Notes on strict concordance relations [1]

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Abstract

The idea of concordance is central to many MCDM techniques. It leads to comparing alternatives by pairs on the basis of a comparison in terms of importance of the coalitions of attributes favoring each element of the pair. Such a way of comparing alternatives has a definite "ordinal" flavor. It is well-know that it may lead to relations that do not possess any remarkable transitivity properties. This paper shows how to use standard conjoint measurement techniques to characterize such relations. Their main distinctive feature is shown to lie in their very crude way to distinguish various levels of preference differences on each attribute.

Keywords: Conjoint measurement, Strict concordance relations, Nontransitive preferences, Noncompensatory preferences.

1 Introduction

Let x and y be two alternatives evaluated on several attributes. A simple way to compare these two alternatives, taking all attributes into account, goes as follows:

- compare the evaluations of x and y on attribute i and decide whether attribute i favors x, favors y or favors none of x and y. Repeat this operation for each attribute. This leads to defining three disjoints subsets of attributes: those favoring x, those favoring y and those for which none of the two alternatives is favored,

- compare the set of attributes favoring x with the set of attributes favoring y in terms of "importance",

- declare that "x is preferred to y" if the set of attributes favoring x is "more important than" the set of attributes favoring y.

This way of comparing alternatives has a definite "ordinal" flavor and several of its particular cases (e.g. weighted majority comparisons) have been advocated by psychologists (see Russo and Dosher, 1983; Tversky, 1969) as simple heuristics for comparing objects using an "intra-dimensional" information processing strategy. It is also at work in several well-known multi-attribute techniques, usually classified under the heading "outranking methods" (see Roy, 1991, 1996; Vansnick, 1986). The purpose of this paper is, within a classical conjoint measurement framework, to characterize the type of preference relations that may arise from such a way of comparing alternatives.

Simple examples inspired by Condorcet's paradox show that this mode of comparing alternatives does not always lead to preference relations, henceforth called *strict concordance relations*, having "nice" transitivity properties. Therefore such preferences appear as quite distinct from the transitive structures usually studied in conjoint measurement (see Krantz et al., 1971; Wakker, 1989), e.g. those representable by an additive utility model. Adopting a framework for conjoint measurement tolerating intransitive preferences (see Bouyssou and Pirlot, 2002b, 2004a) will enable us to characterize strict concordance relations using axioms that will emphasize their main specific feature, i.e. the very crude way in which they isolate various levels of "preference differences" on each attribute. The results presented here extend the preliminary analysis in Bouyssou and Pirlot (2002a).

An earlier study of preference relations induced by ordinal aggregation methods in a conjoint measurement framework is due to Fishburn (1975, 1976) through his definition of *noncompensatory preferences*. It has long been thought that noncompensatory preferences provided the adequate framework for the analysis of preferences generated by ordinal aggregation methods and Fishburn's definition has received much attention in the field of decision analysis with multiple attributes. It will however turn out that noncompensatory preferences *à la* Fishburn are not totally adequate to deal with the whole variety of strict concordance relations.

This paper is organized as follows. We introduce our setting in section 2. Strict concordance relations are defined and illustrated in section 3. Our general framework for conjoint measurement allowing for nontransitive preferences is presented in section 4. Section 5 characterizes strict concordance relations within this general framework. A final section discusses our results and presents directions for future research.

2 Definitions and Notation

In this paper we consider a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements of X will be interpreted as *alternatives* evaluated on a set $N = \{1, 2, \ldots, n\}$ of *attributes*. When $J \subseteq N$, we denote by X_J (resp. X_{-J}) the set $\prod_{i \in J} X_i$ (resp. $\prod_{i \notin J} X_i$). With customary abuse of notation, (x_J, y_{-J}) will denote the element $w \in X$ such that $w_i = x_i$ if $i \in J$ and $w_i = y_i$ otherwise. When $J = \{i\}$, we simply write X_{-i} and (x_i, y_{-i}).

We use $\mathcal{P}$ to denote an asymmetric binary relation on X interpreted as a *strict preference relation* between alternatives. The *symmetric complement* of $\mathcal{P}$ is denoted by $\mathcal{I}$.

