Chain Representations of Nested Families of Biorders [1]

Denis Bouyssou [2] Jean-Paul Doignon [3]

5 November 2019 b

[1]This work has benefited from many helpful discussions with Thierry Marchant.

[2]LAMSADE, UMR 7243, CNRS, Université Paris Dauphine, PSL Research University, 75016 Paris, France. e-mail: bouyssou@lamsade.dauphine.fr.

[3] Université Libre de Bruxelles, Département de Mathématique c.p. 216, Boulevard du Triomphe, 1050 Bruxelles, Belgique. e-mail: doignon@ulb.ac.be.

Abstract

Among the real-valued representations of nested families of biorders some representations reflect the nestedness of the family in a simple way. Calling them chain representations, we prove their existence in the finite and countably infinite cases. For the general case, we obtain chain representations in a well-chosen linearly ordered set. Although the existence of real-valued representations in general remains an open problem, our analysis answers questions left pending in the literature. It also leads to new proofs of classical theorems on the existence of a real representation for a single biorder, as well as for a single interval order. A combinatorial property of the set of all biorders from a finite set to another finite set plays a central role in the new proof; called weak gradedness, it is a particularization of well-gradedness which derives from a simpler argument.

Keywords: biorders, nested relations, interval orders, semiorders, numerical representations, chain representations, well-gradedness, weak gradedness.

MSC (2010): 06A75.

Contents

1 Introduction

This chapter deals with an apparently new problem: the existence of chain representations, possibly using real numbers, of families of biorders. It connects three themes present in the literature. The first is the existence of real (numerical) representations of binary relations. The second is the consideration of binary relations from one set to another set, instead of binary relations on a single set. The third is the consideration of nested families of relations and of real representations reflecting this nestedness.

Understanding which types of binary relations admit a specific form of real representation has a long history. For instance, let us postulate here that a relation $\mathcal{R}$ on a set X admits a representation by real numbers when for a, b in X, there holds $a \mathcal{R} b$ if and only if the real value attributed to a is greater or equal than the real value attributed to b. When X is finite, such a representation exists if and only if the relation $\mathcal{R}$ is a weak order (or a total preorder) on X. For infinite sets X, characterizing the existence of a representation is a more involved problem whose solution essentially goes back to Cantor (for an English version, see Cantor, 1955) and in its modern form to Debreu (1954). Debreu's result has important implications in Economics: it singles out the preference relations that are faithfully described by a utility function assigning real numbers to the items compared (see Bridges and Mehta, 1995, Fishburn, 1970a, for thorough reviews).

Another example comes from the literature in Psychology. A Guttman scale (Guttman, 1944, 1950) amounts to a relation from one set to a second set which is captured by the comparison of real values assigned to the items of each set. Ducamp and Falmagne (1969) provide a mathematical characterization of such a relation (at least in the finite case). Here we use the term "biorder" coined by Doignon, Ducamp, and Falmagne (1984) in a paper which also handles the infinite case by relying on the results of Cantor and Debreu we just alluded to. Biorders contain as particular cases interval orders (Fishburn, 1970b, 1973b, 1985; they are biorders from one set to itself that are irreflexive) and semiorders (Luce, 1956, Scott and Suppes, 1958; they are semitransitive interval orders).

The study of nested families of relations is a classic theme in the literature on probabilistic consistency (Block and Marschak, 1960, Marschak, 1960, Luce and Suppes, 1965, Marley, 1968, Roberts, 1971, Fishburn, 1973a, Roberts, 1979, Ch. 6, Roubens and Vincke, 1985, Ch. 5, Suppes, Krantz, Luce, and Tversky, 1989, Ch. 17). The usual interpretation is that one observes the frequency $p(x, y)$ with which a subject chooses between the two objects x and y. A family of nested relations is then obtained by cutting the probabilistic relation $p(x, y)$ at different levels, usually taken above 1/2. Indeed, if $\lambda_1 > \lambda_2 \geq 1/2$, letting $x \; R_1 \; y$ iff $p(x, y) \geq \lambda_1$ and $x \; R_2 \; y$ iff $p(x, y) \geq \lambda_2$, we obtain a nested family of two relations $R_1 \subseteq R_2$.

A similar interpretation holds with biorders. Classically, the set A is a set of "subjects" and the set Z is a set of "problems" that subjects are asked to solve. Recording the frequency $p(a, x)$ with which subject $a \in A$ solves problem $x \in Z$, we obtain a probabilistic relation from A to Z that can be used, as above, to define a nested family of biorders (Doignon, Monjardet, Roubens, and Vincke, 1988).

In this chapter we extend representation results from single biorders to nested families of biorders, using representations which reflect in a simple way the nestedness of the family. This leads to our main problem: *establishing the existence of chain representations of nested families of biorders*.

We give fairly complete results characterizing the existence of chain representations in some arbitrary linearly ordered set $(E, \geq)$. The case of real chain-representations (E is taken to be $\mathbb{R}$) is more delicate outside the denumerable case (denumerable means finite or countably infinite).

We have two main results. The first says that all nested families of biorders from one finite set to another finite set have a real chain-representation. The second says that the same is true for *all* chains of biorders from an arbitrary set to another arbitrary set if the chain representations are sought in an arbitrary linearly ordered set. Moreover, our results offer some new proofs of results characterizing the existence of representations for a *single* relation.

In the finite case, our sufficiency proof uses combinatorial properties of biorders. A weakening of the property of *well-gradedness*, called *weak gradedness*, plays a central role; both properties concern the family of all biorders from a finite set to a finite set (we refer to Doignon and Falmagne, 1997, for a study of well-gradedness).

The rest of the paper is organized as follows. Section 2 recalls a number of useful facts on biorders. Section 3 introduces weak gradedness and compares it to the stronger property of well-gradedness. Section 4 presents our main results, in the finite case, on chain representations of nested families of biorders. We also relate them to the existing literature. Turning to the general case in Section 5, we analyse the question of the existence of chain representations. We illustrate some of the difficulties of this more delicate problem on a number of examples. A final section discusses our results and lists several open problems that we see as interesting opportunities for future research.

We use a standard vocabulary for binary relations. To avoid any ambiguity, we define the main properties and structures that we use in Appendix A.

2 Biorders

2.1 Notation and framework

Let A and Z be two disjoint sets. A binary relation $\mathcal{R}$ from A to Z is a subset of $A \times Z$.

Remark 1
The disjointness hypothesis on A and Z, which we make throughout the chapter, may seem quite restrictive. This is not so. If A and Z are not disjoint, we build duplications A' of A and Z' of Z with A' and Z' disjoint, and next a new relation $\mathcal{R}'$ from A' to Z' which faithfully encodes $\mathcal{R}$; see details in Doignon et al. (1984, Def. 4, p. 79). All properties and concepts used below are better understood if we work with $\mathcal{R}'$ rather than with $\mathcal{R}$. A similar remark holds for any relation defined on a set X: we replace it, w.l.o.g., with a relation from a duplication of X to another, disjoint duplication of X. •

For a relation $\mathcal{R}$ from A to Z, we denote by $\overline{\mathcal{R}} = (A \times Z) \setminus \mathcal{R}$ its *complement*, by $\mathcal{R}^{-1} = \{(z, a) \in Z \times A : (a, z) \in \mathcal{R}\}$ its *converse* and by $\overline{\mathcal{R}}^{-1} = \{(z, a) \in Z \times A : (a, z) \notin \mathcal{R}\}$ its *dual*. Moreover, if $\mathcal{R}$ is a relation from A to Z and $\mathcal{T}$ is a relation from Z to K, we define the *product* $\mathcal{R}\mathcal{T}$ of $\mathcal{R}$ and $\mathcal{T}$ by letting $a\ \mathcal{R}\mathcal{T}\ \ell$, where $a \in A$ and $\ell \in K$, when $a\ \mathcal{R}\ x$ and $x\ \mathcal{T}\ \ell$ for some $x \in Z$.

Remark 2
Clearly the above definition of the complement $\overline{\mathcal{R}}$ depends on the specification of A and Z. In any case, the context will make clear which complement is denoted by $\overline{\mathcal{R}}$: when $\mathcal{R}$ is a relation from A to Z, the complement is taken with respect to $A \times Z$. •

2.2 Biorders

A binary relation $\mathcal{R}$ from A to Z is a *biorder* if it is Ferrers, that is, for all $a, b \in A$ and all $x, y \in Z$, we have:

$$[a\ \mathcal{R}\ x \text{ and } b\ \mathcal{R}\ y] \Rightarrow [a\ \mathcal{R}\ y \text{ or } b\ \mathcal{R}\ x].$$

More compactly, $\mathcal{R}$ is a biorder when $\mathcal{R}\,\overline{\mathcal{R}}^{-1}\,\mathcal{R} \subseteq \mathcal{R}$.

The following notation and concepts are central in this chapter. Let $\mathcal{R}$ be a relation from A to Z, with A and Z disjoint sets. For a in A, we let $a\mathcal{R} = \{z \in Z\ :\ a\ \mathcal{R}\ z\}$ and $\mathcal{R}z = \{a \in A\ :\ a\ \mathcal{R}\ z\}$. The *left trace* of $\mathcal{R}$ is the binary relation $\succsim^A_{\mathcal{R}}$ on A defined by letting, for all $a, b \in A$ (the second equivalence is trivial),

$$a \succsim^A_{\mathcal{R}} b \Leftrightarrow [b\ \mathcal{R}\ x \Rightarrow a\ \mathcal{R}\ x, \text{ for all } x \in Z] \Leftrightarrow a\mathcal{R} \supseteq b\,\mathcal{R}. \tag{1}$$

Similarly, the *right trace* of $\mathcal{R}$ is the binary relation $\succsim^Z_{\mathcal{R}}$ on Z defined by letting, for all $x, y \in Z$,

$$x \succsim^Z_{\mathcal{R}} y \Leftrightarrow [a\ \mathcal{R}\ x \Rightarrow a\ \mathcal{R}\ y, \text{ for all } a \in A] \Leftrightarrow \mathcal{R}x \subseteq \mathcal{R}y. \tag{2}$$

Whatever $\mathcal{R}$, the relations $\succsim^A_{\mathcal{R}}$ and $\succsim^Z_{\mathcal{R}}$ are, by construction, quasi orders (that is, reflexive and transitive relations).

Here are easy characterizations of biorders (see for instance Doignon et al., 1984, Proposition 2, p. 78; Monjardet, 1978, Theorem 1, p. 60; Rabinovitch, 1978, Theorem 2, p. 52).

Proposition 1
The following assertions about a relation $\mathcal{R}$ from A to Z are equivalent:

(i) $\mathcal{R}$ is a biorder;

(ii) the sets $a\mathcal{R}$, for $a \in A$, form a chain;

(iii) the sets $\mathcal{R}x$, for $x \in Z$, form a chain;

(iv) the left trace $\succsim^A_{\mathcal{R}}$ of $\mathcal{R}$ is complete;

(v) the right trace $\succsim^Z_{\mathcal{R}}$ of $\mathcal{R}$ is complete.

