vincke (1997), Roubens and Vincke (1985), Suppes, Krantz, Luce, and Tversky (1989) for detailed studies of the various properties of semiorders.

A *complete preorder* on X is a complete and transitive relation. A *linear* order (or *total* order) on X is a complete, antisymmetric and transitive relation.

The trace $\succsim_S$ of a semiorder S on X is the relation defined as follows: for all $x,y \in X$, $x \succsim_S y$ if for all $z \in X$, $y\ S\ z$ implies $x\ S\ z$ and $z\ S\ x$ implies $z\ S\ y$. We omit the subscript when there is no ambiguity on the underlying semiorder. It is easy to check that the trace $\succsim_S$ can be equivalently defined using P, i.e., $x \succsim_S y$ if for all $z \in X$, $y\ P\ z$ implies $x\ P\ z$ and $z\ P\ x$ implies $z\ P\ y$.

It is well-known (see, e.g., Aleskerov et al., 2007, Monjardet, 1978, Pirlot and Vincke, 1997) that the trace of a semiorder is a complete preorder (moreover, a semiorder is identical to its trace iff it is a complete preorder). We will use $\sim$, $\succ$, $\precsim$, and $\prec$, as is usual. Two elements $x,y \in X$ such that $x \sim y$ are said *equivalent*, i.e., for all $z \in X$, we have $z\ S\ x$ iff $z\ S\ y$ and $x\ S\ z$ iff $y\ S\ z$.

2.2 Chains

Let R be a binary relation on the set X. An R-chain (we use here the terminology used in the field of ordered sets, see Caspard, Leclerc, and Monjardet, 2012 or Schröder, 2016. Graph theorists may prefer the term "path") is a sequence x_i of elements of X indexed by a subset of consecutive integers $J \subseteq \mathbb{Z}$ and such that any two consecutive elements of the sequence belong to the relation R. Formally, the sequence $(x_i, x_i \in X, i \in J)$, where $J \subseteq \mathbb{Z}$ and $(x_i, x_{i+1}) \in R$, for all $i, i+1 \in J$ is an R-chain. We shall consider P-chains and I-chains in the sequel.

Note that an R-chain needs neither have a first nor a last element. In other terms, it can have an infinite number of elements before or after a given element, but not between two given elements. An R-chain is said to start at $x \in X$, if the set J has a minimum element and x is the element of X indexed by the minimal element in J. In this case, the chain is said to *have a first element*, which is this x. An R-chain is said to terminate at $y \in X$, if the set J has a maximum element and y is the element of X indexed by the maximal element in J. In this case, the chain is said to *have a last element*, which is this y. An R-chain starting at x and terminating at y (we also say "an R-chain from x to y") is finite by definition.

A P-chain $(x_i, i \in J)$ has an *upper* (resp. *lower*) *bound* if there exists $a \in X$ (resp. $b \in X$) such that $a \; P \; x_i$ (resp. $x_i \; P \; b$) for all $i \in J$. If the chain has both an upper bound a and a lower bound b, we say it is *bounded*.

Note that the set $\{x_i : i \in J\} \cup \{a, b\}$ is totally ordered by P, but cannot always be indexed by the elements of a subset J' of $\mathbb{Z}$. It cannot be when the P-chain $(x_i, i \in J)$ has no first or no last element. The elements of a finite subset of X which is totally ordered by P can be indexed by a set J of consecutive integers in order to form a P-chain. If a P-chain $(x_i, i \in J)$ has no last (resp. first) element, then for all $i \in J$, x_{i+k} (resp. x_{i-k}) belongs to the chain, for all $k \in \mathbb{N}$.

2.3 Representations with positive threshold

Let us make precise the definition of a strict representation with positive threshold.

Definition 1

A strict representation with positive threshold of a semiorder $S = (P, I)$ on the set X is a constant $k > 0$ and a function u from X to $\mathbb{R}$, such that, for *all* $x, y \in X$,

$$x \; P \; y \Leftrightarrow u(x) > u(y) + k,$$
$$x \; I \; y \Leftrightarrow -k \leq u(x) - u(y) \leq k. \tag{1}$$

A strict unit representation *of the semiorder* $S = (P, I)$ *is a strict positive threshold representation with* $k = 1$.