Let $J \subseteq N$ be a nonempty set of attributes. We define the *marginal preference* $\mathcal{P}_J$ induced on X_J by $\mathcal{P}$ letting, for all $x_J, y_J \in X_J$:

$$x_J \; \mathcal{P}_J \; y_J \Leftrightarrow (x_J, z_{-J}) \; \mathcal{P} \; (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with symmetric complement $\mathcal{I}_J$. When $J = \{i\}$, we write $\mathcal{P}_i$ instead of $\mathcal{P}_{\{i\}}$. If, for all $J \subseteq N$ and all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \; \mathcal{P} \; (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \; \mathcal{P}_J \; y_J,$$

we say that $\mathcal{P}$ is independent for J. If $\mathcal{P}$ is independent for all nonempty subsets of attributes we say that $\mathcal{P}$ is *independent*.

We say that attribute $i \in N$ is *influent* (for $\mathcal{P}$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$ and *Not*$[(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})]$ and *degenerate* otherwise. It is clear that a degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N.

We say that attribute $i \in N$ is *essential* (for $\mathcal{P}$) if $(x_i, z_{-i}) \; \mathcal{P} \; (y_i, z_{-i})$, for some $x_i, y_i \in X_i$ and some $z_{-i} \in X_{-i}$. It should be clear that any essential attribute is influent, whereas the converse is not true. In order to avoid unnecessary minor complications, we suppose henceforth *all attributes in N are influent*. This does *not* imply that all attributes are essential. This however implies that $\mathcal{P}$ is nonempty.

3 Strict concordance relations

3.1 Definition

The following definition, building on Bouyssou and Pirlot (2002a) and Fargier and Perny (2001), formalizes the idea of a strict concordance relation, i.e. a preference relation that has been obtained comparing alternatives by pairs on the basis of the "importance" of the attributes favoring each element of the pair.

Definition 1 (Strict concordance relations)
Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. We say that $\mathcal{P}$ is a strict concordance relation *(or, more briefly, that $\mathcal{P}$ is a SCR) if there are:*

- *an* asymmetric *binary relation P_i on each X_i ($i = 1, 2, \ldots, n$),*
- *a binary relation $\rhd$ between disjoint subsets of N that is* monotonic *w.r.t. inclusion, i.e. such that for all $A, B, C, D \subseteq N$ with $A \cap B = \varnothing$ and $C \cap D = \varnothing$,*

$$\left.\begin{array}{c} A \rhd B \\ C \supseteq A \text{ and } B \supseteq D \end{array}\right\} \Rightarrow C \rhd D. \tag{1}$$

such that, for all $x, y \in X$,

$$x \,\mathcal{P}\, y \Leftrightarrow P(x, y) \rhd P(y, x), \tag{2}$$

where $P(x, y) = \{i \in N : x_i \; P_i \; y_i\}$. We say that $\langle \rhd, P_i \rangle$ is a representation *of $\mathcal{P}$.*

Hence, when $\mathcal{P}$ is a SCR, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of the asymmetric relations P_i. It does not depend on "preference differences" between the various levels on each attribute besides the distinction between "positive", "negative" and "neutral" attributes as indicated by P_i.

Let $\mathcal{P}$ be a SCR with a representation $\langle \rhd, P_i \rangle$. For all $A, B \subseteq N$, we define the relations $\triangleq$ and $\unrhd$ between disjoint subsets of N letting: $A \triangleq B \Leftrightarrow [Not[A \rhd B]$ and $Not[B \rhd A]]$ and $A \unrhd B \Leftrightarrow [A \rhd B$ or $A \triangleq B]$.

The following proposition takes note of some elementary properties of SCR; it uses the hypothesis that all attributes are influent.