For the record let us spell out the following result from Ducamp and Falmagne (1969, Th. 3, finite case) and Doignon et al. (1984, Prop. 4, p. 79), which is the *representation theorem of biorders* in the denumerable case (meaning finite or countably infinite case).

Proposition 2
Let $\mathcal{R}$ be a binary relation from A to Z. When each of A and Z is denumerable, the following statements are equivalent:

(i) *the relation $\mathcal{R}$ is a biorder;*

(ii) *there are a real-valued function f on A and a real-valued function g on Z such that, for all $a \in A$ and $x \in Z$,*

$$a \, \mathcal{R} \, x \Leftrightarrow f(a) > g(x). \tag{3}$$

Furthermore, the functions f and g can always be chosen in such a way that, for all $a, b \in A$ and $x, y \in Z$,

$$\begin{aligned} a \succsim^A_{\mathcal{R}} b \Leftrightarrow f(a) \geq f(b), \\ x \succsim^Z_{\mathcal{R}} y \Leftrightarrow g(x) \geq g(y). \end{aligned} \tag{4}$$

Let $\mathcal{R}$ be any relation from A to Z. In Proposition 2, it is tempting to replace the real-valued mappings f and g with mappings to some linearly ordered set $(E, \geq)$. As we will see, the equivalence in the above proposition then holds for *all* sets A and Z. We now give a name and a notation for the pair of mappings.

Definition 1 (Representations)
A $>$-*representation in* $(E, \geq)$ of the relation $\mathcal{R}$ from A to Z consists in a linearly ordered set $(E, \geq)$ and in two mappings $f : A \to E$ and $g : Z \to E$ such that, for all a in A and z in Z:

$$a \,\mathcal{R}\, z \Leftrightarrow f(a) > g(z). \tag{5}$$

We then also say that (f, g) is a $>$-representation (also called a *strict representation*) of $\mathcal{R}$ in $(E, \geq)$. The representation is *trace-compatible*, or *respects the traces*, when moreover the following holds, for all $a, b \in A$ and $x, y \in Z$,

$$\begin{aligned} a \succsim^A_{\mathcal{R}} b &\Leftrightarrow f(a) \geq f(b), \\ x \succsim^Z_{\mathcal{R}} y &\Leftrightarrow g(x) \geq g(y). \end{aligned} \tag{6}$$

The representation is *special* if, for all $a \in A$ and all $x \in Z$, we have $f(a) \neq g(x)$.

The definition of $\geq$-*representations* results from changing only the order sign in Equation (5) from $>$ to $\geq$. Such representations are also called *nonstrict representations.* ⌟

As it is easily checked, a relation $\mathcal{R}$ from A to Z admits a $>$-representation in the ordered set $(E, \geq)$ exactly if the relation $\overline{\mathcal{R}}^{-1}$ from Z to A admits a $\leq$-representation in the same linearly ordered set $(E, \geq)$.

A general characterization of the existence of a $>$-representation in some linearly ordered set follows. It is a slight variation on Doignon, Ducamp, and Falmagne (1987, Prop. 1, p. 4).

Proposition 3
A relation $\mathcal{R}$ from A to Z has a $>$-representation in some linearly ordered set $(E, \geq)$ if and only if $\mathcal{R}$ is a biorder. For a biorder, there always exists a representation which is both special and trace-compatible.

PROOF SKETCH
Necessity is easy. We sketch sufficiency. Define a relation $\mathcal{Q}_m$ on $A \cup Z$ by the following table:

$\mathcal{Q}_m$	A	Z
A	$\succsim^A_{\mathcal{R}}$	$\mathcal{R}$
Z	$\overline{\mathcal{R}}^{-1}$	$\succsim^Z_{\mathcal{R}}$

Using Equations (1) and (2), it is routine to show that $\mathcal{Q}_m$ is a weak order. This weak order is such that for all $a \in A$ and all $z \in Z$, it is never true that $a \,\mathcal{Q}_m\, z$ and $z \,\mathcal{Q}_m\, a$. The relation $\mathcal{Q}_m \cap \mathcal{Q}_m^{-1}$ is an equivalence (a reflexive, symmetric and transitive relation) on $A \cup Z$. We build the set E as the quotient of $A \cup Z$ by $\mathcal{Q}_m \cap \mathcal{Q}_m^{-1}$. This set is linearly ordered by the relation induced by $\mathcal{Q}_m$ on the quotient set. The function f associates to each $a \in A$ the equivalence class of $\succsim_{\mathcal{R}}^{A}$ to which it belongs. Similarly, the function g associates to each $x \in Z$ the equivalence class of $\succsim_{\mathcal{R}}^{Z}$ to which it belongs. It is clear that (f, g) is a $\geq$-representation of $\mathcal{R}$ in $(E, \geq)$.

By definition of $\mathcal{Q}_m$, the representation respects the traces. It is also special, so that it is at the same time a $>$-representation and a $\geq$-representation. □

Remark 3
Notice that any representation in a denumerable linearly ordered set $(E, \geq)$ is easily turned into a real representation: it suffices to compose the mappings f and g with any embedding of $(E, \geq)$ into $(\mathbb{R}, \geq)$. Clearly, this is no more true in the general case. •

The study of nested families of biorders led us to new proofs of Propositions 2 and 3 (see Subsection 4.4 and after Corollary 1). The next section introduces a tool useful to investigate the combinatorial properties of biorders on finite sets.

3 Well-gradedness and Weak Gradedness

3.1 Well-gradedness

Consider two relations $\mathcal{R}$ and $\mathcal{S}$ from A to Z, both sets being finite (and disjoint). The distance between these two relations is $d(\mathcal{R}, \mathcal{S}) = |\mathcal{R} \triangle \mathcal{S}|$, with $\mathcal{R} \triangle \mathcal{S} = (\mathcal{R} \setminus \mathcal{S}) \cup (\mathcal{S} \setminus \mathcal{R})$, and $|U|$ denoting the cardinality of the set U.

Let $BO(A, Z)$ (or simply BO, when there is no ambiguity on the underlying sets) be the collection of all biorders from A to Z. Consider two relations $\mathcal{R}, \mathcal{S} \in BO$ such that $d(\mathcal{R}, \mathcal{S}) = \ell$. Doignon and Falmagne (1997) show that there are $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_\ell$ in BO such that $\mathcal{F}_0 = \mathcal{R}$, $\mathcal{F}_\ell = \mathcal{S}$, and $d(\mathcal{F}_{i-1}, \mathcal{F}_i) = 1$, for $i = 1, 2, \ldots, \ell$. In words, for any two biorders at distance ℓ from one another, some sequence of exactly ℓ elementary steps, each consisting in the *addition* or the *removal* of a single (ordered) pair, transforms the first biorder into the second one without never leaving BO. Collections, like BO, having this property are called *well-graded* in Doignon and Falmagne (1997).

The previous paragraph points, for a given biorder, to exceptional pairs (a, x) whose addition to, or deletion from, the biorder produces again a biorder. The following definition and lemma capture the two resulting collections of pairs.

Definition 2 (Inner and outer fringes)
The *inner fringe* and *outer fringe* of a biorder $\mathcal{R}$ are defined respectively by

$$\mathcal{R}^{\mathcal{I}} = \mathcal{R} \setminus \mathcal{R}\overline{\mathcal{R}}^{-1}\mathcal{R}, \qquad \mathcal{R}^{\mathcal{O}} = \overline{\mathcal{R}}\mathcal{R}^{-1}\overline{\mathcal{R}} \setminus \mathcal{R}. \tag{7}$$

⌟

Doignon and Falmagne (1997) prove the following result.

Lemma 1
We have $\mathcal{R}^{\mathcal{I}} = \mathcal{R}\overline{\mathcal{R}}^{-1}\mathcal{R} \subseteq \mathcal{R}$ *and* $\mathcal{R}^{\mathcal{O}} = \overline{\mathcal{R}}\mathcal{R}^{-1}\overline{\mathcal{R}} \subseteq \overline{\mathcal{R}}$*. Moreover:*

$$\mathcal{R}^{\mathcal{I}} = \{p \in \mathcal{R} : \mathcal{R} \setminus \{p\} \textit{ is a biorder}\}, \tag{8}$$

$$\mathcal{R}^{\mathcal{O}} = \{q \in (A \times Z) \setminus \mathcal{R} : \mathcal{R} \cup \{q\} \textit{ is a biorder}\}. \tag{9}$$

Observe, in particular, that if $\mathcal{R}, \mathcal{S} \in BO$ are such that $\mathcal{R} \subseteq \mathcal{S}$ and $d(\mathcal{R}, \mathcal{S}) = \ell$, then there are $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_\ell \in BO$ such that $\mathcal{T}_0 = \mathcal{R}$, $\mathcal{T}_\ell = \mathcal{S}$, $d(\mathcal{T}_{i-1}, \mathcal{T}_i) = 1$, for $i = 1, 2, \ldots, \ell$, and $\mathcal{R} = \mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \cdots \subsetneq \mathcal{T}_\ell = \mathcal{S}$. In words, some sequence of ℓ elementary steps, each one consisting in the addition of a single pair, transforms $\mathcal{R}$ into $\mathcal{S}$ without ever leaving BO (no pair removal is applied). This property, implied by well-gradedness (compare with Proposition 4 in Doignon and Falmagne, 1997), is called here "weak gradedness". It is all we need to establish our results on nested families of biorders in the finite case.

3.2 Weak gradedness

This subsection deals only with the finite case. We first establish a lemma which implies the property of "weak gradedness" from previous paragraph.

Lemma 2
Suppose that the sets A and Z are finite. For any two biorders $\mathcal{R}$ and $\mathcal{S}$ from A to Z with $\mathcal{R} \subsetneq \mathcal{S}$, there exists some pair p in $\mathcal{S} \setminus \mathcal{R}$ such that $\mathcal{R} \cup \{p\}$ is again a biorder.

Proof
According to Proposition 1, the subsets $(a\mathcal{R})_{a \in A}$ form a chain. We may thus list the elements of A as a_1, a_2, ..., a_n in such a way that

$$a_1\mathcal{R} \subseteq a_2\mathcal{R} \subseteq \cdots \subseteq a_n\mathcal{R} \tag{10}$$

(many or even all of the inclusions may be equalities). Notice that for any z in $a_{i+1}\,\mathcal{R} \setminus a_i\,\mathcal{R}$ where $i \in \{1, 2, \ldots, n-1\}$, the new relation $\mathcal{R}' = \mathcal{R} \cup \{(a_i, z)\}$ is again a biorder (the subsets $a_i\,\mathcal{R}'$, for $i = 1$, 2, ..., n, still form a chain).

