Conjoint measurement

A brief introduction

Denis Bouyssou

CNRS–LAMSADE
Paris, France

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Typical problem

Comparing holiday packages

	cost	# of days	travel time	category of hotel	distance to beach	Wifi	cultural interest
A	200 €	15	12 h	***	45 km	Y	++
B	425 €	18	15 h	****	0 km	N	−−
C	150 €	4	7 h	**	250 km	N	+
D	300 €	5	10 h	***	5 km	Y	−

Central problems

- helping a DM choose between these packages
- helping a DM structure his/her preferences

Introduction

Context: decision analysis

- careful analysis of objectives
- careful analysis of attributes
- careful selection of alternatives
- availability of the DM

Context: recommendation systems

- no analysis of objectives
- attributes as available
- alternatives as available
- limited access to the user

Motivation

Introduction

Basic model

- additive value function model

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

x, y : alternatives

x_i : "evaluation" of alternative x on attribute i

$v_i(x_i)$: number

- underlies most existing MCDM techniques

Underlying theory: conjoint measurement

- Economics (Debreu, 1960)
- Psychology (Luce & Tukey, 1964)
- tool to help structure preferences

Part I

Classical theory: conjoint measurement

Aside: measurement of physical quantities

Lonely individual on a desert island

- no tools, no books, no knowledge of Physics
- wants to rebuild a system of physical measures

A collection a rigid straight rods

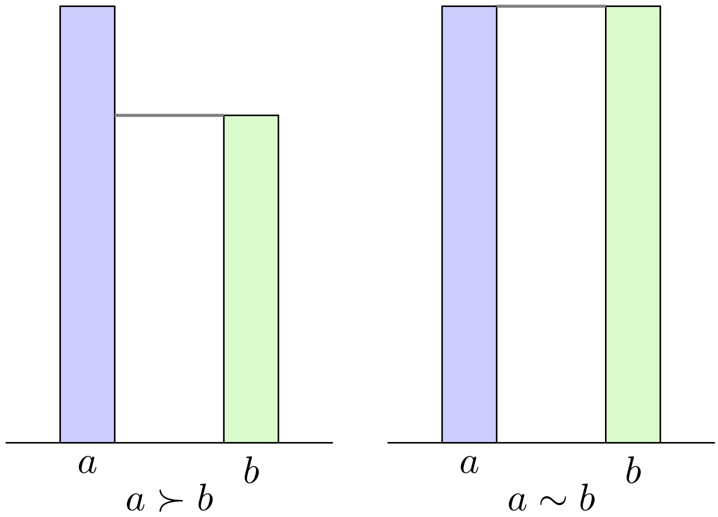
- problem: measuring the length of these rods
  - pre-theoretical intuition
    - length
    - softness, beauty

3 main steps

- comparing objects
- creating and comparing new objects
- creating standard sequences

Step 1: comparing objects

- experiment to conclude which rod has “more length”
- place rods side by side on the same horizontal plane



Comparing objects

Results

- $a \succ b$: extremity of rod a is higher than extremity of rod b
- $a \sim b$: extremity of rod a is as high as extremity of rod b

Expected properties

- $a \succ b$, $a \sim b$ or $b \succ a$
- $\succ$ is asymmetric
- $\sim$ is symmetric
- $\succ$ is transitive
- $\sim$ is transitive
- $\succ$ and $\sim$ combine “nicely”
  - $a \succ b$ and $b \sim c \Rightarrow a \succ c$
  - $a \sim b$ and $b \succ c \Rightarrow a \succ c$

Summary of experiments

- $\succsim = \succ \cup \sim$ is a weak order
  - complete ($a \succsim b$ or $b \succsim a$)
  - transitive ($a \succsim b$ and $b \succsim c \Rightarrow a \succsim c$)

Comparing objects

Consequences

$\succsim = \succ \cup \sim$ is a weak order

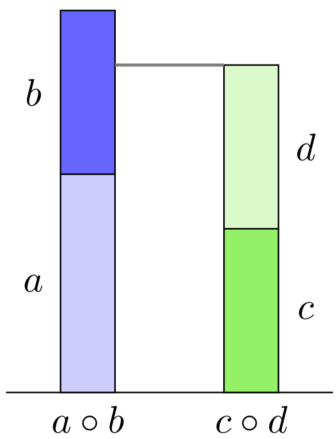
- associate a real number $\Phi(a)$ to each object a
- the comparison of numbers faithfully reflects the results of experiments

$$a \succ b \Leftrightarrow \Phi(a) > \Phi(b) \qquad a \sim b \Leftrightarrow \Phi(a) = \Phi(b)$$

- the function Φ defines an ordinal scale
  - applying an increasing transformation to Φ leads to a scale that has the same properties
  - any two scales having the same properties are related by an increasing transformation

Step 2: creating and comparing new objects

- use the available objects to create new ones
- concatenate objects by placing two or more rods “in a row”



$$a \circ b \succ c \circ d$$

Concatenation

Constraints induced by concatenation

- we want to be able to deduce $\Phi(a \circ b)$ from $\Phi(a)$ and $\Phi(b)$
- simplest requirement

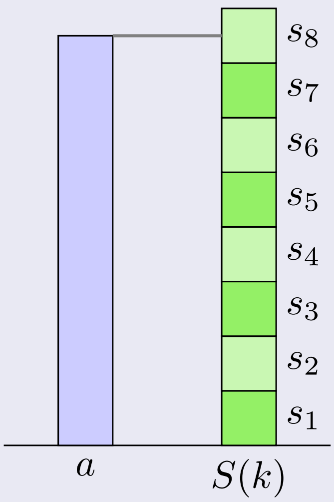
$$\Phi(a \circ b) = \Phi(a) + \Phi(b)$$

- monotonicity constraints

$$a \succ b \text{ and } c \sim d \Rightarrow a \circ c \succ b \circ d$$

Step 3: creating and using standard sequences

- choose a standard rod
- be able to build perfect copies of the standard
- concatenate the standard rod with its perfects copies



$$S(8) \succ a \succ S(7)$$
$$\Phi(s) = 1 \Rightarrow 7 < \Phi(a) < 8$$

Convergence

First method

- choose a smaller standard rod
- repeat the process

Second method

- prepare a perfect copy of the object
- concatenate the object with its perfect copy
- compare the "doubled" object to the original standard sequence
- repeat the process

Summary

Extensive measurement

- Krantz, Luce, Suppes & Tversky (1971, chap. 3)

4 Ingredients

1. well-behaved relations $\succ$ and $\sim$
2. concatenation operation $\circ$
3. consistency requirements linking $\succ$, $\sim$ and $\circ$
4. ability to prepare perfect copies of some objects in order to build standard sequences

Neglected problems

- many!