Proposition 1
If $\mathcal{P}$ is a SCR with a representation $\langle \rhd, P_i \rangle$, then:

1. *for all $i \in N$, P_i is nonempty,*
2. *for all $A, B \subseteq N$ such that $A \cap B = \varnothing$, exactly one of $A \rhd B$, $B \rhd A$ and $A \triangleq B$ holds and we have $\varnothing \triangleq \varnothing$*
3. *for all $A \subseteq N$, $A \unrhd \varnothing$ and $N \rhd \varnothing$,*
4. *$\mathcal{P}$ is independent,*
5. *for all $i \in N$, either $\mathcal{P}_i = P_i$ or $\mathcal{P}_i = \varnothing$,*
6. *$\mathcal{P}$ has a* unique *representation.*

The main objective of this paper is to characterize SCR within a general framework of conjoint measurement, using conditions that will allow to isolate their specific features. Before doing so, it is worth giving a few examples illustrating the variety of SCR and noting the connections between SCR and P. C. Fishburn's *noncompensatory preferences.*

3.2 Examples

The following examples show that SCR arise with a large variety of ordinal aggregation models that have been studied in the literature.

Example 1 (Simple Majority preferences (Sen, 1986))
The binary relation $\mathcal{P}$ is a simple majority preference if there is a strict weak order P_i on each X_i such that:

$$x \,\mathcal{P}\, y \Leftrightarrow |\{i \in N : x_i \, P_i \, y_i\}| > |\{i \in N : y_i \, P_i \, x_i\}| \, .$$

A simple majority preference relation is easily seen to be a strict SCR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$,

$$A \rhd B \Leftrightarrow |A| > |B| \, .$$

In general, $\mathcal{P}$ is neither negatively transitive nor transitive. All influent attributes for $\mathcal{P}$ are essential. $\diamond$

Example 2 (Weak majority preferences (Fishburn, 1973))
The binary relation $\mathcal{P}$ is a weak majority preference if there is a strict weak order P_i on each X_i such that:

$$x \,\mathcal{P}\, y \Leftrightarrow |\{i \in N : x_i \, P_i \, y_i\}| > \frac{|N|}{2}.$$

A weak majority preference relation is easily seen to be a SCR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$:

$$A \rhd B \Leftrightarrow |A| > \frac{|N|}{2}.$$

Note that influent attributes will not be essential. $\diamond$

Example 3 (Weighted majority with threshold (Vansnick, 1986))
The binary relation $\mathcal{P}$ is a weighted majority preference with threshold if there are real numbers $\rho > 1$ and $\varepsilon \geq 0$ and, for all $i \in N$, a strict semiorder P_i on X_i and a positive real number $w_i > 0$, such that:

$$x \,\mathcal{P}\, y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \rho \sum_{j \in P(y,x)} w_j \; + \; \varepsilon.$$

A weighted majority preference with threshold is easily seen to be a SCR defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$:

$$A \rhd B \Leftrightarrow \sum_{i \in A} w_i > \rho \sum_{j \in B} w_j + \varepsilon.$$

As soon as $\varepsilon > 0$, influent attributes may not be essential. $\diamond$

3.3 Noncompensatory preferences *à la* Fishburn

Noncompensatory preferences introduced by Fishburn (1975, 1976) are closely related to—but distinct from— SCR. His definition also starts with an *asymmetric* binary relation $\mathcal{P}$ on $X = \prod_{i=1}^{n} X_i$. Let $\mathcal{P}(x, y) = \{i : x_i \mathcal{P}_i y_i\}$. It is clear that, for all $x, y \in X$, $\mathcal{P}(x, y) \cap \mathcal{P}(y, x) = \varnothing$.

Definition 2 (Noncompensatory Preferences (Fishburn, 1976))
An asymmetric binary relation $\mathcal{P}$ on $X = \prod_{i=1}^{n} X_i$ is said to be noncompensatory *if:*

$$\left.\begin{array}{rcl} \mathcal{P}(x,y) & = & \mathcal{P}(z,w) \\ \mathcal{P}(y,x) & = & \mathcal{P}(w,z) \end{array}\right\} \Rightarrow [x \,\mathcal{P}\, y \Leftrightarrow z \,\mathcal{P}\, w], \qquad (NC)$$

for all $x, y, z, w \in X$.

Hence, when $\mathcal{P}$ is noncompensatory, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of $\mathcal{P}_i$. This is close to the definition of a SCR with $\mathcal{P}_i$ replacing P_i and no monotonicity involved.