For $i = 1$, 2, ..., n, we have by assumption

$$a_1\mathcal{R} \subseteq a_1\mathcal{S}, \qquad a_2\mathcal{R} \subseteq a_2\mathcal{S}, \qquad \ldots \qquad a_n\mathcal{R} \subseteq a_n\mathcal{S}. \tag{11}$$

Although the subsets $a_i\mathcal{S}$, for $i = 1$, 2, ..., n, also form a chain, we do not necessarily have $a_i\mathcal{S} \subseteq a_{i+1}\mathcal{S}$. We may however permute, if necessary, the elements of A in order to have both (10) and the following: for all i, j in $\{1, 2, \ldots, n\}$,

$$[i < j \text{ and } a_i\mathcal{R} = a_j\mathcal{R}] \implies a_i\mathcal{S} \subseteq a_j\mathcal{S}. \tag{12}$$

Now if there exists some i in $\{1, 2, \ldots, n-1\}$ such that $(a_{i+1}\mathcal{R} \setminus a_i\mathcal{R}) \cap a_i\mathcal{S} \neq \varnothing$, then for any z in the intersection the pair $p = (a_i, z)$ makes the thesis true. Moreover, if $a_n\mathcal{R} \subset a_n\mathcal{S}$, then for any element z in $a_n\mathcal{S} \setminus a_n\mathcal{R}$ the pair $p = (a_n, z)$ makes the thesis true.

We are thus left with the situation where we have for any i in $\{1, 2, \ldots, n-1\}$

$$a_{i+1}\mathcal{R} \setminus a_i\mathcal{R} \subseteq Z \setminus a_i\mathcal{S} \tag{13}$$

and moreover

$$a_n\mathcal{R} = a_n\mathcal{S}. \tag{14}$$

We next derive from Equations (13) and (14) the contradiction $\mathcal{S} = \mathcal{R}$. It suffices to prove by induction $a_i\mathcal{S} = a_i\mathcal{R}$, meaning here is $a_i\mathcal{S} \subseteq a_i\mathcal{R}$ for $i = n$, $n-1$, ..., 1. The case $i = n$ is Equation (14). Assuming $a_{i+1}\mathcal{R} = a_{i+1}\mathcal{S}$ for some i with $n - 1 \geq i \geq 1$, we show $a_i\mathcal{R} = a_i\mathcal{S}$ by working out two cases.

If $a_i\mathcal{R} = a_{i+1}\mathcal{R}$, then from Equation (12) we have $a_i\mathcal{S} \subseteq a_{i+1}\mathcal{S} = a_{i+1}\mathcal{R} = a_i\mathcal{R}$ and we are done (remember $a_i\mathcal{R} \subseteq a_i\mathcal{S}$).

If $a_i\mathcal{R} \subsetneq a_{i+1}\mathcal{R}$, take some t in $a_{i+1}\mathcal{R} \setminus a_i\mathcal{R}$. By the induction assumption, we have also $a_{i+1}\ \mathcal{S}\ t$, and by (13) we have $(a_i, t) \notin \mathcal{S}$. Take any y in $a_i\mathcal{S}$. We deduce $a_{i+1}\ \mathcal{R}\ y$ because $\mathcal{R}$ is a biorder. Then $a_i\ \mathcal{R}\ y$ follows from the induction assumption. Now if the pair (a_i, y) would not be in $\mathcal{R}$, then y would contradict (13). □

We have seen above that if $\mathcal{R}$ is a biorder from A to Z and (a, x) is a pair in $\overline{\mathcal{R}} = (A \times Z) \setminus \mathcal{R}$, then $\mathcal{R} \cup \{(a, x)\}$ is again a biorder if and only if it belongs to the outer fringe of $\mathcal{R}$. Hence, Lemma 2 asserts that if $\mathcal{R} \subsetneq \mathcal{S}$ and $d(\mathcal{R}, \mathcal{S}) = 1$, then the outer fringe of $\mathcal{R}$ has at least one of its pairs in $\mathcal{S}$. Alternatively, it says that if $\mathcal{R} \subsetneq \mathcal{S}$ and $d(\mathcal{R}, \mathcal{S}) = 1$, that the inner fringe of $\mathcal{S}$ has at least one of its pairs that does not belong to $\mathcal{R}$.

It is worth taking a more general point of view on Lemma 2. Biorders from A to Z, being relations, are subsets of $A \times Z$. Their collection is thus a collection of subsets of a ground set (here $A \times Z$). Lemma 2 implies a property of this collection, that we now designate in a more general setting.

Definition 3

Let $\mathcal{F}$ be a collection of subsets of a finite ground set E. We say that $\mathcal{F}$ is *weakly graded* when for any two elements F and G of $\mathcal{F}$ with $F \subsetneq G$, there is a sequence F_0, F_1, ..., F_k of elements of $\mathcal{E}$ such that $F = F_0$, $F_k = G$ and for each $i = 1, 2$, ..., k there hold both $F_{i-1} \subsetneq F_i$ and $|F_i \setminus F_{i-1}| = 1$. ⌟

In Definition 3, we necessarily have $k = |G \setminus F|$. Notice that the collection $\mathcal{F}$ is weakly graded as soon as for any two of its elements, say J and K, with $J \subsetneq K$, there exists a third element L such that $J \subsetneq L \subsetneq K$ and moreover $|L \setminus J| = 1$. This is the property obtained for the collection of biorders from A to Z in Lemma 2.

Let us compare "well-gradedness" and "weak gradedness" of the collection BO of biorders from A to Z. We know from Doignon and Falmagne (1997) that BO is well-graded. This clearly implies that it is weakly graded. However, we prefer to stick here with weak gradedness, which is a weaker property. Indeed, the proof of well-gradedness of BO needs more elaborate arguments than those in the proof of Lemma 2. To see why, let $A = \{a, b\}$ and $Z = \{y, z\}$. Consider the two biorders $\mathcal{R} = \{(a, y)\}$ and $\mathcal{S} = \{(b, z)\}$, which are at distance 2 from one another. To transform $\mathcal{R}$ into $\mathcal{S}$ in 2 steps, we cannot start by adding some pair to $\mathcal{R}$. The first step has to consist in the deletion of (a, y). Notice that $\mathcal{R}$ and $\mathcal{S}$ form a collection of relations which is (trivially) weakly graded but not well-graded.

We are now fully equipped to tackle the case of nested families of biorders.

4 Nested Families of Biorders on Finite Sets

This section establishes results on nested families of biorders from A to Z, with A and Z finite sets. We deal with the extension of these results to the infinite case in the next section.

4.1 Relation to the literature

Doignon et al. (1988) have proposed a detailed study of (non-necessarily nested) families of biorders defined on finite sets and their real representations. Their findings consolidate and extend results due to Roberts (1971), Fishburn (1973a), Monjardet (1984, 1988), Roubens and Vincke (1985, Ch. 5), and Doignon (1988). Most of these results deal with situations in which the intersection of the left traces (or of the right traces) of all biorders in the family is complete. This gives rise to what Doignon et al. (1988) have called right (or left) homogeneous families of biorders. The real representation of such families involves using a single function either on A (for right-homogeneous families) or on Z (for left-homogeneous families)[1]. This function is a real representation of the intersection of the traces on this set that is supposed to be complete.

Doignon et al. (1988) also particularize their results to the case of nested families of biorders (see Section IV of their paper) through the study of the cuts of a valued relation. Nested families of biorders are the subject of the present text. We study real-valued representations of such families without supposing that they are left or right homogeneous.

[1]This is no mistake since our use of left and right trace does not conform to the terminology of Doignon et al. (1988), as explained in Bouyssou and Marchant (2011).

4.2 Results

Definition 4

A *family of relations* from A to Z consists in an integer $k \geq 1$ and k relations $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$. It is *nested* if

$$\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots \subseteq \mathcal{R}_k.$$

If all the relations $\mathcal{R}_i$, $i = 1, 2, \ldots, k$ are biorders, we speak of a *nested family of biorders.* ⌟

Doignon et al. (1988, Proposition 1, p. 460–461) state that, if $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$ is a nested family of biorders between the finite sets A and Z, then there are real-valued functions f_i on A and g_i on Z, for $i = 1, 2, \ldots, k$, such that, for all $a \in A$, $x \in Z$, and $i = 1, 2, \ldots, k$,

$$a \; \mathcal{R}_i \; x \Leftrightarrow f_i(a) > g_i(x),$$

and, for all $i, j \in \{1, 2, \ldots, k\}$ with $i > j$,

$$f_i(a) > g_i(x) \Rightarrow f_j(a) > g_j(x).$$

We use a different notion of representation, called a "chain representation", that makes obvious the nested character of the family (see Figure 1).

Definition 5

A *real chain->-representation* of a family of relations $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$ consists in real-valued functions $f_1, f_2, \ldots, f_k$ on A and $g_1, g_2, \ldots, g_k$ on Z such that, for all $a \in A$, $x \in Z$ and $i \in \{1, 2, \ldots, k\}$,

$$\begin{aligned} a \; \mathcal{R}_i \; x &\Leftrightarrow f_i(a) > g_i(x), \\ f_k(a) &\geq f_{k-1}(a) \geq \cdots \geq f_1(a), \\ g_1(x) &\geq g_2(x) \geq \cdots \geq g_k(x). \end{aligned} \tag{15}$$

The chain->-representation *respects the traces*, or is *trace-compatible*, when for all $a, b \in A$, $x, y \in Z$ and $i \in \{1, 2, \ldots, k\}$

$$a \succsim^A_{\mathcal{R}_i} b \Leftrightarrow f_i(a) \geq f_i(b), \tag{16}$$

$$x \succsim^Z_{\mathcal{R}_i} y \Leftrightarrow g_i(x) \geq g_i(y). \tag{17}$$

It is *special* when for all $a \in A$, $x \in Z$ and $i \in \{1, 2, \ldots, k\}$

$$f_i(a) \neq g_i(z). \tag{18}$$

⌟

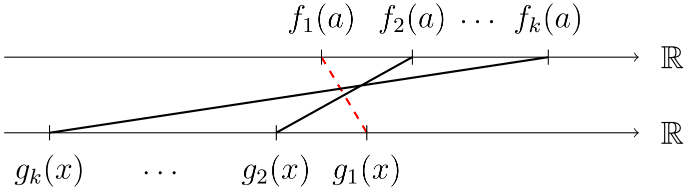


Figure 1: Illustration of the real values assigned to a and x in a chain representation: here, $a\ \mathcal{R}_2\ x$, $a\ \mathcal{R}_3\ x$, ..., $a\ \mathcal{R}_k\ x$ hold but $a\ \mathcal{R}_1\ x$ does not hold. We have $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots \subseteq \mathcal{R}_k$, $f_k \geq f_{k-1} \geq \cdots \geq f_1$, and $g_1 \geq g_2 \geq \cdots \geq g_k$

We illustrate a chain representation in Figure 1.

Our main purpose in this section is to prove the following proposition that gives necessary and sufficient conditions for the existence of real chain $>$-representations. This clearly tightens the result from Doignon et al. (1988) recalled above.

Proposition 4
Let $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$ be a family of relations from the finite set A to the finite set Z. This family has a real chain-$>$-representation $f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_k$ if and only if it is a nested family of biorders.

Any nested family of biorders admits some real $>$-representation which is both special and trace-compatible.

It is clear that the desired real representation in Proposition 4 implies that $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$ is a nested family of biorders. The proof of the converse implication in Subsection 4.3 relies on the following lemma.