Question

Can this be applied outside Physics?

- no concatenation operation
  - intelligence!
  - pain!

What is conjoint measurement?

Conjoint measurement

- mimicking the operations of extensive measurement
  - when there are no concatenation operation readily available
  - when several dimensions are involved

Seems overly ambitious

- let us start with a simple example

Example: Hammond, Keeney & Raiffa, 1999

Choice of an office to rent

- five locations have been identified
- five attributes are being considered
  - *Commute* time (minutes)
  - *Clients*: percentage of clients living close to the office
  - *Services*: ad hoc scale
    - A (all facilities), B (telephone and fax), C (no facility)
  - *Size*: square feet ($\simeq$ 0.1 m^2)
  - *Cost*: $ per month

Attributes

- *Commute*, *Size* and *Cost* are natural attributes
- *Clients* is a proxy attribute
- *Services* is a constructed attribute

Data

	a	b	c	d	e
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Hypotheses and context

- a single cooperative DM
- choice of a single office
- ceteris paribus reasoning seems possible
  Commute: decreasing *Clients*: increasing
  Services: increasing *Size*: increasing
  Cost: decreasing
- dominance has meaning

	a	*b*	*c*	*d*	*e*
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	*A*	*B*	*C*	*A*	*C*
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

- b dominates alternative e
- d is "close" to dominating a
- divide and conquer: dropping alternatives
  - drop a and e

	b	*c*	*d*
Commute	25	20	25
Clients	80	70	85
Services	*B*	*C*	*A*
Size	700	500	950
Cost	1700	1500	1900

- no more dominance
- assessing tradeoffs
- all alternatives except c have a common evaluation on *Commute*
- modify c in order to bring it to this level
  - starting with c, what is the gain on *Clients* that would exactly compensate a loss of 5 min on *Commute*?
  - difficult but central question
  - bracketing

	c	c'
Commute	20	**25**
Clients	70	$\mathbf{70} + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

find δ such that $c' \sim c$

Answer

- for $\delta = 8$, I am indifferent between c and c'
- replace c with c'

	b	c'	d
Commute	25	25	25
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- all alternatives have a common evaluation on *Commute*
- divide and conquer: dropping attributes
  - drop attribute *Commute*

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

- check again for dominance
- unfruitful
- assess new tradeoffs
  - neutralize *Services* using *Cost* as reference

	b	c'	d
Clients	80	78	85
Services	B	C	A
Size	700	500	950
Cost	1700	1500	1900

Questions

- what maximal increase on *Cost* would you be prepared to pay to go from C to B on *Services* for c'?
  - answer: 250 $
- what minimal decrease on *Cost* would you ask if we go from A to B on *Services* for d?
  - answer: 100 $

	b	c'	c''	d	d'
Clients	80	78	78	85	85
Services	B	C	**B**	A	**B**
Size	700	500	500	950	950
Cost	1700	1500	1500 + **250**	1900	1900 − **100**

- replacing c' with c''
- replacing d with d'
- dropping *Services*

	b	c''	d'
Clients	80	78	85
Size	700	500	950
Cost	1700	1750	1800

- checking for dominance: c'' is dominated by b
- c'' can be dropped

- dropping c''

	b	d'
Clients	80	85
Size	700	950
Cost	1700	1800

- no dominance
- question: starting with b what is the additional amount on *Cost* that you would be prepared to pay to increase *Size* by 250?
    - answer: 250 $

	b	b'	d'
Clients	80	80	85
Size	700	**950**	950
Cost	1700	1700 + **250**	1800

- replace b with b'
- drop *Size*

	b'	d'
Clients	80	85
Size	950	950
Cost	1950	1800

	b'	d'
Clients	80	85
Cost	1950	1800

- check for dominance
- d' dominates b'

Conclusion

- Recommend d as the final choice

Summary

Remarks

- very simple process
- process entirely governed by $\succ$ and $\sim$
- no question on “intensity of preference”
- notice that importance is not even mentioned
  - it is there but in a more complex form than just “weights”
- why be interested in something more complex?

Problems

- set of alternative is small
  - many questions otherwise
- output is not a preference model
  - if new alternatives appear, the process should be restarted
- what are the underlying hypotheses?

Monsieur Jourdain doing conjoint measurement

Similarity with extensive measurement

- $\succ$: preference, $\sim$: indifference
- we have implicitly supposed that they combine nicely

Recommendation: d

- we should be able to prove that $d \succ a$, $d \succ b$, $d \succ c$ and $d \succ e$
- dominance: $b \succ e$ and $d \succ a$
- tradeoffs + dominance: $b \succ c''$, $c'' \sim c'$, $c' \sim c$, $d' \sim d$, $b' \sim b$, $d' \succ b'$

$$d \succ a, b \succ e$$
$$c'' \sim c', c' \sim c, b \succ c''$$
$$\Rightarrow b \succ c$$
$$d \sim d', b \sim b', d' \succ b'$$
$$\Rightarrow d \succ b$$

Monsieur Jourdain doing conjoint measurement

OK... but where are the standard sequences?