Some useful properties of noncompensatory preferences are summarized in the following:

Lemma 1
If an asymmetric relation $\mathcal{P}$ on $X = \prod_{i=1}^{n} X_i$ is noncompensatory, then:

1. *$\mathcal{P}$ is independent,*
2. *$x_i \,\mathcal{I}_i\, y_i$ for all $i \in N \Rightarrow x \,\mathcal{I}\, y$,*
3. *$x_j \,\mathcal{P}_j\, y_j$ for some $j \in N$ and $x_i \,\mathcal{I}_i\, y_i$ for all $i \in N \setminus \{j\} \Rightarrow x \,\mathcal{P}\, y$,*
4. *all attributes are essential.*

As shown in the following example, there are SCR relations violating *all* conditions in lemma 1 except independence.

Example 4
Let $X = X_1 \times X_2 \times X_3$ with $X_1 = \{x_1, y_1\}$, $X_2 = \{x_2, y_2\}$ and $X_3 = \{x_3, y_3\}$. Let $x_1 \,P_1\, y_1$, $x_2 \,P_2\, y_2$ and $x_3 \,P_3\, y_3$. Define $\mathcal{P}$ letting, for all $x, y \in X$,

$$x \,\mathcal{P}\, y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j \;+\; \varepsilon.$$

with $w_1 = w_2 = 1$, $w_3 = 2$ and $\varepsilon = 1$. By construction, $\mathcal{P}$ is a SCR. It is clear that attributes 1 and 2 are not essential contrarily to attribute 3. These two attributes nevertheless are influent since $(x_1, x_2, y_3) \,\mathcal{P}\, (y_1, y_2, y_3)$ but neither $(x_1, y_2, y_3) \,\mathcal{P}\, (y_1, y_2, y_3)$ nor $(y_1, x_2, y_3) \,\mathcal{P}\, (y_1, y_2, y_3)$.

Although, $x_1 \,\mathcal{I}_1\, y_1$, $x_2 \,\mathcal{I}_2\, y_2$ and $y_3 \,\mathcal{I}_3\, y_3$, we have $(x_1, x_2, y_3) \,\mathcal{P}\, (y_1, y_2, y_3)$. Note that $(y_1, y_2, x_3) \,\mathcal{I}\, (x_1, x_2, y_3)$, although $y_1 \,\mathcal{I}_1\, x_1$, $y_2 \,\mathcal{I}_2\, x_2$ and $x_3 \,\mathcal{P}_3\, y_3$. Hence $\mathcal{P}$ violates all conditions in lemma 1 except independence. $\diamond$

Several authors have used the definition of noncompensation, or several variants of it, as an axiom with the aim of characterizing preferences that can be obtained with ordinal aggregation methods (see Bouyssou, 1992; Bouyssou and Vansnick, 1986; Dubois et al., 2003; Fargier and Perny, 2001). The above example shows that these results only deal with SCR in which all attributes are essential. Furthermore, these results use a condition (NC) that is quite different from the usual cancellation conditions invoked in conjoint measurement. Therefore, they are not very helpful in order to understand the specific features of SCR when compared to other types of binary relations, e.g. the ones that can be represented by an additive utility model. The route that we follow below seems to avoid these difficulties.

4 A general framework for nontransitive conjoint measurement

This section follows the analysis in Bouyssou and Pirlot (2002b, 2004a) using asymmetric relations instead of reflexive relations. We envisage here binary relations $\mathcal{P}$ on X that can be represented as:

$$x \,\mathcal{P}\, y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) > 0, \tag{M}$$

where p_i are real-valued functions on X_i^2 that are *skew symmetric* (i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$) and F is a real-valued function on $\prod_{i=1}^n p_i(X_i^2)$ being *odd* (i.e. such that $F(\mathbf{x}) = -F(-\mathbf{x})$, abusing notation in an obvious way) and *nondecreasing* in all its arguments.