It is clear that a strict representation with positive threshold exists iff a strict unit representation exists. We focus on strict unit representations below.

Because we will be dealing with infinite sets, it is important to keep in mind that other kinds of numerical representation using a positive threshold can be envisaged. Let us simply mention here the case of a *nonstrict unit representation* in which, for all $x, y \in X$,

$$x \; P \; y \Leftrightarrow u(x) \geq u(y) + 1,$$
$$x \; I \; y \Leftrightarrow -1 < u(x) - u(y) < 1. \tag{2}$$

In the finite case, it is well know that strict and nonstrict unit representations are equivalent (see Pirlot and Vincke, 1997, Ch. 3 & 4 or Roberts, 1979, Ch. 6). The same is true in the countably infinite case (Beja and Gilboa, 1992, Th. 3.8, p. 436). In the general case, the two types of representations are distinct. For instance, the canonical nonstrict semiorder on $\mathbb{R}$ ($x \; P \; y \Leftrightarrow x \geq y + 1$, $x \; I \; y \Leftrightarrow |x - y| < 1$) has a trivial nonstrict unit representation but has no strict unit representation [2]. A similar phenomenon occurs with biorders (Doignon, Ducamp, and Falmagne, 1984) and, hence, interval orders (Fishburn, 1973, 1985).

We shall only consider strict unit representations that assign the same value to equivalent elements of X w.r.t. the trace $\succsim$. Consequently, we shall assume henceforth that the equivalence class of each element of X w.r.t. the trace of the semiorder is reduced to a singleton. In other words, for all $x, y \in X$, $x \succsim y$ and $y \succsim x$ imply $x = y$. Therefore, the trace $\succsim$ is a linear order on X. Its asymmetric part is denoted by $\succ$ and its symmetric part by $\sim$. This is not restrictive (see Candeal and Induráin, 2010, Lemma 3.2).

[2] Indeed, in this semiorder, for all $x \in \mathbb{R}$, the ordered pair $(x+1, x)$ is a nose, as defined in Definition 4. Hence, we have an uncountable number of noses, while the existence of a strict numerical representation implies that the number of noses must be finite or countably infinite. See Remark 2 below.

2.4 Axioms and previous results

2.4.1 Bounded P-chain condition

A necessary condition for the existence of a strict unit representation of a semiorder is the bounded P-chain condition. It says that every bounded P-chain is finite.

This condition was introduced by Manders (1981) and Beja and Gilboa (1992) under slightly different forms (we use here the version in Candeal and Induráin, 2010). It is simple to check (Manders, 1981, Prop. 8, p. 237) that if there is an I-chain joining any two elements of X, then the bounded P-chain condition holds. This condition is called *regularity* in Candeal and Induráin (2010).

The bounded P-chain condition has the flavor of an Archimedean axiom. It sounds like "Every bounded standard sequence is finite" (Krantz, Luce, Suppes, and Tversky, 1971, p. 25). Here, the sequence of pairs of objects in P plays a role that resembles that of equally spaced preference intervals used in standard sequences (see, in particular, the *strong standard sequences* defined in Gonzales, 2003, p. 51). Such properties are required for enabling representations using real numbers.

The bounded P-chain condition is clearly necessary for the existence of a strict unit representation as well as for a nonstrict unit representation. Consider, e.g., the case of strict representation. Suppose it has an infinite increasing chain $(x_i, x_i \in X, i \in J)$, indexed by the set of consecutive integers $\mathbb{N}$, such that $(x_{i+1}\ P\ x_i$, for all $i \in \mathbb{N}$ and such that $\omega\ P\ x_i$, for all $i \in \mathbb{N}$. This would imply

$$\omega\ P \ldots P\ x_{j+1}\ P\ x_j\ P \ldots P\ x_2\ P\ x_1$$

for all $j \in \mathbb{N}$, implying $u(x_j) > u(x_1) + j$, so that $u(\omega) - u(x_1) > n$, for all $n \in \mathbb{N}$. This is clearly impossible.