- hidden... but really there!
- standard sequence for length: objects that have exactly the same length
- tradeoffs: preference intervals on distinct attributes that have the same length
  - $c \sim c'$
  - $[25, 20]$ on *Commute* has the same length as $[70, 78]$ on *Client*

	c	c'	f	f'
Commute	20	**25**	*20*	**25**
Clients	70	**78**	*78*	**82**
Services	*C*	*C*	*C*	*C*
Size	500	500	500	500
Cost	1500	1500	1500	1500

$[70, 78]$ has the same length $[78, 82]$ on *Client*

Setting

- $N = \{1, 2, \ldots, n\}$ set of attributes
- X_i: set of possible levels on the ith attribute
- $X = \prod_{i=1}^{n} X_i$: set of all conceivable alternatives
  - X include the alternatives under study… and many others
- $J \subseteq N$: subset of attributes
- $X_J = \prod_{j \in J} X_j$, $X_{-J} = \prod_{j \notin J} X_j$
- $(x_J, y_{-J}) \in X$
- $(x_i, y_{-i}) \in X$
- $\succsim$: binary relation on X: "at least as good as"
- $x \succ y \Leftrightarrow x \succsim y$ and $Not[y \succsim x]$
- $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$

What will be ignored today

Important issues ignored

- structuring of objectives
- from objectives to attributes
- adequate family of attributes
- risk, uncertainty, imprecision

Marginal preference and independence

Marginal preferences

- $J \subseteq N$: subset of attributes
- $\succsim_J$ marginal preference relation induced by $\succsim$ on X_J

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J}$$

Independence

- J is independent for $\succsim$ if
$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J$$
- common levels on attributes other than J do not affect preference

Independence

Definition

- for all $i \in N$, $\{i\}$ is independent, $\succsim$ is weakly independent
- for all $J \subseteq N$, J is independent, $\succsim$ is independent

Proposition (Folk)

Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then:

- $\succsim_i$ is a weak order on X_i
- $[x_i \succsim_i y_i, \text{ for all } i \in N] \Rightarrow x \succsim y$
- $[x_i \succsim_i y_i, \text{ for all } i \in N \text{ and } x_j \succ_j y_j \text{ for some } j \in N] \Rightarrow x \succ y$

for all $x, y \in X$

Dominance

- as soon as I have a weakly independent weak order
- dominance arguments apply

Independence in practice

Independence

- it is easy to imagine examples in which independence is violated
  - Main course and Wine example

$$(\text{Fish}, \text{WW}) \succ (\text{Meat}, \text{WW})$$
$$(\text{Meat}, \text{RW}) \succ (\text{Fish}, \text{RW})$$

- it is nearly hopeless to try to work if weak independence is not satisfied
- some (e.g., R. L. Keeney) think that the same is true for independence
- in all cases if independence is violated, things get complicated

Outline of theory: 2 attributes

Question

- suppose I can "observe" $\succsim$ on $X = X_1 \times X_2$
  - asking questions to the DM
- what must be supposed to guarantee that I can represent $\succsim$ in the additive value function model

$$v_1 : X_1 \to \mathbb{R}$$
$$v_2 : X_2 \to \mathbb{R}$$
$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

- $\succsim$ must be an independent weak order

Method

- try building standard sequences and see if it works!

Uniqueness

Important observation

Suppose that there are v_1 and v_2 such that

$$(x_1, x_2) \succsim (y_1, y_2) \Leftrightarrow v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2)$$

Take $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ with $\alpha > 0$

$$w_1 = \alpha v_1 + \beta_1 \quad w_2 = \alpha v_2 + \beta_2$$

is also a valid representation

Consequences

- fixing $v_1(x_1) = v_2(x_2) = 0$ is harmless
- fixing $v_1(y_1) = 1$ is harmless if $y_1 \succ_1 x_1$

Standard sequences

Preliminaries

- choose arbitrarily two levels $x_1^0, x_1^1 \in X_1$
- make sure that $x_1^1 \succ_1 x_1^0$
- choose arbitrarily one level $x_2^0 \in X_2$
- $(x_1^0, x_2^0) \in X$ is the reference point (origin)
- the preference interval $[x_1^0, x_1^1]$ is the unit

Building a standard sequence on X_2

- find a "preference interval" on X_2 that has the same "length" as the reference interval $[x_1^0, x_1^1]$
- find x_2^1 such that

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$

$$v_1(x_1^0) + v_2(x_2^1) = v_1(x_1^1) + v_2(x_2^0) \text{ so that}$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0)$$

- the structure of X_2 has to be "rich enough"

Standard sequences

Consequences

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$
$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0)$$

- it can be supposed that

$$v_1(x_1^0) = v_2(x_2^0) = 0$$
$$v_1(x_1^1) = 1$$

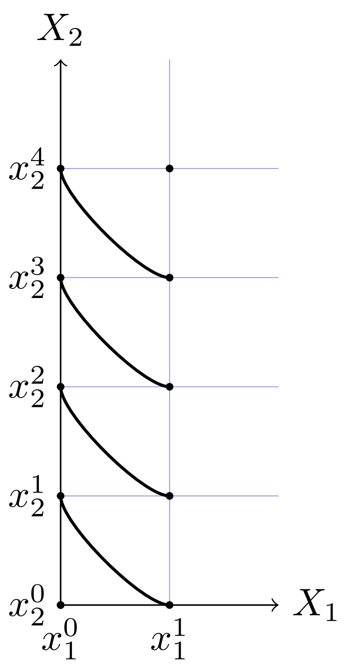
$$\Rightarrow v_2(x_2^1) = 1$$

Going on

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$$
$$(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$$
$$(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$$
$$\dots$$
$$(x_1^0, x_2^k) \sim (x_1^1, x_2^{k-1})$$

$$v_2(x_2^1) - v_2(x_2^0) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$v_2(x_2^2) - v_2(x_2^1) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$v_2(x_2^3) - v_2(x_2^2) = v_1(x_1^1) - v_1(x_1^0) = 1$$
$$\dots$$
$$v_2(x_2^k) - v_2(x_2^{k-1}) = v_1(x_1^1) - v_1(x_1^0) = 1$$

$$\Rightarrow v_2(x_2^2) = 2, v_2(x_2^3) = 3, \dots, v_2(x_2^k) = k$$



Standard sequence

Archimedean

- implicit hypothesis for length
  - the standard sequence can reach the length of any object

$$\forall x, y \in \mathbb{R}, \exists n \in \mathbb{N} : ny > x$$

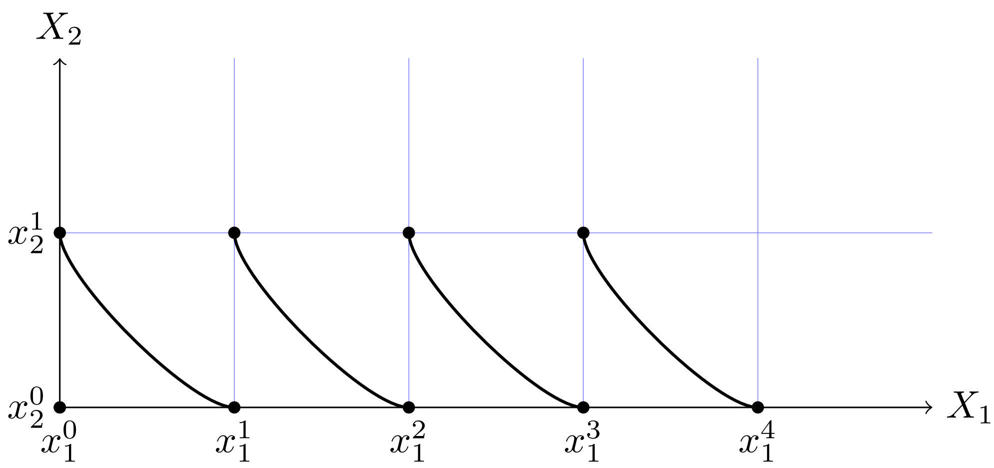
- a similar hypothesis has to hold here
- rough interpretation
  - there are not "infinitely" liked or disliked consequences