Definition 3 (Conditions $\boldsymbol{ARC1}$ and $\boldsymbol{ARC2}$)
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:

$$ARC1_i \textit{ if } \quad \left.\begin{array}{c} (x_i, a_{-i}) \,\mathcal{P}\, (y_i, b_{-i}) \\ \textit{and} \\ (z_i, c_{-i}) \,\mathcal{P}\, (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, c_{-i}) \,\mathcal{P}\, (y_i, d_{-i}) \\ \textit{or} \\ (z_i, a_{-i}) \,\mathcal{P}\, (w_i, b_{-i}), \end{array}\right.$$

$$ARC2_i \textit{ if } \quad \left.\begin{array}{c} (x_i, a_{-i}) \,\mathcal{P}\, (y_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i}) \,\mathcal{P}\, (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i, a_{-i}) \,\mathcal{P}\, (w_i, b_{-i}) \\ \textit{or} \\ (w_i, c_{-i}) \,\mathcal{P}\, (z_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $ARC1$ (resp. $ARC2$) if it satisfies $ARC1_i$ (resp. $ARC2_i$) for all $i \in N$.

Condition $ARC1_i$ (Asymmetric inteR-attribute Cancellation) strongly suggests that either the difference (x_i, y_i) is at least as large as the difference (z_i, w_i) or *vice versa.* Condition $ARC2_i$ suggests that the preference difference (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i). Taking $x_i = y_i$, $z_i = w_i$, $a_{-i} = c_{-i}$ and $b_{-i} = d_{-i}$ shows that $ARC2_i$ implies that $\mathcal{P}$ is independent for $N \setminus \{i\}$. Hence, $\mathcal{P}$ is independent when $ARC2$ holds. Simple examples show that condition $ARC1$ and $ARC2$ are independent in the class of asymmetric relations. These two conditions allow to characterize model (M) when X is finite or countably infinite.

Theorem 1
Let $\mathcal{P}$ be a binary relation on finite or countably infinite set $X = \prod_{i=1}^{n} X_i$. Then $\mathcal{P}$ has a representation (M) *iff it is asymmetric and satisfies $ARC1$ and $ARC2$.*

It should be observed that model (M) is sufficiently general to contain as particular cases most conjoint measurement models, when interpreted in terms of an asymmetric binary relation, including the classical additive utility model (see Krantz et al., 1971; Wakker, 1989) and the additive difference model (see Tversky, 1969). We show in section 5 that SCR relations form a subclass of the binary relations having a representation in model (M).

5 A new characterization of SCR

Consider a binary relation $\mathcal{P}$ having a representation in (M) with all functions p_i taking at most three distinct values, i.e., a binary relation for which the induced relations comparing preference differences on each attribute involve at most three distinct equivalence classes. Defining the relation P_i letting $x_i \; P_i \; y_i$ when $p_i(x_i, y_i) > 0$, intuition suggests that such a binary relation is quite similar to a SCR. We formalize this intuition below and show how to characterize SCR within the framework provided by model (M).

Definition 4 (Conditions $MAJ1$ and $MAJ2$)
Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$MAJ1_i \textit{ if } \quad \left.\begin{array}{c} (x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \\ \textit{and} \\ (z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i}) \\ \textit{and} \\ (z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i}) \\ \textit{or} \\ (x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i}), \end{array}\right.$$

$$MAJ2_i \textit{ if } \quad \left.\begin{array}{c} (x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \\ \textit{and} \\ (w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i}) \\ \textit{or} \\ (z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $MAJ1$ (resp. $MAJ2$) if it satisfies $MAJ1_i$ (resp. $MAJ2_i$) for all $i \in N$.

Lemma 2
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$.

1. *If $\mathcal{P}$ is a SCR then it satisfies $ARC1$ and $ARC2$.*

2. *If $\mathcal{P}$ is a SCR then it satisfies $MAJ1$ and $MAJ2$.*

3. *In the class of asymmetric relations, conditions $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$ are independent.*

Our central result says that conditions $MAJ1$ and $MAJ2$ isolate within model (M) the class of all SCR. We have:

Theorem 2
Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\mathcal{P}$ is a SCR iff it is asymmetric and satisfies $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$.

A related result was already presented in Bouyssou and Pirlot (2002a). Instead of conditions $MAJ1$ and $MAJ2$, we used a condition (called coarseness) amounting to bluntly saying that all functions p_i can take at most three distinct values. As explained in Bouyssou and Pirlot (2003), the use of such a condition is problematic since it destroys the independence between $ARC1$ and $ARC2$. In presence of $ARC1$ and $ARC2$, conditions $MAJ1$ and $MAJ2$ imply that SCR have a similar effect without destroying independence.