Lemma 3
Let $\mathcal{R}$ and $\mathcal{S}$ be two biorders from the finite set A to the finite set Z such that $\mathcal{S} \setminus \{p\} = \mathcal{R}$ for some pair p in $\mathcal{S} \setminus \mathcal{R}$. Assume the real-valued functions f on A and g on Z are such that for all b in A, y in Z,

$$b\ \mathcal{R}\ y \Leftrightarrow f(b) > g(y), \tag{19}$$

and suppose furthermore that the $>$-representation f, g is special and trace-compatible.

Then there are also real-valued functions f^* on A and g^* on Z such that, for all b in A, y in Z,

$$b\ \mathcal{S}\ y \Leftrightarrow f^*(b) > g^*(y), \tag{20}$$

$$f^*(b) \geq f(b), \tag{21}$$

$$g(y) \geq g^*(y), \tag{22}$$

and moreover the $>$-representation f^*, g^* of $\mathcal{S}$ is special and trace-compatible.

The following subsection collects the proofs.

4.3 Proofs

The proof of Lemma 3 which we now give focuses on the representations themselves. Some additional comments of combinatorial nature appear after the proof.

Proof of Lemma 3

Assume $p = (a, x)$, with thus $\mathcal{R} \cup \{(a, x)\} = \mathcal{S}$. We are given a special representation (f, g) of $\mathcal{R}$ which respects the traces of $\mathcal{R}$. If $b \in A \setminus \{a\}$ and $y \in Z \setminus \{z\}$, then $(b, y) \in \mathcal{R}$ if and only if $(b, y) \in \mathcal{S}$; we are thus tempted to set $f^*(b) = f(b)$ and $g^*(y) = g(y)$, which we do. In view on the assumptions on f and g, the required inequalities (20)–(22) are satisfied for all $b \in A \setminus \{a\}$ and $y \in Z \setminus \{z\}$.

There remains only to assign values to $f^*(a)$ and $g^*(x)$. To this aim, select some c in A and some z in Z such that

$$\begin{aligned} f(c) &= \min\{f(b) \; : \; b \in A \text{ and } f(b) > g(x)\} = \min f(\mathcal{R}x), && (23)\\ g(z) &= \max\{g(y) \; : \; z \in Z \text{ and } f(a) > g(z)\} = \max\, g(a\mathcal{R}). && (24) \end{aligned}$$

Notice that c is well defined in A except if $\mathcal{R}x$ is empty, and similarly z is well defined in Z except if $a\mathcal{R}$ is empty. We leave to the reader the two cases $\mathcal{R}x = \varnothing$ and $a\mathcal{R} = \varnothing$, and in the sequel assume that c and z are well defined. Notice $g(z) < f(a) < g(x) < f(c)$. There is more to be said on the values of f and g (compare with Figure 2).

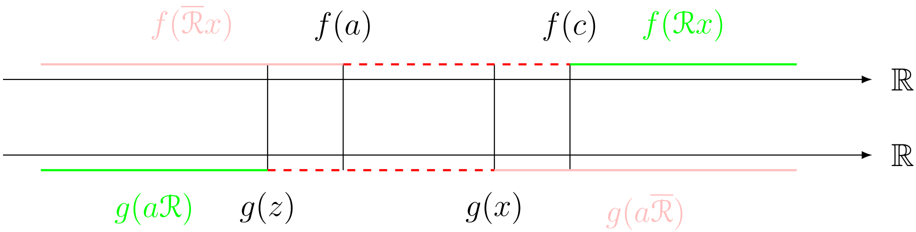


Figure 2: Illustration of the proof of Lemma 3.

No value $f(b)$, for $b \in A$, can be in the interval $]\, f(a), f(c)\, [$. Indeed: (i) if we had $f(a) < f(b) < g(x)$, there would exist some y in Z such that $f(a) < g(y) < f(b)$ (because f respects the trace on A we must have $a \succ^A_{\mathcal{R}} b$). Then $a\, \overline{\mathcal{R}}\, y\, \mathcal{R}^{-1}\, b\, \overline{\mathcal{R}}\, x$ contradicts the assumption that $\mathcal{R} \cup \{(a, x)\}$ is a biorder; (ii) $f(b) = g(x)$ contradicts that f, g is a special $>$-representation; (iii) $g(x) < f(b) < f(c)$ would give $b \in \mathcal{R}x$ in contradiction with the definition of $f(c)$. Because the $>$-representation f, g respects the traces, we derive that $\{b \in A \; : \; f(b) = f(c)\}$ is the equivalence class just above the one of a in A w.r.t. the trace $\succsim^A_{\mathcal{R}}$ (the class of a is *just above* the class of b when $a \succ^A_{\mathcal{R}} b$ and for no element c in A do we have $a \succ^A_{\mathcal{R}} c \succ^A_{\mathcal{R}} b$).

Similarly, no value $g(y)$, for $y \in Z$, can be in $]\,g(z), g(x)\,[$. Indeed, (i) $g(z) < g(y) < f(a)$ would contradict the definition of z; (ii) $g(y) = f(a)$ cannot hold because the $>$-representation f, g is special; (iii) $f(a) < g(y) < g(x)$ would imply the existence of b in A such that $y \; \mathcal{R}^{-1} \; b \; \overline{\mathcal{R}} \; x$ which together with $a \; \overline{\mathcal{R}} \; y$ contradicts that $\mathcal{R} \cup \{(a,x)\}$ is a biorder. Consequently, $\{y \in Z \; : \; g(y) = g(z)\}$ is the equivalence class just below the one of x in Z (w.r.t. the trace $\succsim^Z_{\mathcal{R}}$).

Because $\mathcal{S}$ differs from $\mathcal{R}$ only by the addition of the pair (a,x), we must assign values to $f^*(a)$ and $g^*(x)$ in such a way that $g(z) \leq g(x) < f(a) \leq f(c)$; moreover, all pairs (a,y) and (b,x), for $b \in A$ and $y \in Z$, are then correctly represented. Hence any such assignment of $f^*(a)$ and $g^*(x)$ delivers a $>$-representation f^*, g^* of $\mathcal{S}$. However, to make sure that the representation respects the traces of $\mathcal{S}$, we must take care. Remember that the trace $\succsim^A_{\mathcal{R}}$ reflects comparisons among themselves of the subsets $b\mathcal{R}$ of Z—see Equation (1); also, $\succsim^Z_{\mathcal{R}}$ reflects comparisons among themselves of the subsets $\mathcal{R}y$ of A. The addition of (a,x) to $\mathcal{R}$ modifies only two such sets, namely $a\mathcal{R}$ to which x is added, and $\mathcal{R}x$ to which a is added. For the element c defined in Equation (23), we derive that $a\mathcal{S} = a\mathcal{R} \cup \{x\}$ forms a subset of $c\mathcal{R} = c\mathcal{S}$, but we can have either $a\mathcal{S} = c\mathcal{S}$ or $a\mathcal{S} \subsetneq c\mathcal{S}$.

Let us now define $f^*(a)$ (see Figure 3): if $a\mathcal{S} = c\mathcal{S}$, then a and c become equivalent in the trace $\succsim^A_{\mathcal{S}}$ of $\mathcal{S}$, and we set $f^*(a) = f(c)$; otherwise, we set $f^*(a) = (f(a) + 2\,g(x))/3$. In a similar way, if $\mathcal{S}x = \mathcal{S}z$, we set $g^*(x) = g(z)$, otherwise $g^*(x) = (2\,f(a) + g(x))/3$.

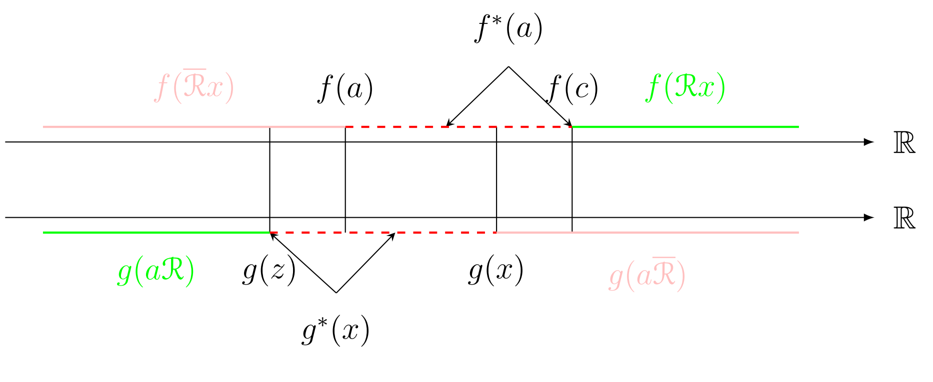


Figure 3: Assignations of $f^*(a)$ and $g^*(x)$ in the proof of Lemma 3.

With the above assignments of $f^*(a)$ and of $g^*(x)$, the $>$-representation f^*, g^* respects the traces of $\mathcal{S}$. Moreover, the representation is special in view of the assumption that the $>$-representation f, g is special and the use of strict inequalities in the definitions of $f^*(a)$ and $g^*(x)$. □

The above proof takes care of the evolution of the collection of equivalence classes when the biorder $\mathcal{R}$ changes into the biorder $\mathcal{S}=\mathcal{R}\cup\{(a,x)\}$. Note that only four classes are susceptible of modification: the class of a and the class just above it in A, the class of x and the class just below it in Z (the meaning of "just above" is explained in the proof). The next example illustrates various ways in which these four classes can evolve.

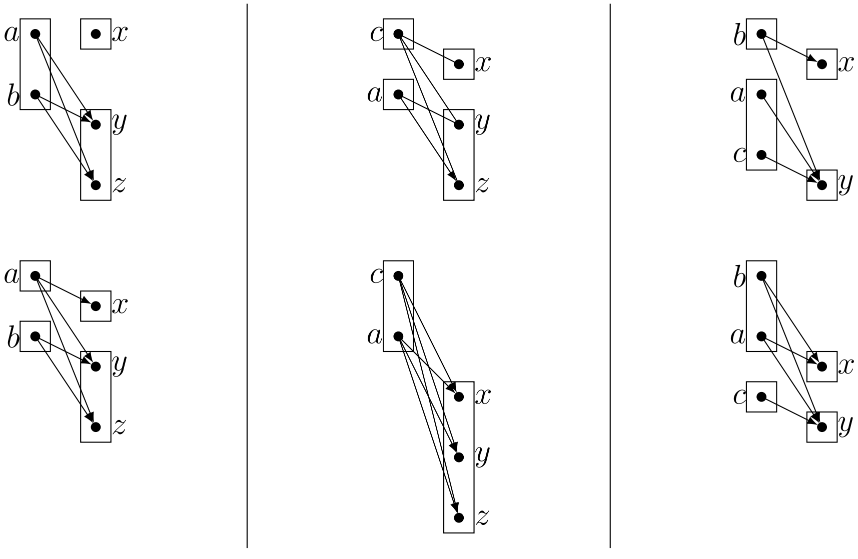


Figure 4: Three examples of evolution of the equivalence classes of the traces, when the addition of the single pair (a, x) to the upper biorder produces the lower biorder (Example 1).