Building a standard sequence on X_1

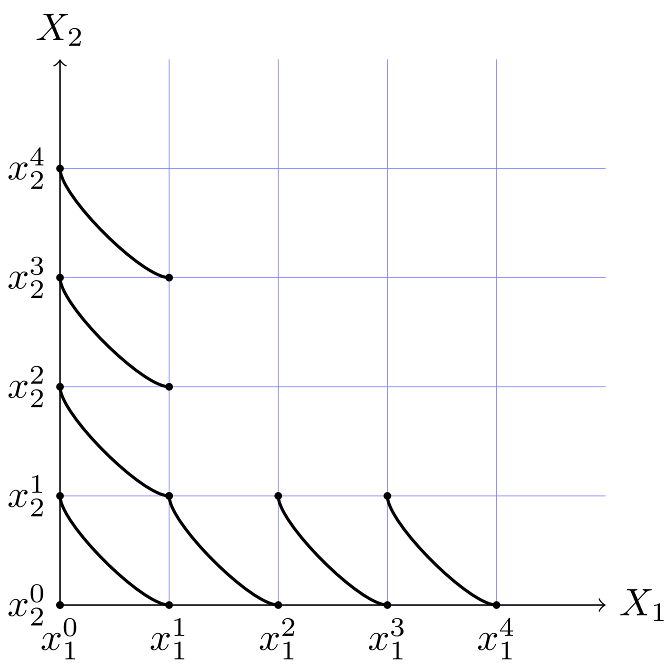
$$(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$$
$$(x_1^3, x_2^0) \sim (x_1^2, x_2^1)$$
$$\dots$$
$$(x_1^k, x_2^0) \sim (x_1^{k-1}, x_2^1)$$

$$v_1(x_1^2) - v_1(x_1^1) = v_2(x_2^1) - v_2(x_2^0) = 1$$
$$v_1(x_1^3) - v_1(x_1^2) = v_2(x_2^1) - v_2(x_2^0) = 1$$
$$\dots$$
$$v_1(x_1^k) - v_1(x_1^{k-1}) = v_2(x_2^1) - v_2(x_2^0) = 1$$

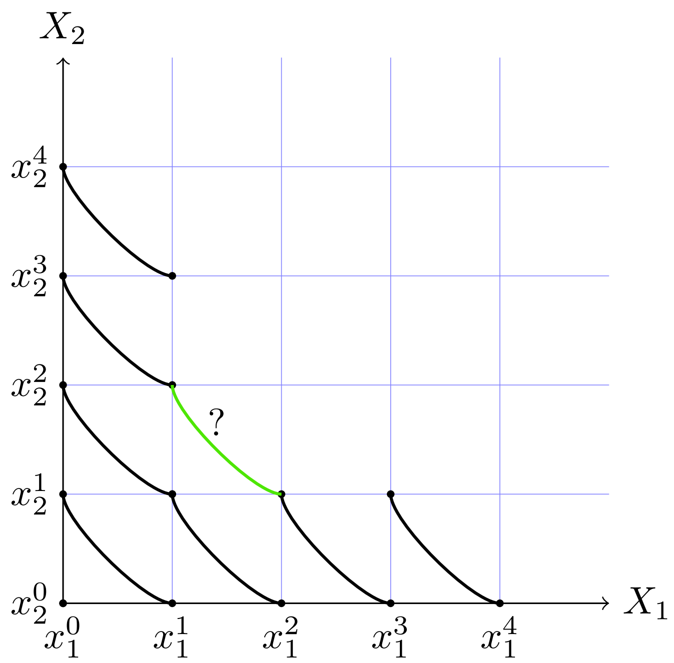
$$v_1(x_1^2) = 2, v_1(x_1^3) = 3, \dots, v_1(x_1^k) = k$$



X_2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4

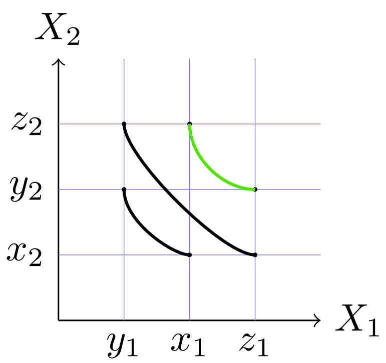


X_2
x_2^4
x_2^3
x_2^2
x_2^1
x_2^0
X_1
x_1^0
x_1^1
x_1^2
x_1^3
x_1^4



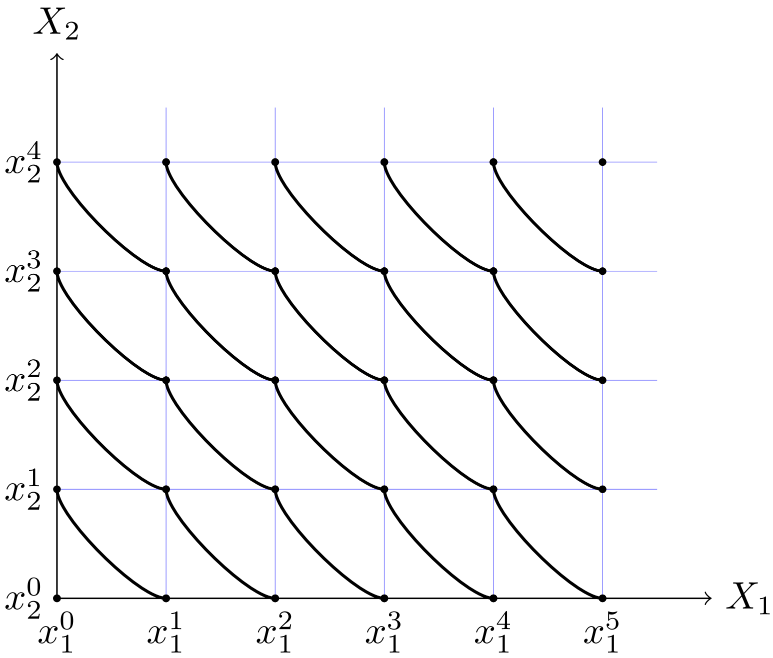
Thomsen condition

$$\begin{array}{c}(x_1, x_2) \sim (y_1, y_2)\\ \text{and}\\ (y_1, z_2) \sim (z_1, x_2)\end{array} \Rightarrow (x_1, z_2) \sim (z_1, y_2)$$