2.4.2 s-separability

Candeal and Induráin (2010) introduce the following condition that they call semiorder-separability, ("s-separability", for short).

Definition 2
A semiorder $S = (P, I)$ on X is semiorder-separable if there is a denumerable

set E, $E \subseteq X$, such that, for all $a, b \in X$ with $a\ P\ b$, there are

$$c \in E \text{ such that } a\ P\ c \succsim b \text{ and}$$
$$d \in E \text{ such that } a \succsim d\ P\ b$$

The fact that this condition is necessary for the existence of a strict unit representation is shown in Candeal and Induráin (2010, p. 488, 2nd col.)

2.4.3 The main result in Candeal and Induráin (2010)

The main result in Candeal and Induráin (2010) can be rephrased as follows.

Theorem 1 (Candeal and Induráin, 2010, Theorem 3.6)
A semiorder $S = (P, I)$ on a set X has a strict unit representation iff it satisfies the bounded P-chain condition and is s-separable.

Notice that the trace of an s-separable semiorder is Debreu-separable (see Candeal and Induráin, 2010, Lemma 3.4. We often say d-separable instead of Debreu-separable, for short). This is a condition guaranteeing the existence of a numerical representation of the trace (i.e., the existence of a function $v : X \to \mathbb{R}$ such that $x \succsim y$ iff $v(x) \geq v(y)$). We recall the definition of Debreu-separability below. We refer to Bridges and Mehta (1995) for a detailed analysis of this condition and several equivalent formulations found in the literature.

Definition 3
A semiorder $S = (P, I)$ is d-separable if its trace $\succsim$ is d-separable. The trace is d-separable if it has a denumerable order-dense set, i.e., there is a denumerable set D, $D \subseteq X$, such that, for all $a, b \in X$ with $a \succ b$, there is $d \in D$, such that $a \succsim d \succsim b$.

3 Results

We revisit the s-separability condition. Our aim is to factorize it into d-separability and another condition. The latter is expressed in terms of the *noses* of the semiorder.

3.1 Noses and half-noses

The notion of "nose" of a semiorder has been introduced in Pirlot (1990, 1991). When X is finite, it is instrumental to build synthetic representations of a semiorder (Pirlot, 1991) as well as proving that it has a minimal representation and building it (Pirlot, 1990). It was shown in Doignon and Falmagne (1997) that the noses of a semiorder are exactly the ordered pairs of the *inner fringe* of the semiorder $S = (P, I)$. The inner fringe of semiorder consists in the set of ordered pairs belonging to P that can be removed from P and turned into I, while remaining in the set of semiorders.

Definition 4
The ordered pair $(a, b) \in X^2$ is a nose *of the semiorder $S = (P, I)$ if $a\ P\ b$ and there is no $c \in X$ such that $a\ P\ c \succ b$ and there is no $d \in X$ such that $a \succ d\ P\ b$.*

Noses play a special role w.r.t. s-separability as shown by the following lemma.

Lemma 1
If the semiorder $S = (P, I)$ on X (for which $\succsim$ is antisymmetric) is s-separable by the denumerable set E, then a and b belong to E whenever (a, b) is a nose.

Proof
Let (a, b) be a nose, so that $a\ P\ b$. By the s-separability property, there is $c \in E$ such that $a\ P\ c \succsim b$. By definition of a nose, we have $c = b$ and therefore, $b \in E$. Using s-separability, there is also $d \in E$ such that $a \succsim d\ P\ b$, which implies $a = d$ and $a \in E$ since (a, b) is a nose. $\square$

Remark 1
For later use, let us observe that if (a, b) is a nose then there cannot exist a nose (a, c) with $b \neq c$. Indeed, if $b \succ c$, we have aPc, $b \neq c$ and aPb, violating the definition of a nose. Hence if a is the left endpoint of a nose there is unique right endpoint b associated to it, so that (a, b) is a nose. A similar observation holds in the opposite direction: if b is the right endpoint of a nose there is unique left endpoint a associated to it, so that (a, b) is a nose. $\bullet$

We will also need to care about *half-noses*, as defined below.