6 Discussion

The main contribution of this paper was to propose a characterization of SCR within the framework of a general model for nontransitive conjoint measurement. This characterization shows that, beyond surface, SCR have a lot in common with the usual structures manipulated in conjoint measurement. It emphasizes the main specific feature of SCR, i.e. the option not to distinguish a rich preference difference relation on each attribute. Our results were shown to be more general than earlier ones based on the idea of noncompensation *à la* Fishburn. The results in this paper can be extended in several directions.

1. It is not difficult to extend our results to deal with SCR that would be strictly monotonic, i.e., such that, $A \cap B = \varnothing$, $Not[B \rhd A]$, $C \cap D = \varnothing$, $C \supseteq A$ and $B \supseteq D$ with at least one strict inclusion, imply $C \rhd D$. This requires strengthening conditions $ARC1$ and $ARC2$.

2. Our definition of SCR does not require the relations P_i to possess any remarkable property besides asymmetry. This is at variance with what is done in most ordinal aggregation methods (see the examples in section 3.2). Often, P_i is supposed to be a strict weak order. When it is desirable to model imperfect discrimination on each attribute, P_i is supposed to be a strict semiorder. As shown in Bouyssou and Pirlot (2003), it is not difficult to tackle this case using, instead of model (M), the following general model

$$x \,\mathcal{P}\, y \Leftrightarrow F(\varphi_1(u_1(x_1), u_1(y_1)), \ldots, \varphi_n(u_n(x_n), u_n(y_n))) > 0,$$

where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i)^2$ that are *skew symmetric* and *nondecreasing* in their first argument and F is a real-valued function on $\prod_{i=1}^{n} \varphi_i(u_i(X_i)^2)$ being *odd* and *nondecreasing* in all its arguments. This analysis requires the addition of conditions that are essentially independent from the one introduced above. This is in line with the fact stressed by Saari (1994, 1998) that "ordinal" aggregation models make little use of the transitivity properties of the relations that are aggregated.

3. Our definition of SCR does not require the relations $\rhd$ to have any remarkable property besides being monotonic. Again this is at variance with what is usually done in most aggregation techniques in which $\rhd$ is supposed to be transitive or even, is supposed to have an additive representation. We show in Bouyssou and Pirlot (2003) how to tackle such situations within the framework introduced here.

4. As first noted in Fishburn (1975, 1976), using condition NC simply allows to understand the conditions under which $\mathcal{P}$ may possess "nice transitivity properties". This is not surprising since NC is very much like a "single profile" analogue of Arrow's Independence of Irrelevant Alternatives (Arrow, 1963). Therefore, as soon as the structure of X is "sufficiently" rich, imposing nice transitivity properties on a noncompensatory relation $\mathcal{P}$ leads to a very uneven distribution of "power" between the various attributes (see Bouyssou, 1992; Fishburn, 1976). We show in Bouyssou and Pirlot (2003) that similar results hold for SCR even when they violate (NC).

5. We restricted our attention here to an asymmetric relation $\mathcal{P}$ interpreted as strict preference. It is not difficult to extend our analysis, using the results in Bouyssou and Pirlot (2002b), to cover the case studied in Fargier and Perny (2001) and Greco et al. (2001) in which:

$$x \; \mathcal{S} \; y \Leftrightarrow S(x, y) \trianglerighteq S(y, x),$$

where $\mathcal{S}$ is a reflexive binary relation on X, S_i is a *complete* binary relation on X_i, $\trianglerighteq$ is a *reflexive* binary relation on 2^N and $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$. Such an analysis, requiring to distinguish "indifference" from "incomparability" is performed in Bouyssou and Pirlot (2004b).

The analysis in Greco et al. (2001) deserves special attention. They study a version of SCR adapted to reflexive relations that is closely related to ELECTRE I (Roy, 1968). A major advantage of their analysis is that it allows to analyze discordance effects. This remains an open question within our framework.

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