

A note on ELECTRE TRI-nB with few limiting profiles^a

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Abstract

ELECTRE TRI-nB is a method designed to sort alternatives evaluated on several attributes among ordered categories. It is an extension of ELECTRE TRI-B that uses several limiting profiles, instead of one, to delimit each category. In a companion paper we have characterized the partitions that can be obtained with ELECTRE TRI-nB, using a simple axiom called linearity. The simplicity of this characterization crucially depends on the possibility to use as many limiting profiles as we like to delimit a category. This is not completely realistic and there is a need to study models in which the number of limiting profiles delimiting each category is restricted. This note starts such a study. We investigate the case of ELECTRE TRI-nB models based on unanimity and using no more than two limiting profiles per category. Our results show that the constraints on the number of limiting profiles make the problem difficult.

Keywords: Multiple criteria analysis, Sorting models, ELECTRE TRI.

1 Introduction

This paper is about ELECTRE TRI (often abbreviated as ETRI in what follows). The original method called ETRI-B (Roy and Bouyssou, 1993, Yu, 1992) uses one limiting profile to delimit each category. Fernández, Figueira, Navarro, and Roy

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(2017) have suggested an extension of this method, called ETRI-nB, that uses several limiting profiles to delimit each category¹.

In a companion paper (Bouyssou, Marchant, and Pirlot, 2020), we have given a theoretical look at ETRI-nB. We presented a simple characterization of the partitions that can be obtained using the pseudo-conjunctive version of ETRI-nB (ETRI-nB-pc). This characterization uses a condition, linearity, that is familiar (Bouyssou and Marchant, 2007, Goldstein, 1991, Greco, Matarazzo, and Słowiński, 2001, Słowiński, Greco, and Matarazzo, 2002) in the study of sorting models and that has a transparent interpretation. This allowed us to precisely position ETRI-nB-pc within the larger family of sorting model for alternatives evaluated on several attributes.

A limitation of the analysis in Bouyssou et al. (2020) is that the simple characterization is obtained when there is no restriction on the number of profiles delimiting each category. If practical applications of ETRI-nB are sought for, this is problematic. Hence, it is useful to study models in which the number of profiles used to delimit each category is constrained. Such constraints will surely facilitate the elicitation of the parameters of the model (Fernández, Figueira, and Navarro, 2019). This is the purpose of this note.

Our initial aim was to study the expressiveness gain brought by increasing the size of the set of profiles in ETRI-nB. When there are two categories, going from a single profile, the case studied in Bouyssou and Marchant (2007), to an arbitrarily large number of profiles, the case studied in Bouyssou et al. (2020), leads to a *huge* expressiveness gain. We are interested to know whether this gain is already present going from one to a small number of profiles. The problem turns out to be more difficult than the one studied in Bouyssou et al. (2020). We have no complete answer at this time. We will illustrate these difficulties with the study of “unanimous” ETRI-nB models (meaning that the only winning coalition is the set of all attributes; such models are obviously without discordance) with two categories.

The rest of this text is organized as follows. Section 2 introduces our notation and framework. Section 3 presents some background results on ETRI-B and ETRI-nB. Section 4 presents our main results. A final section discusses our findings.

2 Notation and framework

As in Bouyssou et al. (2020), we restrict our attention to the case of two categories. This allows us to use a simple framework while not concealing any important difficulty. For the same reasons, we suppose throughout that the set of objects

¹In the rest of this text, we write “profile” instead of “limiting profile”. All the profiles used below are limiting profiles.

to be sorted is *finite*. Our setting and vocabulary is exactly as in Bouyssou et al. (2020)

2.1 The setting

Let $n \geq 2$ be an integer and $X = X_1 \times X_2 \times \cdots \times X_n$ be a *finite* set of objects. Elements x, y, z, \dots of X are interpreted as alternatives evaluated on a set $N = \{1, 2, \dots, n\}$ of attributes. For $x, y \in X$, $i \in N$ and $J \subseteq N$, we use X_J , X_{-J} , X_i , X_{-i} , (x_J, y_{-J}) , and (x_i, y_{-i}) , as is usual.

Our primitives consist in a twofold *partition* $\langle \mathcal{A}, \mathcal{U} \rangle$ of the set X . We interpret the set \mathcal{A} as containing sAtisfactory objects, while \mathcal{U} contains Unsatisfactory ones.

We say that an attribute $i \in N$ is influential for $\langle \mathcal{A}, \mathcal{U} \rangle$ if there are $x_i, y_i \in X_i$ and $a_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \in \mathcal{A}$ and $(y_i, a_{-i}) \in \mathcal{U}$. We say that an attribute is degenerate if it is not influential. Suppressing, if necessary, degenerate attributes, it is not restrictive to suppose that all attributes are influential for $\langle \mathcal{A}, \mathcal{U} \rangle$. We do so in what follows.