Definition 5
The ordered pair $(a, b) \in X^2$ is a lower half-nose *(l-h-nose) of the semiorder $S = (P, I)$ if $a\ P\ b$ and there is no $c \in X$ such that $a\ P\ c \succ b$. The ordered pair (a, b) is a* proper l-h-nose *if it is an l-h-nose that is not a nose, i.e., there is $d \in X$ such that $a \succ d\ P\ b$.*

The ordered pair $(a, b) \in X^2$ is an upper half-nose *(u-h-nose) of the semiorder $S = (P, I)$ if $a\ P\ b$ and there is no $d \in X$ such that $a \succ d\ P\ b$. The pair (a, b) is a* proper u-h-nose *if it is a u-h-nose that is not a nose, i.e., there is $c \in X$ such that $a\ P\ c \succ b$.*

We denote by $\mathcal{N}_{plh}$ (resp. $\mathcal{N}_{puh}$) the set of right endpoints b (resp. left endpoints a) of proper l-h-noses (resp. u-h-noses) (a, b).

We have the following result.

Lemma 2
If the semiorder $S = (P, I)$ is d-separable, then the sets $\mathcal{N}_{plh}$ and $\mathcal{N}_{puh}$ are denumerable.

Proof
We give the proof for $\mathcal{N}_{plh}$, the case of $\mathcal{N}_{puh}$ being similar.

Let (a, b) be a proper l-h-nose. Hence, we know that:
i) $a\ P\ b$,
ii) there is no $c \in X$ such that $a\ P\ c$ and $c \succ b$,
iii) there is a $d \in X$ such that $a \succ d$ and $d\ P\ b$.

We define the set $N(b) = \{x \in X : x\ P\ b$, and there is no $c \in X$ such that $x\ P\ c$ and $c \succ b\}$. In other words, for all $x \in N(b)$, (x, b) is a l-h-nose.

By hypothesis, $a \in N(b)$. We know that there is an element $d \in X$ such that $a \succ d\ P\ b$. We claim that d belongs to $N(b)$. Indeed, we have $d\ P\ b$. Suppose that there is $c \in X$ such that $d\ P\ c$ and $c \succ b$. Since $a \succ d$ and $d\ P\ c$, we obtain $a\ P\ c$. So that we have $a\ P\ c$ and $c \succ b$, a contradiction. Hence, the claim is proved and $N(b)$ contains at least two elements a and d.

We claim that $N(b)$ is an interval w.r.t. $\succ$. To prove this, let $x, y \in N(b)$. Take any z such that $x \succ z \succ y$. Let us show that we have $z \in N(b)$, which will prove the claim. Because, $z \succ y$ and $y\ P\ b$, we have $z\ P\ b$. Suppose that there is $c \in X$ such that $z\ P\ c$ and $c \succ b$. Because $x \succ z$, $z\ P\ c$ implies $x\ P\ c$. Hence, we have $x\ P\ c$ and $c \succ b$, contradicting the fact that $x \in \mathcal{N}_{plh}$. Hence, for all $b \in \mathcal{N}_{plh}$, $N(b)$ is a nondegenerate interval of $\succsim$.

Now, let (a, b) and (e, f) be two proper l-h-noses, so that $b, f \in \mathcal{N}_{plh}$. Suppose that $b \neq f$, We claim that the associated intervals $N(b)$ and $N(f)$

are disjoint. Indeed, $x \in N(b)$ implies $x \; P \; b$, and there is no $c \in X$ such that $x \; P \; c$ and $c \succ b$. Similarly, $x \in N(f)$ implies $x \; P \; f$, and there is no $d \in X$ such that $x \; P \; d$ and $d \succ f$. Because, $b \neq f$ we have either $b \succ f$ or $f \succ b$. Suppose that $f \succ b$, the other case being similar. We have $x \; P \; f$ and $f \succ b$, violating the fact that $x \in N(b)$.