Consequence

- there is an additive value function on the grid

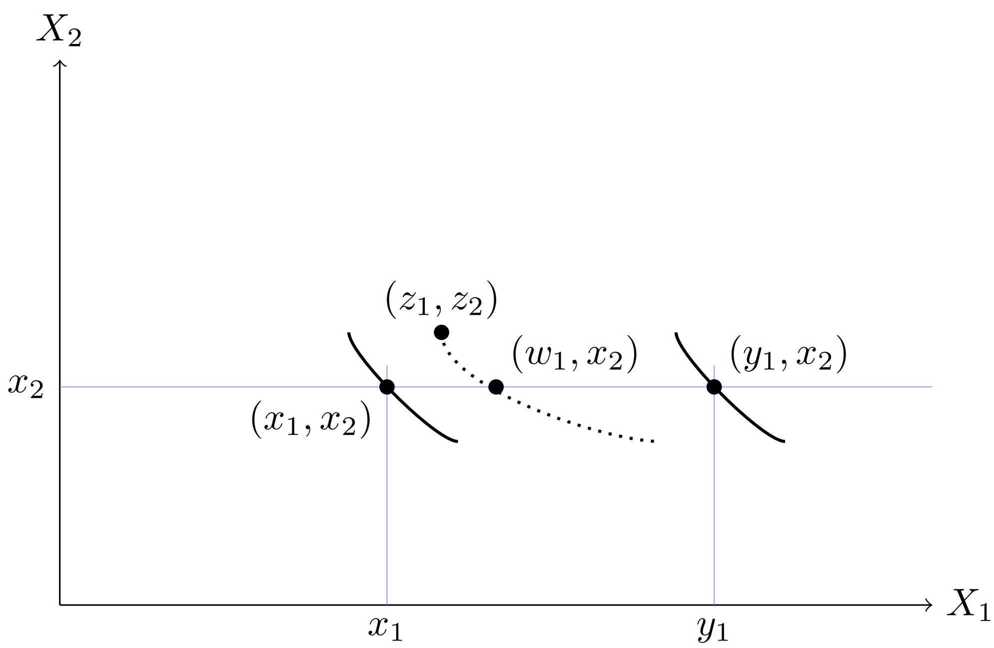


Summary

- we have defined a "grid"
- there is an additive value function on the grid
- iterate the whole process with a "denser grid"

Hypotheses

- Archimedean: every strictly bounded standard sequence is finite
- essentiality: both $\succ_1$ and $\succ_2$ are nontrivial
- restricted solvability



$$\left.\begin{array}{l}(y_1, x_2) \succ (z_1, z_2)\\ (z_1, z_2) \succ (x_1, x_2)\end{array}\right\} \Rightarrow \exists w_1 \text{ such that } (z_1, z_2) \sim (w_1, x_2)$$

Basic result

Theorem (2 attributes)

If

- restricted solvability holds
- each attribute is essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions

The representation is unique up to scale and location

General case

Good news

- entirely similar…
- with a very nice surprise: Thomsen can be forgotten
  - if $n = 2$, independence is identical with weak independence
  - if $n > 3$, independence is much stronger than weak independence

Basic result

Theorem (more than 2 attributes)

If

- restricted solvability holds
- at least three attributes are essential

then
the additive value function model holds
if and only if
$\succsim$ is an independent weak order satisfying the Archimedean condition

The representation is unique up to scale and location

Independence and even swaps

Even swaps technique

- assessing tradeoffs...
- after having suppressed attributes

Implicit hypothesis

- what happens on these attributes do not influence tradeoffs
- this is another way to formulate independence

Assessing value functions

Direct technique

- check independence
- build standard sequences
  - "weights" (importance) has no explicit rôle
  - do not even pronounce the word!!

Problems

- many questions
- questions on fictitious alternatives
- rests on indifference judgments
- discrete attributes
- propagation of "errors"

Indirect techniques

Principle

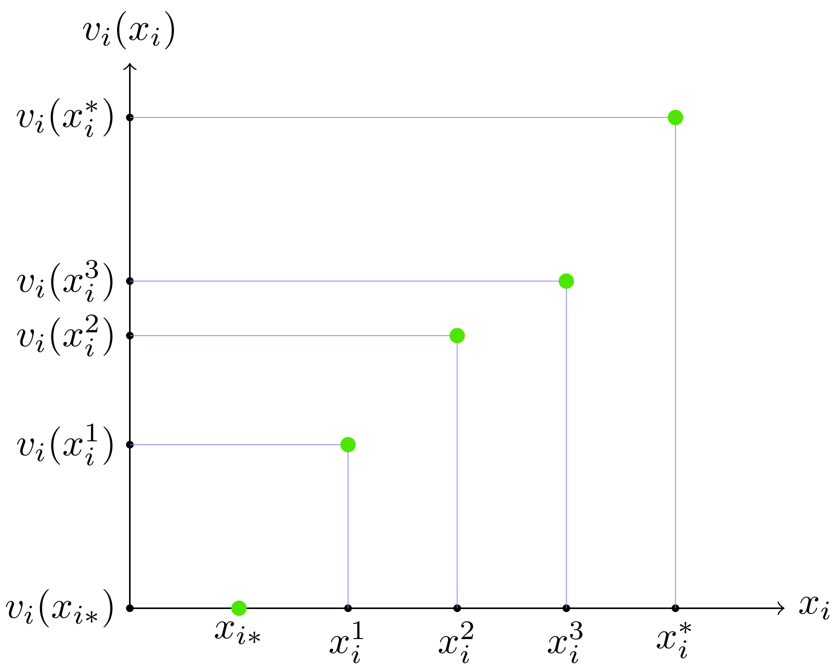
- select a number of reference alternatives that the DM knows well
- rank order these alternatives
- test, using LP, if this information is compatible with an additive value function
  - if yes, present a central one
    - interact with the DM
    - apply the resulting function to the whole set of alternatives
  - if not
    - interact with the DM