Example 1

On the upper part of Figure 4 there are three biorders. The addition of the single pair (a, x) produces the corresponding biorders on the lower part. ◇

PROOF OF PROPOSITION 4

Let $\mathcal{R}_1$, $\mathcal{R}_2$, ..., $\mathcal{R}_k$ be a nested family of biorders from a finite set A to a finite set Z. We may assume $\mathcal{R}_1=\varnothing$, because otherwise we may add $\varnothing$ as the very first biorder (and renumber the other biorders). If $|\mathcal{R}_{i+1}\setminus\mathcal{R}_i|>1$, we apply Lemma 2 and insert a new biorder in between $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$. Repeating the construction while it remains possible, we end up with a nested family of biorders such that two consecutive biorders differ by exactly one pair (finiteness comes into play here); all biorders $\mathcal{R}_1=\varnothing$, $\mathcal{R}_2$, ..., $\mathcal{R}_k$ belong to the constructed family.

Clearly, there exists a trivial real $>$-representation of the empty biorder that is special and trace-compatible (for an example, take $f_1(b)=0$ for all $b\in A$, and $g_1(y) = 1$ for all $y \in Z$). Then repeated applications of Lemma 3 produces a real chain-$>$-representation of the constructed family, thus also a real chain-$>$-representation of the given family $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$. Moreover, the resulting representation is special and trace-compatible. $\square$

4.4 Remarks and extensions

As announced above, we observe that Proposition 4 offers an alternative proof of the fact that any biorder $\mathcal{R}$ from a finite set A to a finite set Z has a real representation which is special and trace-compatible. This is obvious observing that the single biorder $\mathcal{R}$ forms a nested family of biorder(s), to which Proposition 4 applies. As a matter of fact, we do not even need Lemma 2 here: it suffices to check that each nonempty biorder $\mathcal{R}$ from A to Z contains at least one pair (a, x) such that $\mathcal{R} \setminus \{(a, x)\}$ is again a biorder (in other words, the inner fringe of $\mathcal{R}$ is nonempty). To get such a pair (a, x), first select in A any element a with $a\mathcal{R}$ minimal for the left trace among the nonempty sets $b\mathcal{R}$, where $b \in A$ (minimal for the trace means minimal for the inclusion), and next pick any element x in $a\mathcal{R}$: that $\mathcal{R} \setminus \{(a, x)\}$ is again a biorder follows at once from Proposition 1(ii).

This new proof for the existence of a real representation is quite different from the previous ones given in the literature: it is in some sense "algorithmic", the representation with the desired property being built up step by step, with the number of steps equal to the number of pairs in $\mathcal{R}$.

Notice that, because our method of proof starts with the empty biorder to end up with the given biorder by addition of a single pair at each step, it also applies to interval orders. Indeed, an interval order $\mathcal{P}$ is nothing but a biorder from a set X to the same set X that is irreflexive. Our proof builds the real representation of the interval order $\mathcal{P}$ by forging the real representation of the biorder $\mathcal{R}$ from X' to X'' which is the duplication of $\mathcal{P}$ (remember Remark 1). Because the pairs (x', x''), for $x \in X$, are outside of both biorders $\varnothing$ and $\mathcal{R}$, they are never added to the current biorder (in terms of X, the pairs (x, x) are never added and the proof works only with irreflexive biorders).

For the record, we spell out the following:

Proposition 5
Let $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$ be a family of relations on the finite set X. There are real-valued functions $f_1, f_2, \ldots, f_k$ on A and $g_1, g_2, \ldots, g_k$ on Z such that, for all $x, y \in$

X and $i \in \{1, 2, \ldots, k\}$,

$$\begin{gathered} x \,\mathcal{P}_i\, y \Leftrightarrow f_i(x) > g_i(y), \\ f_k(x) \geq f_{k-1}(x) \geq \cdots \geq f_1(x), \\ g_1(x) \geq g_2(x) \geq \cdots \geq g_k(x), \\ f_i(x) \leq g_i(x), \end{gathered}$$

if and only if $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$ is a nested family of interval orders on X.

For a nested family of interval orders, there always exists a real $>$-representation which is special and trace-compatible.

Semiorders are the particular interval orders which are semitransitive, or equivalently the interval orders for which the left and right traces are never contradictory (Aleskerov, Bouyssou, and Monjardet, 2007, Monjardet, 1978). The following proposition immediately follows from Proposition 5.

Proposition 6
Let $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k)$ be a family of relations on the finite set X. There are real-valued functions $f_1, f_2, \ldots, f_k$ on A and $g_1, g_2, \ldots, g_k$ on Z such that, for all $x, y \in X$ and $i \in \{1, 2, \ldots, k\}$,

$$\begin{gathered} x \,\mathcal{S}_i\, y \Leftrightarrow f_i(x) > g_i(y), \\ f_k(x) \geq f_{k-1}(x) \geq \cdots \geq f_1(x), \\ g_1(x) \geq g_2(x) \geq \cdots \geq g_k(x), \\ f_i(x) \leq g_i(x), \\ f_i(x) > f_i(y) \Rightarrow g_i(x) \geq g_i(y) \end{gathered}$$

if and only if $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k)$ is a nested family of semiorders.

For any nested family of semiorders, the mappings f_1, f_2, ..., f_k and g_1, g_2, ..., g_k can be selected in order to moreover form a special $>$-representation which respects the traces.

A more difficult question asks whether any nested family of semiorders on a finite set admits some representation 'with no nesting' (Aleskerov et al., 2007, Fishburn, 1970a). The latter means a representation as in Proposition 6 which furthermore satisfies $f_i(x) \geq f_i(y) \Rightarrow g_i(x) \geq g_i(y)$, for all $x, y \in X$ and $i \in \{1, 2, \ldots, k\}$. An even more advanced question asks for the existence of some representation with constant thresholds, in the sense that $g_i = f_i + \tau_i$ for some positive constant τ_i, for $i = 1, 2, \ldots, k$ (Scott and Suppes, 1958). While we know that any semiorder defined on a denumerable set admits a representation with no nesting, and also that a semiorder defined on a finite set admits a constant threshold representation, extending the results to the case of nested families of semiorders looks as a delicate problem. We leave the latter questions on semiorders for further study. Crucial steps leading to positive answers would be proofs of lemmas similar to Lemma 3.

5 The Infinite Case

5.1 Definitions

When the sets A and Z are not restricted to be finite as in the preceding section, the situation becomes more difficult. Observe first that biorders can form finite or infinite families, which we now call 'chains of biorders' to emphasize the fact that they generalize the finite families studied until here. The notation BO designates again the collection of all biorders from A to Z.

Definition 6 (Chain of biorders)
A chain of biorders from A to Z consists in a nonempty index set I and a mapping from I to the collection BO, which we denote as $(\mathcal{R}_i)_{i\in I}$, such that for all i, j in I, either $\mathcal{R}_i \supset \mathcal{R}_j$ or $\mathcal{R}_j \supset \mathcal{R}_i$. ⌟

Clearly, when I is finite, a chain a biorders is nothing more than a nested family of biorders (up to an adequate renumbering of the biorders).

Definition 7 (Chain >-Representations)
A *chain representation* $(f_i, g_i)_{i\in I}$ of a chain $(\mathcal{R}_i)_{i\in I}$ of biorders consists in a linearly ordered set $(E, \geq)$ and, for each i in I, in two mappings $f_i : A \to E$ and $g_i : Z \to E$ such that

1. for all i in I, a in A and z in Z:
$$a \,\mathcal{R}_i\, z \Leftrightarrow f_i(a) > g_i(z);$$
2. for all i, j in I,
$$\mathcal{R}_i \subseteq \mathcal{R}_j \Rightarrow f_j \geq f_i \text{ and } g_i \geq g_j.$$

The representation is *real* when $(E, \geq) = (\mathbb{R}, \geq)$. The definition of *chain $\geq$-representations* is similar (with $\geq$ instead of $>$). ⌟

We first characterize chains of biorders that admit a chain >-representation in some linearly ordered set $(E, \geq)$. For real representations we have more questions than answers. From now on we concentrate on chain >-representations (results on $\geq$-representations are easily derived).

5.2 Chain representations in a linearly ordered set

According to Proposition 3, any biorder admits a representation in some linearly ordered set $(E, \geq)$. Hence, it is natural to ask whether the same holds for any chain of biorders. To derive a positive answer in Proposition 7 below, we introduce techniques different from the ones in Section 4.3 (in the finite case, we handled nested families of biorders with the crucial tool of weak gradedness). The need for a new technique can be grasped from the following counterexample showing that several statements on finite biorders in Doignon and Falmagne (1997) do not extend to the infinite setting.

Example 2
Here is a biorder whose inner and outer fringes are both empty. Let $A = Z = \mathbb{Q}$. Take any three real numbers which are linearly independent over $\mathbb{Q}$, for instance 1, $\sqrt{2}$ and $\sqrt{3}$. Setting $f(a) = \sqrt{2}\,a + \sqrt{3}$ for $a \in A$, and $g(z) = z$ for $z \in Z$, we obtain the biorder $\mathcal{R} = \{(a, z) \in A \times Z : \sqrt{2}\,a + \sqrt{3} > z\}$ for which (f, g) is a $>$-representation in $(\mathbb{Q}, \geq)$. Let us prove that the inner fringe $\mathcal{R}^{\mathcal{I}}$ is empty. For any pair (a, z) in $\mathcal{R}$, we have $\sqrt{2}\,a + \sqrt{3} > z$. There thus exist some rational number y such that $\sqrt{2}\,a + \sqrt{3} > y > z$ and then some rational number b such that $y \geq \sqrt{2}\,b + \sqrt{3} > z$. This gives $(a, y) \in \mathcal{R}$, $(b, y) \notin \mathcal{R}$, $(b, z) \in \mathcal{R}$. We obtain $(a, z) \in \mathcal{R}\overline{\mathcal{R}}^{-1}\mathcal{R}$, and thus $\mathcal{R}^{\mathcal{I}} = \varnothing$. One proves in a similar way $\mathcal{R}^{\mathcal{O}} = \varnothing$. $\diamond$

Proposition 7
Any chain of biorders has a chain $>$-representation in some linearly ordered set.