Now each of these intervals contains at least two distinct points and therefore at least an element from the denumerable set D that d-separates $S = (P, I)$. Consequently, the set $\mathcal{N}_{plh}$ is denumerable. $\square$

3.2 Main results

We are now in position to propose the announced factorization for the s-separability condition.

Proposition 1
A semiorder $S = (P, I)$ on X is s-separable iff it is d-separable and its set of noses is denumerable.

Proof
[$\Rightarrow$] By Lemma 1, the set of noses is denumerable. The s-separability property implies that $\succsim$ is d-separable (see Candeal and Induráin, 2010, Lemma 3.4). We include the proof for completeness. Let $x, y \in X$ be such that $x \succ y$. There is $z \in X$ such that $x \; P \; z$ and $z \; S \; y$ and/or $w \in X$ such that $w \; P \; y$ and $x \; S \; w$. In the former case, s-separability entails that there is $d \in E$ such that $x \succsim d \; P \; z$ and, since $z \; S \; y$, we have $x \succsim d \succ y$. In the latter case, there is $c \in E$ such that $w \; P \; c \succsim y$ and, since $x \; S \; w$, we have $x \succ c \succsim y$.

[$\Leftarrow$] Let D be a denumerable set that d-separates $\succsim$. Let $x, y \in X$ be such that $x \; P \; y$. If (x, y) is not a nose, there are two cases:

1. either there is $y' \succ y$ such that $x \; P \; y'$,

2. or there is $x' \prec x$ such that $x' \; P \; y$.

In the first case, by the d-separability of $\succ$, there is $c \in D$ such that $y' \succsim c \succsim y$. Therefore we have $x \; P \; c \succsim y$. Further, either there is $x' \prec x$ such that $x' \; P \; y$ or, for all $x' \prec x$, we have $Not[x' \; P \; y]$. In the former case, d-separability implies that there is $d \in D$ such that $x' \precsim d \precsim x$. Otherwise, (x, y) is a proper u-h-nose. In order to have $d \in E$ such that $x \succsim d \; P \; y$, we set $d = x$ and include, using Lemma 2, the denumerable set $\mathcal{N}_{puh}$ of left endpoints of the proper u-h-noses in E.

In the second case, by the d-separability of $\succ$, there is $d \in D$ such that $x' \precsim d \precsim x$. Therefore we have $x \succsim d\ P\ y$. Further, there are two cases. Either there is $y' \succ y$ such that $x\ P\ y'$ or, for all $y' \succ y$, we have $Not[x\ P\ y']$. In the former case, d-separability implies that there is $c \in D$ such that $y' \succsim c \succsim y$. Then, we have $x\ P\ c \succsim y$. Otherwise, (x, y) is a proper l-h-nose. In order to have $c \in E$ such that $x\ P\ c \succsim y$, we set $c = y$ and include, using Lemma 2, the denumerable set $\mathcal{N}_{plh}$ of right endpoints of the proper l-h-noses in E.

Finally, by considering E as the union of D, $\mathcal{N}_{plh}$, $\mathcal{N}_{puh}$ and the set of elements a, b such that (a, b) is a nose, which is denumerable by hypothesis, we obtain a denumerable set E, which s-separates the semiorder (P, I). $\square$

Remark 2

It is easy to show that having a denumerable set of noses is a necessary condition for a semiorder to have a strict unit representation. Indeed, assume that f is a unit representation of the semiorder $S = (P, I)$ (and, consequently is a numerical representation of $\succsim$, since we have supposed $\succsim$ to be antisymmetric). Suppose that (a, b) is a nose of S. Since $a\ P\ b$, we have $f(a) > f(b) + 1$. Let ε_{ab} be the positive number $f(a) - f(b) - 1$. By definition of a nose, there is no element $c \neq b$ such that $a\ P\ c \succ b$ and therefore, there is no c such that $f(c) \in (f(b), f(a) - 1]$, an interval of length $\varepsilon_{ab} > 0$. To each nose (a, b) is associated such an interval of positive length and all these intervals are disjoint. Since there is only a denumerable number of disjoint intervals of positive length in $\mathbb{R}$, the number of noses is denumerable. $\bullet$

Combining Remark 2 and Proposition 1 with Theorem 1 leads to our main result.