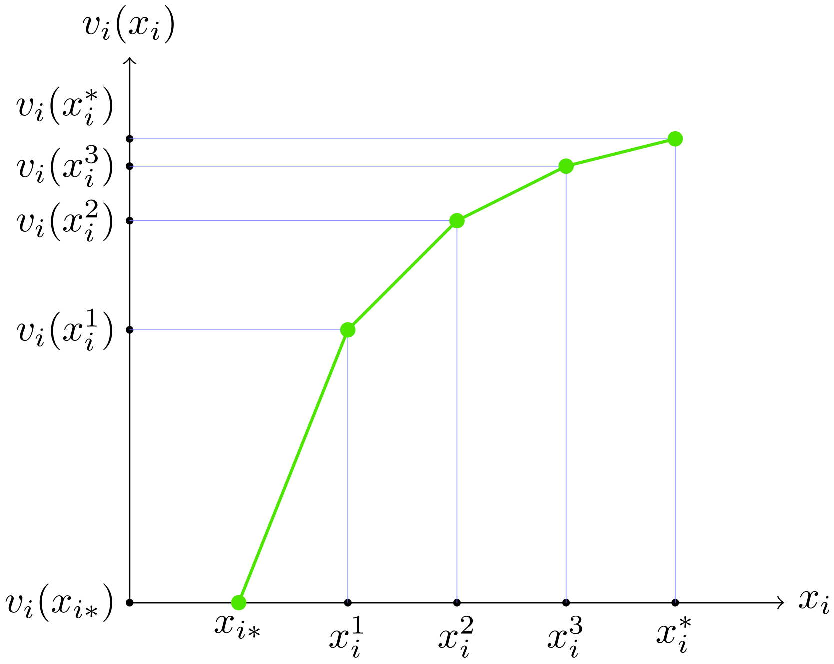
UTA: decision variables

Aim

- assess $v_1, v_2, \ldots, v_n$
- normalization
  - x_{i*}: worst level on attribute i
  - x_i^*: best level on attribute i
  - $v_1(x_{1*}) = v_2(x_{2*}) = \ldots = v_n(x_{n*}) = 0$
  - $\sum_{i=1}^{n} v_i(x_i^*) = 1$
- if the attribute is discrete
  - take as many variables as there are levels
- if the attribute is not discrete
  - consider a piecewise linear approximation

- discrete attribute
  - $X_i = \{x_{i*}, x_i^1, x_i^2, \ldots, x_i^{r_i}, x_i^*\}$
- continuous attribute
  - choose the number of linear pieces $r_i + 1$
  - $[x_{i*}, x_i^1], [x_i^1, x_i^2], \ldots, [x_i^{r_i-1}, x_i^{r_i}], [x_i^{r_i}, x_i^*]$





UTA: constraints

Using these conventions

- for all x, $v(x) = \sum_{i=1}^{n} v_i(x_i)$ can be expressed as a linear combination of the $\sum_i^n (r_i + 1)$ variables

$$x \succ y \Leftrightarrow v(x) > v(y)$$
$$v(x) - v(y) + \sigma^+(xy) - \sigma^-(xy) \geq \varepsilon$$

$$x \sim y \Leftrightarrow v(x) = v(y)$$
$$v(x) - v(y) + \sigma^+(xy) - \sigma^-(xy) = 0$$

UTA: LP

$$\text{minimize } Z = \sum_{\text{constraints}} \sigma^+(xy) + \sigma^-(xy)$$

s.t.