Proof
Let $(\mathcal{R}_i)_{i \in I}$ be a chain of biorders from A to Z. To build a chain $>$-representation $(f_i, g_i)_{i \in I}$ in the linearly ordered set $(E, \geq)$, we first specify a set E and, for each i in I, two mappings $f_i : A \to E$ and $g_i : Z \to E$ by letting

$$\begin{aligned} E &= (A \times I) \cup (Z \times I), \\ f_i(a) &= (a, i), \quad \text{for } a \in A, \\ g_i(z) &= (z, i), \quad \text{for } z \in Z. \end{aligned}$$

It remains to equip E with an adequate linear ordering. To this end, consider first the following set of pairs of E:

$$\begin{aligned} \mathcal{X} = &\{((a, j), (a, i)) \ : \ a \in A,\ \mathcal{R}_i \subsetneq \mathcal{R}_j\} \cup \\ &\{((z, i), (z, j)) \ : \ z \in Z,\ \mathcal{R}_i \subsetneq \mathcal{R}_j\} \cup \\ &\{((a, i), (z, i)) \ : \ a\, \mathcal{R}_i\, z\} \cup \\ &\{((z, i), (a, i)) \ : \ a\, \overline{\mathcal{R}_i}^{-1}\, z\}. \end{aligned}$$

Remark that $\mathcal{X}$ consists exactly of the pairs forced in the linear ordering $\geq$ of E by any representation of the given chain in $(E, \geq)$. Now we check that $\mathcal{X}$ is acyclic, that is, $\mathcal{X}$ has no cycle. Here we view a cycle of $\mathcal{X}$ as a finite sequence e_1, e_2, $\dots$, e_k of elements in E such that (e_1, e_2), (e_2, e_3), $\dots$, (e_{k-1}, e_k), (e_k, e_1) are all in $\mathcal{X}$. The length of the latter cycle is its number k of elements.

Here are properties of such a cycle $\mathcal{C}$, where we always assume a, $b \in A$ and y, $z \in Z$:

(i) no four successive elements of $\mathcal{C}$ can have a same second component i: if not, we would meet, say, either (a,i), (y,i), (b,i) and (z,i) along the cycle, or (y,i), (a,i), (z,i) and (b,i) along the cycle. In the first case we have $a\ \mathcal{R}_i\ y\ \overline{\mathcal{R}}^{-1}\ b\ \mathcal{R}_i\ z$ which implies $a\ \mathcal{R}_i\ z$ and we could shorten the cycle by replacing (a,i), (y,i), (b,i) and (z,i) with (a,i), (z,i), a contradiction. A similar contradiction occurs in the second case.

(ii) no three successive elements of $\mathcal{C}$ can have a same second component i: if not, we would meet, say, either (a,i), (z,i) and (b,i) along the cycle, or (y,i), (a,i) and (z,i) along the cycle. In the first case, by (i), we must have some (a,j), with $\mathcal{R}_j \supset \mathcal{R}_i$, before (a,i) in the cycle and some (b,h) with $\mathcal{R}_i \supset \mathcal{R}_h$ after (b,i) in the cycle. Then replacing (a,j), (a,i), (z,i), (b,i) (b,h) with (a,i), (z,j), (z,h), (b,h) shortens the cycle, a contradiction. The second case also leads to a contradiction.

(iii) if (a,i) and (z,i) are successive elements of the cycle $\mathcal{C}$, then by (ii) the cycle contains successive elements (a,j), (a,i), (z,i) and (z,k) with $\mathcal{R}_j \supset \mathcal{R}_i$ and $\mathcal{R}_k \supset \mathcal{R}_i$. We must then have $j = k$, otherwise the cycle could be shortened (use (a,k) or (z,j)).

(iv) similarly, if (z,i) and (b,i) are successive elements in $\mathcal{C}$, then we have also successive elements (z,h), (z,i), (b,i) and (b,h) in the cycle.

Properties (i)–(iv) of $\mathcal{C}$ imply that $\mathcal{C}$ must be of the form (a,j), (a,i), (z,i), (z,k), (a,j) with $\mathcal{R}_k \supset \mathcal{R}_i$, which is impossible. So we have proved that $\mathcal{X}$ is an acyclic relation on E. By the following lemma, there exists a strict linear ordering $>$ on E extending $\mathcal{X}$. After taking the reflexive closure of $>$, we obtain a linear order $\geq$. Then $(f_i, g_i)_{i \in I}$ is a chain $>$-representation of the chain $(\mathcal{R}_i)_{i \in I}$ of biorders in $(E, \geq)$. $\square$

The following lemma is a well-known fact. We sketch its proof for completeness.

Lemma 4

Let $\mathcal{X}$ be an irreflexive relation on an arbitrary nonempty set E. There exists a strict linear ordering $>$ on E extending $\mathcal{X}$ (that is, $x\ \mathcal{X}\ y$ implies $x > y$) if and only if the relation $\mathcal{X}$ is acyclic.

PROOF

Necessity being obvious, we prove only sufficiency. First define a relation $\mathcal{R}$ (the transitive closure of $\mathcal{X}$) by letting, for a, b in E,

$$a \,\mathcal{R}\, b \Leftrightarrow \text{there exist } n \text{ in } \mathbb{N} \text{ and } c_1, c_2, \ldots, c_n \text{ in } E \text{ such that}$$
$$a \,\mathcal{X}\, c_1,\ c_1 \,\mathcal{X}\, c_2,\ c_2 \,\mathcal{X}\, c_3,\ \ldots,\ c_{n-1} \,\mathcal{X}\, c_n,\ c_n \,\mathcal{X}\, b.$$

The acyclicity of $\mathcal{X}$ implies that $\mathcal{R}$ is a strict partial order on E. By the main result of Szpilrajn (1930), $\mathcal{R}$ is contained in some strict linear ordering of E. □

Here is an easy reinforcement of Proposition 7.

Corollary 1

Any chain $(\mathcal{R}_i)_{i \in I}$ of biorders from A to Z has a chain $>$-representation in some linearly ordered set, such that the representation of any biorder $\mathcal{R}_i$ in the chain is special and trace-compatible.

PROOF

In the proof of Proposition 7, we modify the definition of E. Using the equivalence classes and quotient sets of the equivalence relations $\sim^A_{\mathcal{R}_i}$ and $\sim^Z_{\mathcal{R}_i}$ for each biorder $\mathcal{R}_i$ in the chain, we let

$$\begin{aligned} E &= \bigcup_{i \in I} (A/\!\sim^A_{\mathcal{R}_i}, i) \cup (Z/\!\sim^Z_{\mathcal{R}_i} \times I), \\ f_i(a) &= (\sim^A_{\mathcal{R}_i} a, i), \quad \text{for } a \in A, \\ g_i(z) &= (\sim^Z_{\mathcal{R}_i} z, i), \quad \text{for } z \in Z. \end{aligned}$$

The definition of the relation $\mathcal{X}$ on E is now

$$\begin{aligned} \mathcal{X} \;=\; & \{((\sim^A_{\mathcal{R}_j} a, j), (\sim^A_{\mathcal{R}_i} a, i)) \;:\; a \in A,\ \mathcal{R}_i \subsetneq \mathcal{R}_j\} \cup \\ & \{((\sim^Z_{\mathcal{R}_i} z, i), (\sim^Z_{\mathcal{R}_j} z, j)) \;:\; z \in Z,\ \mathcal{R}_i \subsetneq \mathcal{R}_j\} \cup \\ & \{((\sim^A_{\mathcal{R}_i} a, i), (\sim^Z_{\mathcal{R}_i} z, i)) \;:\; a \,\mathcal{R}_i\, z\} \cup \\ & \{((\sim^Z_{\mathcal{R}_i} z, i), (\sim^A_{\mathcal{R}_i} a, i)) \;:\; a \,\overline{\mathcal{R}_i}^{-1}\, z\}. \end{aligned}$$

The rest of the proof is similar to the one of Proposition 7. □

Proposition 7 is quite general since it covers any chain of biorders (the index set I may be finite, countably infinite or uncountable), defined from any set A to any set Z. Now, if we restrict attention to denumerable chains of nested biorders with A and Z denumerable sets, it is clear that the set E built in the proof of Proposition 7 is denumerable. Then in the statement we may replace E with $\mathbb{R}$.

Corollary 2
Any denumerable chain of biorders from a denumerable set to another denumerable set has a real chain $>$-representation which is special and trace-compatible.

PROOF
The domain of the linearly ordered set $(E, \geq)$ built in the proof of Corollary 1 is the disjoint union of copies of $A \cup Z$, and the number of copies is just the number of biorders in the actual chain. Hence for any denumerable chain of biorders on denumerable sets A and Z, the set E is itself denumerable. Consequently, there is an order embedding of $(E, \geq)$ in $(\mathbb{R}, \geq)$ and by composing the $>$-representation from Proposition 7 with the embedding we get a real $>$-representation. □

Notice that the way we obtained Corollary 2 provides another proof of Proposition 4, a proof which moreover applies to the countable case. Also, Corollary 1 does not extend to all chains of biorders on countable domains. The reason lies in the existence of uncountable chains of biorders on infinite countable domains, as the following example shows.

Example 3
Here is an uncountable chain of biorders on countable domains (a similar example thus exists for any pair of countable domains). As in Example 2, let $A = Z = \mathbb{Q}$. Then, for any r in $\mathbb{R}$, set $f_r(a) = a$ and $g_r(z) = z + r$ (where $a \in A$ and $z \in Z$). The pair (f_r, g_r) of real-valued mappings defines the biorder $\mathcal{R}_r = \{(a, z) \in A \times Z : a > z + r\}$. Notice that if the real numbers r and s differ, then $\mathcal{R}_r \neq \mathcal{R}_s$ (in geometric terms: given two parallel lines in the real plane, there are points with rational coordinates lying strictly in between the two lines). We conclude that the biorders $\mathcal{R}_r$, for $r \in \mathbb{R}$, form an uncountable chain. ◇

5.3 Real chain representations

We now turn to the study of real chain $>$-representations of chains of biorders in the general case. Having few results, we mainly offer examples and open problems. It is useful to examine first the existence of real representations of a single biorder in the general case.

5.3.1 Single biorders

Proposition 3 shows that any biorder has a special $>$-representation respecting the traces in some linearly ordered set $(E, \geq)$. This is clearly no more true when we take E to be $\mathbb{R}$. Indeed, a special $>$-representation is also a $\geq$-representation. But there are biorders on $\mathbb{R}$ having a $>$-representation and no $\geq$-representation and, hence, no special representation.

Example 4
Let $A = Z = \mathbb{R}$, after taking adequate disjoint duplications, and $\mathcal{R} = \{(a, z) \in A \times Z \ : \ a \geq z\}$. The relation $\mathcal{R}$ is a biorder which does not admit any real $>$-representation. That $\mathcal{R}$ is a biorder is clear because it admits a $\geq$-representation. Now suppose that $\mathcal{R}$ admits some real $>$-representation in $(\mathbb{R}, \geq)$ using functions f and g, that is $a \geq z \Leftrightarrow f(a) > g(z)$ (for all a in A and z in Z). Notice that with $a = z$ we get $f(a) > g(a)$. Moreover, two open intervals $]f(a), g(a)[$ and $]f(b), g(b)[$, where a, $b \in \mathbb{R}$ with $a \neq b$, are disjoint. Indeed, if $a < b$, we get $a \, \overline{\mathcal{R}} \, b$, and so $f(a) \leq g(b)$. Now selecting some rational number q_a in $]f(a), g(a)[$, we form a collection $(q_a)_{a \in \mathbb{R}}$ of distinct rational numbers, in contradiction with the countability of $\mathbb{Q}$. $\diamond$

Here is a necessary condition for real $\geq$-representability. It is taken from Doignon et al. (1984) (other equivalent conditions can be found in Nakamura, 2002).