Theorem 2

A semiorder $S = (P, I)$ on a set X has a strict unit representation iff it satisfies the bounded P-chain condition, is d-separable and has a set of noses that is denumerable. These three conditions are independent.

PROOF

The first part immediately follows from Remark 2 and Proposition 1, in view of Theorem 1. To prove the second part, we need three examples.

Example 1
Let $X = \mathbb{R}^2$. Consider the binary relation S such that $S = P \cup I$ with $(x_1, x_2)\ P\ (y_1, y_2)$ if $x_1 > y_1 + 1$ or $[x_1 = y_1 + 1$ and $x_2 > y_2]$, while I is the symmetric complement of P (i.e., $x\ I\ y \Leftrightarrow Not[x\ P\ y]$ and $Not[y\ P\ x]$).

It is not difficult to show that S is a semiorder (that is not a complete preorder). It is clear that for all $x, y \in X$, there is an I-chain joining them, so that the bounded P-chain condition holds. The set of noses of S is easily seen to be empty. The trace of S is the lexicographic preorder on $\mathbb{R}^2$. Hence, d-separability is violated (see Beardon, Candeal, Herden, Induráin, and Mehta, 2002, Bridges and Mehta, 1995). $\diamond$

Example 2
Let $X = \mathbb{R}$. We consider the binary relation S on X such that $x\ S\ y \Leftrightarrow x \geq y + 1$. It is clear that this relation is a semiorder. For all $x, y \in X$, there is an I-chain joining them, so that the bounded P-chain condition holds. The trace $\succsim$ of S is $\geq$, so that S is d-separable. All ordered pairs $(x, y) \in \mathbb{R}^2$ such that $x = y + 1$ are noses. $\diamond$

Example 3
Let $X = \mathbb{N} \cup \{\omega\}$. Consider the binary relation S such that $\omega\ P\ x$, for all $x \in \mathbb{N}$ and $x\ P\ y$ iff $x > y + 1$, for all $x, y \in \mathbb{N}$, while I is the symmetric complement of P. Since X is denumerable, d-separability and the condition on noses trivially hold. The bounded P-chain condition is violated. $\diamond$

$\square$

Remark 3
In Bouyssou and Pirlot (2019b), for proving the existence of a numerical representation, we use d-separability and the condition that the number of noses is denumerable, instead of s-separability. In this proof, we only use the denumerable set D that is dense in the trace $\succsim$ and the denumerable set of noses endpoints. We do not need to add the proper half-noses as in the proof of Proposition 1. In other words, we do not use all the points in the set E involved in the s-separability condition. $\bullet$

4 Discussion

We have exhibited a set of three independent conditions that are necessary and sufficient for the existence of a strict unit representation of a semiorder.

This has been achieved by factorizing s-separability into d-separability and the condition that the set of noses is denumerable. We feel that these three conditions have a clear interpretation.

The bounded P-chain condition deals with the fact that the threshold is constant and positive. As already noted, it resembles an Archimedean condition. It applies as soon as the set X is infinite, even countably infinite. It is not specific to strict unit representations. It is easy to check that it is also a necessary condition for nonstrict unit representations.

The d-separability condition ensures that the trace of the semiorder, which is a complete preorder, has a numerical representation. This is clearly necessary for strict unit representations but is not specific to them. As can be easily checked, d-separability is also necessary for nonstrict unit representations.