one constraint per pair of compared alternatives

normalization constraints

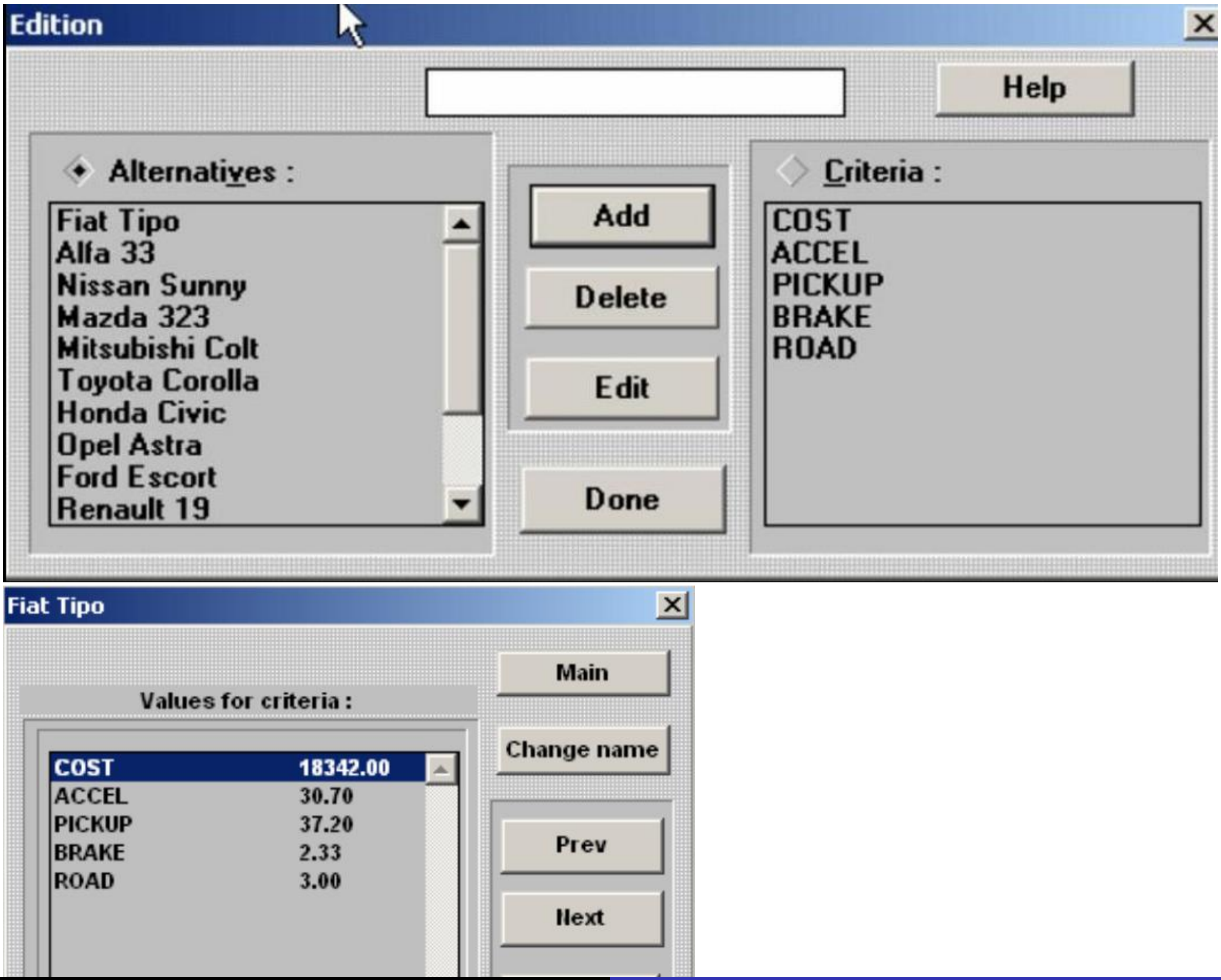
UTA: analyzing results

If $Z^* = 0$

- there is one additive value function compatible with the given information
- there are infinitely many (identically normalized) compatible additive value functions $v \in \mathcal{V}$
- use post-optimality analysis and/or interaction to explore $\mathcal{V}$

If $Z^* > 0$

- there is no additive value function compatible with the given information
- interact
  - increase the number of linear pieces
  - decrease ε
  - modify ranking
  - diagnostic a failure of independence
  - use approximate function



Summary

Conjoint measurement

- highly consistent theory
- together with practical assessment techniques

Why consider extensions?

- hypotheses may be violated
- assessment is demanding
    - time
    - cognitive effort

Part II

A glimpse at possible extensions

Summary

Additive value function model

- requires independence
- requires a finely grained analysis of preferences

Two main types of extensions

1. models with interactions
2. more ordinal models

Interactions

Two extreme models

- additive value function model
  - independence
- decomposable model
  - only weak independence

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots v_n(x_n)] \geq F[v_1(y_1), \ldots v_n(y_n)]$$

Decomposable models

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots v_n(x_n)] \geq F[v_1(y_1), \ldots v_n(y_n)]$$

F increasing in all arguments

Result

Under mild conditions, any weakly independent weak order may be represented in the decomposable model

Problem

- all possible types of interactions are admitted
- assessment is a very challenging task

Two main directions

Extensions

1. work with the decomposable model
   - rough sets
2. find models “in between additive” and decomposable
   - CP-nets, GAI
   - fuzzy integrals

Rough sets

Basic ideas

- work within the general decomposable model
- use the same principle as in UTA
- replacing the numerical model by a symbolic one
- infer decision rules

IF

$x_1 \geq a_1, \dots, x_i \geq a_i, \dots, x_n \geq a_n$ and

$y_1 \leq b_1, \dots, y_i \leq b_i, \dots, y_n \leq b_n$

THEN

$x \succsim y$

- many possible variants
- Greco, Matarazzo, Słowiński

GAI: Example

Choice of a meal: 3 attributes

$X_1 = \{\text{Steak}, \text{Fish}\}$
$X_2 = \{\text{Red}, \text{White}\}$
$X_3 = \{\text{Cake}, \text{sherBet}\}$

Preferences

$$x^1 = (S,R,C) \quad x^2 = (S,R,B) \quad x^3 = (S,W,C) \quad x^4 = (S,W,B)$$
$$x^5 = (F,R,C) \quad x^6 = (F,R,B) \quad x^7 = (F,W,C) \quad x^8 = (F,W,B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

- the important is to match main course and wine
- I prefer Steak to Fish
- I prefer Cake to sherBet if Fish
- I prefer sherBet to Cake if Steak

Models with interactions | GAI networks

Example

$$x^1 = (S,R,C) \quad x^2 = (S,R,B) \quad x^3 = (S,W,C) \quad x^4 = (S,W,B)$$
$$x^5 = (F,R,C) \quad x^6 = (F,R,B) \quad x^7 = (F,W,C) \quad x^8 = (F,W,B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

Independence

$$x^1 \succ x^5 \Rightarrow v_1(S) > v_1(F)$$
$$x^7 \succ x^3 \Rightarrow v_1(F) > v_1(S)$$

Grouping main course and wine?

$$x^7 \succ x^8 \Rightarrow v_3(C) > v_3(B)$$
$$x^2 \succ x^1 \Rightarrow v_3(B) > v_3(C)$$

Example

$$x^1 = (S,R,C) \quad x^2 = (S,R,B) \quad x^3 = (S,W,C) \quad x^4 = (S,W,B)$$
$$x^5 = (F,R,C) \quad x^6 = (F,R,B) \quad x^7 = (F,W,C) \quad x^8 = (F,W,B)$$

$$x^2 \succ x^1 \succ x^7 \succ x^8 \succ x^4 \succ x^3 \succ x^5 \succ x^6$$

Model

$$x \succsim y \Leftrightarrow u_{12}(x_1,x_2) + u_{13}(x_1,x_3) \geq u_{12}(y_1,y_2) + u_{13}(y_1,y_3)$$

$$u_{12}(S,R) = 6 \quad u_{12}(F,W) = 4 \quad u_{12}(S,W) = 2 \quad u_{12}(F,R) = 0$$
$$u_{13}(S,C) = 0 \quad u_{13}(S,B) = 1 \quad u_{13}(F,C) = 1 \quad u_{13}(F,S) = 0$$

Generalized Additive Independence

GAI (Gonzales & Perny)

- axiomatic analysis
- if interdependences are known
  - assessment techniques
  - efficient algorithms (compactness of representation)

What R. L. Keeney would probably say

- the attribute “richness” of meal is missing

GAI

- interdependence within a framework that is quite similar to that of classical theory
- powerful generalization of recent models in Computer Science