Definition 8
Let $\mathcal{R}$ be a relation from A to Z. A countable subset M^* of $A \cup Z$ is *widely dense* for $\mathcal{R}$ when, for all a in A and x in Z, there follows from $a \, \mathcal{R} \, x$ the existence of some $m^* \in M^*$ such that

$$m^* \in Z, \ a \, \mathcal{R} \, m^*, \ \text{and} \ m^* \succsim_{\mathcal{R}}^{Z} x,$$

or

$$m^* \in A, \ a \succsim_{\mathcal{R}}^{A} m^*, \ \text{and} \ m^* \, \mathcal{R} \, d.$$ ⌟

The following statement is Proposition 9 in Doignon et al. (1984).

Proposition 8
A biorder $\mathcal{R}$ has a real $\geq$-representation if and only if there is a countable subset M^* of $A \cup Z$ that is widely dense for $\mathcal{R}$. This representation can always be chosen so as to respect the traces.

The proof of Proposition 8 in Doignon et al. (1984) uses a construction that differs from (but resembles) the one used in Proposition 3 with the relation $\mathcal{Q}_m$ on $A \cup Z$. This is because the relation $\mathcal{Q}_m$ always produces a special representation, and any real special representation is at the same time a $>$-representation and a $\geq$-representation. But there are biorders having a $>$-representation while having no $\geq$-representation (take $\overline{\mathcal{R}}^{-1}$ with $\mathcal{R}$ as in Example 4). Now the proof of Doignon et al. (1984) uses a different construction dating back to Bouchet (1981) and Cogis (1982a,b). Let $\mathcal{R}$ be a relation (not necessarily a biorder here) from A to Z. Among all quasi orders (that is, reflexive and transitive relations) $\mathcal{W}$ on $A \cup Z$ such that $\mathcal{W} \cap (A \times Z) = \mathcal{R}$, there is one which includes all the other ones; it is

Table 1: The four parts of the maximum quasi order $\mathcal{Q}_M$ attached to a relation $\mathcal{R}$.

$\mathcal{Q}_M$	A	Z
A	$\overline{\overline{\mathcal{R}}\ \mathcal{R}^{-1}}$	$\mathcal{R}$
Z	$\overline{\mathcal{R}^{-1}\ \overline{\mathcal{R}}\ \mathcal{R}^{-1}}$	$\overline{\mathcal{R}^{-1}\ \overline{\mathcal{R}}}$

the *maximum quasi order* $\mathcal{Q}_M$ attached to $\mathcal{R}$. Then $\mathcal{Q}_M$ is the union of the four relations in the cells of Table 1. Moreover, the quasi order $\mathcal{Q}_M$ is complete if and only if the relation $\mathcal{R}$ is a biorder. The proof of Proposition 8 by Doignon et al. (1984) then consists in showing that the existence of a widely dense subset for $\mathcal{R}$ implies that $\mathcal{Q}_M$ has a real representation $h: A \times Z \to \mathbb{R}$ as a weak order (meaning $\alpha \ \mathcal{Q}_M \ \beta \Leftrightarrow h(\alpha) \geq h(\beta)$). This directly leads to a real $\geq$-representation f, g of $\mathcal{R}$ respecting the traces (just take for f and g the restrictions of h to respectively A and Z). Observe that the representation f, g is in general not special: we may have $f(a) = g(x)$, which happens exactly when the pair (a, x) is in the outer fringe of $\overline{\mathcal{R}}^{-1}$ as is easily checked. To obtain a real $>$-representation, it suffices to apply the same process to $\overline{\mathcal{R}}^{-1}$ instead of $\mathcal{R}$.

Bosi, Candeal, Induráin, Oloriz, and Zudaire (2001) compare methods for establishing real representations of interval orders (the latter are particular cases of biorders).

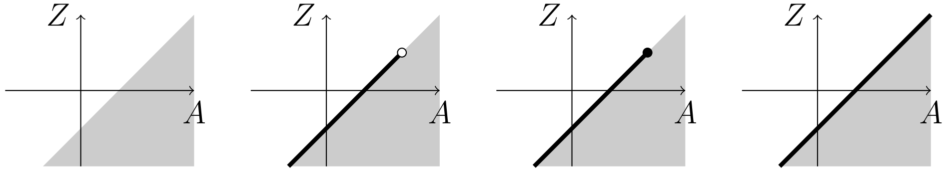
5.3.2 Chains of biorders on infinite sets: open problems

We now turn to the investigation of real chain representations of nested families of biorders in the infinite case. Notice that this means not only that the sets A and Z may be infinite, but also that the chain may contain infinitely many biorders.

We have almost no result here, even under the assumption that the chain of biorders is finite (in other words, it is a nested family of biorders). We will simply illustrate the difficulty of the problem by giving two (related) examples. Our first example is a chain of biorders for which all relations in the chain have empty fringes.

Example 5

Here is a chain of biorders which all have empty inner fringes. Let $A = Z = \mathbb{R}$ and, for $r \in \mathbb{R}$, set $f_r(a) = a$ and $g_r(z) = z + r$ (where $a \in A$ and $z \in Z$). The pair (f_r, g_r) of real-valued mappings defines the biorder $\mathcal{R}_r = \{(a, z) \in A \times Z : a > z + r\}$. For any pair (a, z) in $\mathcal{R}_r$, there are real numbers y and b such that $a > y + r > z + r$, and then $y + r \geq b > z + r$, and so $(a, y) \in \mathcal{R}_r$, $(b, y) \notin \mathcal{R}_r$, $(b, z) \in \mathcal{R}_r$. We obtain $(a, z) \in \mathcal{R}_r \overline{\mathcal{R}_r}^{-1} \mathcal{R}_r$, thus $\mathcal{R}_r^{\mathcal{I}} = \varnothing$. For $r, s \in \mathbb{R}$, notice $\mathcal{R}_r \subsetneq \mathcal{R}_s$ if and only if $r > s$. Thus $(\mathcal{R}_r)_{r \in \mathbb{R}}$ is a chain of biorders, all satisfying $\mathcal{R}_r = \cup\{\mathcal{R}_s : s \in \mathbb{R}, r > s\}$. However $(\mathcal{R}_r)_{r \in \mathbb{R}}$, even after the addition of the empty and the full biorders, is not a maximal chain (see next example). ◇



$\mathcal{R}_r$ $\quad$ $\mathcal{R}_{r,s}$ $\quad$ $\mathcal{R}^+_{r,s}$ $\quad$ $\mathcal{R}^+_r$

Figure 5: Some biorders from the chain in Example 6.

It is not difficult to show that any chain of biorders is included in some maximal chain of biorders (refering to the Axiom of Choice). Working with maximal chains families does not change the picture revealed by the preceding example, as shown below.

Example 6
We build one of the (infinitely many) maximal chains of biorders containing the family $(\mathcal{R}_r)_{r \in \mathbb{R}}$ from the preceding example. A geometric viewpoint is helpful here. Each biorder $\mathcal{R}_r$, consisting of pairs (a, z) from $A \times Z$, is a subset of $\mathbb{R}^2$ (remember $A = Z = \mathbb{R}$); more precisely, $\mathcal{R}_r$ is the open half-plane defined by the inequality $a > z + r$ (we keep the letters a and z to denote the two coordinates of the point (a, z) in $\mathbb{R}^2$). Now consider the additional following subsets, where $(r, s) \in \mathbb{R}^2$ (see Figure 5 for an illustration):

$$\begin{aligned}
\mathcal{R}_{r,s} &= \mathcal{R}_r \cup \{(a + r, a) : a < s\}, \\
\mathcal{R}^+_{r,s} &= \mathcal{R}_{r,s} \cup \{(s + r, s)\}, \\
\mathcal{R}^+_r &= \mathcal{R}_r \cup \{(a + r, a) : a \in \mathbb{R}\}.
\end{aligned}$$

It is easily checked that the whole collection $\{\mathcal{R}_r,\ \mathcal{R}_{r,s},\ \mathcal{R}^+_{r,s},\ \mathcal{R}^+_r\colon (r, s) \in \mathbb{R}^2\}$ is a chain of biorders. Moreover, augmented by the empty and the full biorders, the chain becomes a maximal chain. ◇

Although all biorders forming the chain in Example 5 have a real representation, this is not true for all biorders in the extended, maximal chain in Example 6 (remember Example 4). Because of the last fact, it seems difficult to find conditions for the real representability of infinite chains of biorders—even for those chains of biorders which all admit individually a real representation.

Also for the apparently simpler question of characterizing finite nested chains of biorders having a real representation, we have no clear answer. An important open problem is the following: *given a nested finite family of biorders all having a real $>$-representation, what are the conditions allowing to build a real chain representation of this nested family?*

This will eventually lead us to work out a notion of weak gradedness (or well-gradedness) in more details on infinite sets. Some results on a type of well-gradedness on infinite sets are available in Doignon and Falmagne (2011, Ch. 4).

6 Discussion

We summarize the (apparently) new results established above and list open problems for further investigation. All results concern the existence of chain representations of nested families, or more generally chains, of biorders; they fall into two types.

Any chain of biorders from a set A to a set Z has a chain $>$-representation in some adequately chosen linearly ordered set $(E, \geq)$ (Proposition 7). The proof builds an acyclic relation on the union of disjoint duplications of A and Z, as many duplication as there are biorders in the chain. Applied to the special family $(\varnothing, \mathcal{R}, A \times Z)$ of biorders, it delivers a new argument for the existence of a $>$-representation (or $\geq$-representation) of any biorder $\mathcal{R}$ in some linearly ordered set $(E, \geq)$ (Proposition 3, established in Subsection 4.4); the argument is quite different from the ones existing in the literature (as for instance in Doignon et al., 1984, 1987).

Assume now that the target $(E, \geq)$ is the linearly ordered set of the reals, and that biorders are from a finite set A to a finite set Z. Each nested family of biorders has a real representation (Proposition 4). The proof is based on the notion of weak gradedness (a weakening of well-gradedness, Doignon and Falmagne, 1997). Weak gradedness asserts that for any two biorders $\mathcal{R}$ and $\mathcal{S}$ from A to Z such that $\mathcal{R} \subsetneq \mathcal{S}$, there is a sequence of elementary transformations, each one consisting of the addition of a single pair, which transforms $\mathcal{R}$ into $\mathcal{S}$ while producing a biorder at each step (weak gradedness follows from Lemma 3). For a single biorder from A to Z, the argument for the existence of a real representation in Subsection 4.4 is quite different from those in the literature (Doignon et al., 1984, Ducamp and Falmagne, 1969). Exactly as for a nested family of biorders, it is a constructive argument. The proof directly extends to the construction of a chain representation for any nested family of interval orders or semiorders (Propositions 5 and 6) again on finite sets; in the case of semiorders, the representations obtained are without proper nesting (Aleskerov et al., 2007).

Quite a few problems are left unsolved. Here are the most intriguing ones in our view.