Our final condition states that the set of noses is denumerable. It is specific to strict unit representations. We show in Bouyssou and Pirlot (2019a,b) how to deal with the case of nonstrict representations. This involves replacing our condition on "noses" by a condition on the dual notion of "hollows" (Pirlot, 1990, 1991). We do not develop this point here.

Our results are also linked to the discussion in Candeal and Induráin (2010, Sec. 4, p. 489) of Beja and Gilboa (1992, Th. 4.5(a), p. 439). This theorem asserts that a "Generalized Numerical Representation" (GNR) with $\mathscr{S}$ open exists iff S is a semiorder (for which $\succsim$ is antisymmetric) satisfying d-separability and the bounded P-chain condition and such that the set of P-gap-edge-points is denumerable. An element $x \in X$ is a P-gap-edge-point if there is a $y \in X$ such that $y \; P \; x$ and there is no $z \in X$ for which $y \; P \; z \succ x$.

As noted by Candeal and Induráin (2010), the proof of this result (see Beja and Gilboa, 1992, p. 446–448) refers to "positive threshold GNR in which $\mathscr{S}$ is open". This is tantamount to what we have called a strict unit representation. Hence, Candeal and Induráin (2010) wonder whether Beja and Gilboa (1992) were the first to characterize semiorders having a strict unit representation. They state (p. 489, last par. of 2nd col.) that the result in Beja and Gilboa (1992) should be amended by the addition of a condition stating, in our terms, that the set of all right endpoints of lower-half noses should be denumerable.

Our results allow to be more specific. It is clear that if x is a P-gap-edge-point, there is a y such that (y, x) is a nose or a proper lower-half nose (see Definitions 4 and 5). In other terms, x is the right endpoint of a l-h-nose. A l-h-nose, is either a proper l-h nose or a nose. Whenever d-separability is in force, we do not have to ensure the fact that the set of right endpoints of proper l-h noses is denumerable (Lemma 2). We only have to require that the set of noses is denumerable, which is clearly implied by the requirement that the set of all right endpoints of l-h noses is denumerable: requiring that the set of P-gap-edge-points is denumerable therefore implies that the set of right endpoints of noses as well as the set of right endpoints of proper l-h noses are denumerable. Proposition 1 and Theorem 2 show that this is sufficient to guarantee the existence of a strict unit representation. This condition can be weakened however since, as shown in Lemma 2, d-separability implies that the set of proper l-h-noses is denumerable. Hence, our result sharpens Beja and Gilboa's result discussed in Candeal and Induráin (2010, Sec. 4), while ensuring its correctness. To bring our result closer to the one of Beja and Gilboa, we could require that the set of all right endpoints of noses is denumerable instead of requiring that the set of all noses is denumerable. Clearly, these two conditions are equivalent: to the right endpoint of a nose corresponds a unique nose (see Remark 1).

Let us finally mention two directions for future research.

The first is to relate our analysis to the several equivalent formulations of s-separability analyzed in Candeal et al. (2012). Candeal et al. (2012, Th. 4.11, p. 449) state that, for a semiorder, s-separability is equivalent to any of the conditions ensuring separability for interval orders, as introduced in Oloriz, Candeal, and Induráin (1998) and detailed in Bosi, Candeal, Induráin, Oloriz, and Zudaire (2001) (see also the analysis of separability conditions for biorders in Doignon et al., 1984 and Nakamura, 2002). It would be useful to investigate if one of these conditions, equivalent to s-separability for semiorders, allows us to obtain sharper results.

The second and more difficult question is to tackle the case in which representations are neither strict nor nonstrict, as suggested in Nakamura (2002). The related problem of characterizing interval graphs using mixed intervals, i.e., using intervals that are not necessarily all closed or all open, has recently attracted attention in the Graph Theory community (Dourado, Le, Protti, Rautenbach, and Szwarcfiter, 2012, Rautenbach and Szwarcfiter, 2011, 2013), while the classic result of Frankl and Maehara (1987) may be consider as a dual to Beja and Gilboa (1992, Th. 3.8, p. 436) stating the equivalence of strict and nonstrict representations for denumerable sets.

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