Fuzzy integrals

Origins

- decision making under uncertainty
  - homogeneous Cartesian product
- mathematics
  - integrating w.r.t. a non-additive measure
- game theory
  - cooperative TU games
- multiattribute decisions
  - generalizing the weighted sum

Example

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$a \succ b \quad d \succ c$$

Preferences

a is fine for Engineering d is fine for Economics

Interpretation: interaction

- having good grades in both
  - Math *and* Physics or
  - Maths *and* Economics
- better than having good grades in both
  - Physics *and* Economics

Weighted sum

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$a \succ b \Rightarrow 18w_1 + 12w_2 + 6w_3 > 18w_1 + 7w_2 + 11w_3 \Rightarrow w_2 > w_3$$
$$d \succ c \Rightarrow 5w_1 + 17w_2 + 8w_3 > 5w_1 + 12w_2 + 13w_3 \Rightarrow w_3 > w_2$$

Choquet integral

Capacity

$$\mu : 2^N \to [0,1]$$
$$\mu(\varnothing) = 0, \mu(N) = 1$$
$$A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$$

Choquet integral

$$0 = x_{(0)} \leq x_{(1)} \leq \cdots \leq x_{(n)}$$

$$\begin{array}{rr} x_{(1)} - x_{(0)} & \mu(\{(1),(2),(3),(4)\ldots,(n)\}) \\ x_{(2)} - x_{(1)} & \mu(\{(2),(3),(4)\ldots,(n)\}) \\ x_{(3)} - x_{(2)} & \mu(\{(3),(4)\ldots,(n)\}) \\ \ldots & \ldots \\ x_{(n)} - x_{(n-1)} & \mu(\{(n)\}) \end{array}$$

$$\mathcal{C}_\mu(x) = \sum_{i=1}^{n} \left[x_{(i)} - x_{(i-1)}\right] \mu(A_{(i)})$$
$$A_{(i)} = \{(i),(i+1),\ldots,(n)\}$$

Application

	Physics	Maths	Economics
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

$$\mu(M) = 0.1, \mu(P) = 0.5, \mu(E) = 0.5$$
$$\mu(M,P) = 1 > \mu(M) + \mu(P)$$
$$\mu(M,E) = 1 > \mu(M) + \mu(E)$$
$$\mu(P,E) = 0.6 < \mu(P) + \mu(E)$$

$$\mathcal{C}_\mu(a) = 6 \times 1 + (12-6) \times 1 + (18-12) \times 0.5 = 15.0$$
$$\mathcal{C}_\mu(b) = 7 + (11-7) \times 0.6 + (18-11) \times 0.5 = 12.9$$
$$\mathcal{C}_\mu(c) = 5 + (8-5) \times 1 + (17-8) \times 0.1 = 8.9$$
$$\mathcal{C}_\mu(d) = 5 + (12-5) \times 1 + (13-12) \times 0.5 = 12.5$$

Choquet integral in MCDM

Properties

- monotone, idempotent, continuous
- preserves weak separability
- tolerates violation of independence
- contains many other aggregation functions as particular cases

Capacities

Fascinating mathematical object:

- Möbius transform
- Shapley value
- interaction indices

Questions

Hypotheses

- I can compare x_i with x_j
  - attributes are (level) commensurable

Classical model

- I can indirectly compare $[x_i, y_i]$ with $[x_j, y_j]$

Central research question

- how to assess $u : \bigcup_{i=1}^{n} X_i \to \mathbb{R}$ so that the levels are commensurate?

Choquet integral

Assessment

- variety of mathematical programming based approaches

Extensions

- Choquet integral with a reference point (statu quo)
- Sugeno integral (median)
- axiomatization as aggregation functions
- k-additive capacities

Observations

Classical model

- deep analysis of preference that may not be possible
  - preference are not well structured
  - several or no DM
  - prudence

Idea

- it is not very restrictive to suppose that levels on each X_i can be ordered
- aggregate these orders
- possibly taking importance into account

Social choice

- aggregate the preference orders of the voters to build a collective preference

Outranking methods

ELECTRE

$x \succsim y$ if

- Concordance: a "majority" of attributes support the assertion
- Discordance: the opposition of the minority is not "too strong"

$$x \succsim y \Leftrightarrow \begin{cases} \sum_{i: x_i \succsim_i y_i} w_i \geq s \\ Not[y_i \; V_i \; x_i], \forall i \in N \end{cases}$$

Problem

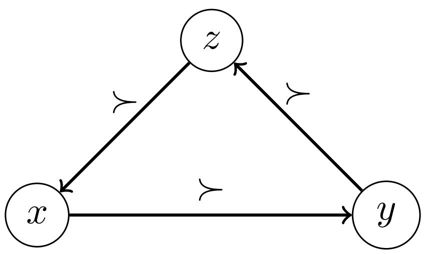
- $\succsim$ may not be complete
- $\succsim$ may not be transitive
- $\succ$ may have cycles

Condorcet's paradox

$$x \succsim y \Leftrightarrow |\{i \in N : x_i \succsim_i y_i\}| \geq |\{i \in N : y_i \succsim_i x_i\}|$$

$$\begin{aligned} 1 &: x_1 \succ_1 y_1 \succ_1 z_1 \\ 2 &: z_2 \succ_2 x_2 \succ_2 y_2 \\ 3 &: y_3 \succ_3 z_3 \succ_3 x_3 \end{aligned}$$

$$\begin{aligned} x &= (x_1, x_2, x_3) \\ y &= (y_1, y_2, y_3) \\ z &= (z_1, z_2, z_3) \end{aligned}$$



Arrow's theorem

Theorem

The only ways to aggregate weak orders while remaining ordinal are not very attractive. . .

- dictator (weak order)
- oligarchy (transitive $\succ$)
- veto (acyclic $\succ$)

Ways out

Accepting intransitivity

- find way to extract information in spite of intransitivity
  - ELECTRE I, II, III, IS
  - PROMETHEE I, II

Do not use paired comparisons

- only compare x with carefully selected alternatives
  - ELECTRE TRI
  - methods using reference points

Conclusion

Fascinating field

- theoretical point of view
  - measurement theory
  - decision under uncertainty
  - social choice theory
- practical point of view
  - rating firms from a social point of view
  - evaluating H_2-propelled cars

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