(i) Working with chain representations of chains of biorders benefits from a better understanding of the collection of all representations of a *single* biorder. We obtained several results on the structure of the collection, whether or not the representations are required to be special and/or to respect the traces. More remains to be done however.

(ii) Among the chain representations of a nested family of semiorders (as in Proposition 6), do there exist representations which, for each of the semiorders, avoid proper nesting, and even are constant threshold representations? See Aleskerov et al. (2007), Doignon et al. (1987) for the latter notions in the case of a single semiorder.

(iii) Finally, we do not have much knowledge about real chain representations in the general case (that is, with no restrictions on the cardinalities of the sets A and Z), even for finite nested families of biorders. We repeat an important open question: let $(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k)$ be a nested family of $k \geq 1$ biorders from the set A to the set Z. Suppose furthermore that each of these biorders has a real $>$-representation (or $\geq$-representation). When is it true that this finite family of nested biorders has a chain representation?

As far as we know, papers on nested families of relations assume that the left or the right traces are compatible (meaning that the intersection of the traces is complete, see for instance Doignon et al., 1988). Working without any compatibility assumption proves to be more difficult but also quite rewarding: for instance, new proofs for the existence of a representation for a single relation are important by-products of our investigations.

Appendix

A Binary relations on a set

A binary relation S on a set X is a subset of $X \times X$. For $x, y \in X$, we often write, as is usual, $x\ S\ y$ instead of $(x, y) \in S$. Whenever the symbol $\succsim$ denotes a binary relation, $\succ$ stands for its asymmetric part ($x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$) and $\sim$ stands for its symmetric part ($x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$). A similar convention holds when subscripts or superscripts appear to $\succsim$.

A binary relation S on X is

(i) *reflexive* if $x \; S \; y$,

(ii) *irreflexive* if $Not[x \; S \; y]$,

(iii) *complete* if $x \; S \; y$ or $y \; S \; x$,

(iv) *symmetric* if $x \; S \; y$ implies $y \; S \; x$,

(v) *asymmetric* if $x \; S \; y$ implies $Not[y \; S \; x]$,

(vi) *antisymmetric* if $x \; S \; y$ and $y \; S \; x$ imply $x = y$,

(vii) *transitive* if $x \; S \; y$ and $y \; S \; z$ imply $x \; S \; z$,

(viii) *Ferrers* if $[x \; S \; y$ and $z \; S \; w] \Rightarrow [x \; S \; w$ or $z \; S \; y]$,

(ix) *semitransitive* if $[x \; S \; y$ and $y \; S \; z] \Rightarrow [x \; S \; w$ or $w \; S \; z]$,

for all $x, y, z, w \in X$.

We list below a number of remarkable structures. A binary relation S on X is

(i) a *quasi order* if it is reflexive and transitive,

(ii) a *weak order* or a *complete preorder* if it is complete and transitive,

(iii) a *linear order* if it is an antisymmetric weak order,

(iv) a *strict linear order* if it is the asymmetric part of a linear order,

(v) an *equivalence* if it is reflexive, symmetric, and transitive,

(vi) a *strict partial order* if it is irreflexive and transitive,

(vii) *interval order* if it is irreflexive and Ferrers;

(viii) *semiorder* if it is irreflexive, Ferrers and semitransitive.

It is well known that an equivalence relation S partitions the set X into equivalence classes. The equivalence class of an element x of X is denoted Sx. The set X/S of all equivalence classes of X under S is the *quotient* of X by S. Any weak order S on X leads to the equivalence relation $S \cap S^{-1}$ on X, whose classes are also the *classes* of S.

A relation S generates a *left trace* $\succsim_S^\ell$ and a *right trace* $\succsim_S^r$ on X, which are the binary relations on X such that

$$x \succsim_S^\ell y \Leftrightarrow [y \; S \; z \Rightarrow x \; S \; z],$$
$$x \succsim_S^r y \Leftrightarrow [z \; S \; x \Rightarrow z \; S \; y]$$

for all $x, y, z \in X$. It is clear that $\succsim_S^\ell$ and $\succsim_S^r$ are always quasi orders.

A characteristic feature of interval orders combines their irreflexivity with the facts that their left and right traces are complete (and therefore weak orders).

A characteristic feature of semiorders is that they are interval orders with noncontradictory left and right traces. The last condition requires that $x \succ_S^\ell y$ and $y \succ_S^r x$ never occur together, or equivalently that the relation defined as $\succsim_S = \succsim_S^\ell \cap \succsim_S^r$ is complete. The relation $\succsim_S$ is thus a weak order, called the *trace* of the semiorder.

A characteristic feature of weak orders is that they are identical to their trace.

References

F. Aleskerov, D. Bouyssou, and B. Monjardet. *Utility Maximization, Choice and Preference*. Springer-Verlag, Berlin, 2nd edition, 2007.

H. D. Block and J. Marschak. Random orderings and stochastic theories of responses. In I. Olkin, S.G. Ghurye, W. Hoeffding, W.G. Madow, and H.B. Mann, editors, *Contibutions to Probability and Statistics*, Stanford studies in mathematics and statistics, pages 97–132. Stanford University Press, Stanford, 1960.

B. Bosi, J. C. Candeal, E. Induráin, E. Oloriz, and M. Zudaire. Numerical representations of interval orders. *Order*, 18(2):171–190, 2001.

A. Bouchet. *Étude combinatoire des ordonnés finis. Applications*. Thèse, Université Scientifique et Médicale de Grenoble, Grenoble, France, 1981.

D. Bouyssou and Th. Marchant. Biorders with frontier. *Order*, 28(1):53–87, 2011.

D. S. Bridges and G. B. Mehta. *Representations of Preferences Orderings*. Number 422 in Lecture Notes in Economics and Mathematical Systems. Springer-Verlag Berlin Heidelberg, 1st edition, 1995.

G. Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers*. Dover Publications, 1955. Beiträge zur Begründung der transfiniten Mengenlehre, Translation of the original text in German dated 1895–1897.

O. Cogis. On the Ferrers dimension of a digraph. *Discrete Mathematics*, 38(1):47–52, 1982a.

O. Cogis. Ferrers digraphs and threshold graphs. *Discrete Mathematics*, 38(1):33–46, 1982b.

G. Debreu. Representation of a preference ordering by a numerical function. In R. Thrall, C. H. Coombs, and R. Davies, editors, *Decision Processes*, pages 159–175. Wiley, New York, 1954.

J.-P. Doignon. Partial structures of preference. In J. Kacprzyk and M. Roubens, editors, *Non-conventional Preference Relations in Decision Making*, number 301 in Lecture Notes in Economics and Mathematical Systems, pages 22–35. Springer-Verlag, Berlin, 1988.

J.-P. Doignon and J.-Cl. Falmagne. Well-graded families of relations. *Discrete Mathematics*, 173(1–3):35–44, 1997.

J.-P. Doignon and J.-Cl. Falmagne. *Learning Spaces*. Springer, Berlin Heidelberg, 2011.

J.-P. Doignon, A. Ducamp, and J.-Cl. Falmagne. On realizable biorders and the biorder dimension of a relation. *Journal of Mathematical Psychology*, 28(1):73–109, 1984.

J.-P. Doignon, A. Ducamp, and J.-Cl. Falmagne. On the separation of two relations by a biorder or a semiorder. *Mathematical Social Sciences*, 13(1):1–18, 1987.

J.-P. Doignon, B. Monjardet, M. Roubens, and Ph. Vincke. Biorder families, valued relations and preference modelling. *Journal of Mathematical Psychology*, 30(1):435–480, 1988.

A. Ducamp and J.-Cl. Falmagne. Composite measurement. *Journal of Mathematical Psychology*, 6(3):359–390, 1969.

P. C. Fishburn. *Utility Theory for Decision-making*. Wiley, New York, 1970a.

P. C. Fishburn. Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology*, 7(1):144–149, 1970b.

P. C. Fishburn. Binary choice probabilities: On the varieties of stochastic transitivity. *Journal of Mathematical Psychology*, 10(4):327–352, 1973a.

P. C. Fishburn. Interval representations for interval orders and semiorders. *Journal of Mathematical Psychology*, 10(1):91–105, 1973b.

P. C. Fishburn. *Interval Orders and Intervals Graphs*. Wiley, New York, 1985.

L. Guttman. A basis for scaling qualitative data. *American Sociological Review*, 9(2): 139–150, 1944.

L. Guttman. The basis for scalogram analysis. In S. A. Stouffer, L. Guttman, E. A. Suchman, P. F. Lazarsfeld, S. A. Star, and J. A Clausen, editors, *Measurement and Prediction*, volume IV of *Studies in Social Psychology in World War II*, pages 60–90, Princeton, New Jersey, 1950. Princeton University Press.

R. D. Luce. Semiorders and a theory of utility discrimination. *Econometrica*, 24(2): 178–191, 1956.

R. D. Luce and P. Suppes. Preference, utility, and subjective probability. In R. D. Luce, R. R. Bush, and E. Galanter, editors, *Handbook of Mathematical Psychology*, vol. 3. Wiley, New York, 1965.

A. A. J. Marley. Some probabilistic models of simple choice and ranking. *Journal of Mathematical Psychology*, 5(2):311–332, 1968.

J. Marschak. Binary-choice constraints and random utility indicators. In K. J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, pages 312–329. Stanford University Press, Stanford, 1960.

B. Monjardet. Axiomatiques et propriétés des quasi-ordres. *Mathématiques et Sciences Humaines*, 63:51–82, 1978.

B. Monjardet. Probabilistic consistency, homogeneous families of relations and linear Λ-relations. In E. Degreef and J. van Buggenhaut, editors, *Trends in Mathematical Psychology*, pages 271–281. North-Holland, Amsterdam, 1984.

B. Monjardet. A generalization of probabilistic consistency: linearity conditions for valued preference relation. In J. Kacprzyk and M. Roubens, editors, *Non-conventional Preference Relations in Decision Making*, number 301 in LNEMS, pages 36–53. Springer-Verlag, Berlin, 1988.

Y. Nakamura. Real interval representations. *Journal of Mathematical Psychology*, 46 (2):140–177, 2002.

I. Rabinovitch. The dimension of semiorders. *Journal of Combinatorial Theory*, 25(1): 50–61, 1978.

F. S. Roberts. Homogeneous families of semiorders and theory of probabilistic consistency. *Journal of Mathematical Psychology*, 8(2):248–263, 1971.

F. S. Roberts. *Measurement Theory with Applications to Decision Making, Utility and the Social Sciences*. Number 7 in Encyclopedia of Mathematics and its Applications. Addison-Wesley, Reading, 1979.

M. Roubens and Ph. Vincke. *Preference Modelling*. Number 250 in LNEMS. Springer-Verlag, Berlin, 1985.

D. Scott and P. Suppes. Foundational aspects of theories of Measurement. *Journal of Symbolic Logic*, 23(2):113–128, 1958.

P. Suppes, D. H. Krantz, R. D. Luce, and A. Tversky. *Foundations of measurement*, volume 2: Geometrical, threshold, and probabilistic representations. Academic Press, New York, 1989.

E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16(1):386–389, 1930.