Remark 1

Suppose that I is a proper nonempty subset of N and $i \in I$. It is clear that, if $i \in N$ is influential, then I is influential, i.e., there are $x_I, y_I \in X_I$ and $a_{-I} \in X_{-I}$ such that $(x_I, a_{-I}) \in \mathcal{A}$ and $(y_I, a_{-I}) \in \mathcal{U}$. Our hypothesis concerning the influence of each attribute $i \in N$ therefore implies that all proper nonempty subsets I of attributes are influential. •

A twofold partition $\langle \mathcal{A}, \mathcal{U} \rangle$ induces on each $i \in N$ a binary relation defined letting, for all $i \in N$ and all $x_i, y_i \in X_i$,

$$x_i \sim_i y_i \text{ if } [\forall a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Leftrightarrow (x_i, a_{-i}) \in \mathcal{A}].$$

This relation is always reflexive, symmetric and transitive, i.e., is an equivalence². As in Bouyssou et al. (2020), it is not restrictive to suppose that all equivalence classes of \sim_i are trivial, i.e., reduced to a single element.

2.2 Models (E) , (E^c) , and (E^u)

The following definition is taken from Bouyssou et al. (2020, Def. 6).

Definition 2

We say that a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E) if:

²Our conventions concerning binary relations are standard. They are *exactly* as in Bouyssou et al. (2020).

- for all $i \in N$, there is a semiorder S_i on X_i (with asymmetric part P_i and symmetric part I_i),
- for all $i \in N$, there is a strict semiorder V_i on X_i that is included in P_i and is the asymmetric part of a semiorder U_i ,
- for all $i \in N$, (S_i, U_i) is a *homogeneous* nested chain of semiorders and $W_i = S_i^{wo} \cap U_i^{wo}$ is a weak order that is compatible with both S_i and U_i ,
- there is a set of subsets of attributes $\mathcal{F} \subseteq 2^N$ such that, for all $I, J \in 2^N$, $[I \in \mathcal{F} \text{ and } I \subseteq J] \Rightarrow J \in \mathcal{F}$,
- there is a binary relation S on X (with symmetric part I and asymmetric part P) defined by:

$$x S y \Leftrightarrow [S(x, y) \in \mathcal{F} \text{ and } V(y, x) = \emptyset],$$

- there is a finite set $\mathcal{P} = \{p^1, \dots, p^k\} \subseteq X$ of profiles, such that for all $p, q \in \mathcal{P}$, $\text{Not}[p P q]$,

such that

$$x \in \mathcal{A} \Leftrightarrow \begin{cases} x S p & \text{for some } p \in \mathcal{P} \\ \text{Not}[q P x] & \text{for all } q \in \mathcal{P}, \end{cases} \quad (E)$$

where,

$$S(x, y) = \{i \in N : x_i S_i y_i\},$$

and

$$V(x, y) = \{i \in N : x_i V_i y_i\}.$$

We then say that $\langle (S_i, V_i)_{i \in N}, \mathcal{F}, \mathcal{P} \rangle$ is a representation of $\langle \mathcal{A}, \mathcal{U} \rangle$ in Model (E). Model (E^c) is the particular case of Model (E), in which there is a representation that shows no discordance effects, i.e., in which all relations V_i are empty. Model (E^u) is the particular case of Model (E), in which there is a representation that requires unanimity, i.e., such that $\mathcal{F} = \{N\}$.

We say that a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ can be represented in Model ($E_{\leq k}$) (resp. (E_k)) if it has a representation in Model (E) that uses no more than k (resp. exactly k) profiles. We define Models ($E_{\leq k}^c$), (E_k^c), ($E_{\leq k}^u$) and (E_k^u) accordingly. \perp

The relations between Model (E) and ETRI-nB-pc were analyzed in detail in Bouyssou et al. (2020, Remarks 8 and 17). To keep this note short, we do not repeat this analysis here.

Ideally we would like to characterize Model ($E_{\leq k}$) for all “small” values of k . Clearly, we expect models using a small number of profiles to be assessed more

easily than models using a larger number of profiles. Moreover, our hope is that this characterization would help us understand the expressiveness gain that is obtained when using 1, 2, 3, ... profiles. Unfortunately, we are still far from this objective. In the sequel we will characterize Model $(E_{\leq 2}^u)$, i.e., the particular case of Model (E) based on unanimity and using no more than two profiles. Before we do so, it will be useful to recall a number of background results.

3 Background results

This section presents a few existing results that will later be helpful.

3.1 A characterization of Model (E)

Following Bouyssou et al. (2020), we define on each X_i the binary relation \succsim_i letting, for all $x_i, y_i \in X_i$,

$$x_i \succsim_i y_i \text{ if [for all } a_{-i} \in X_{-i}, (y_i, a_{-i}) \in \mathcal{A} \Rightarrow (x_i, a_{-i}) \in \mathcal{A}].$$

This relation is, by construction, reflexive and transitive. The symmetric part of the relation \succsim_i clearly coincides with the relation \sim_i defined above.

We say that the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is linear on attribute $i \in N$ (condition i -linear) if, for all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$,

$$\left. \begin{array}{c} (x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (y_i, a_{-i}) \in \mathcal{A}, \\ \text{or} \\ (x_i, b_{-i}) \in \mathcal{A}. \end{array} \right. \quad (i\text{-linear})$$

The partition is said to be linear if it is i -linear, for all $i \in N$. This condition was first proposed in Goldstein (1991) and generalized in Greco et al. (2001) and Słowiński et al. (2002). It is easy to check that a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is i -linear iff the relation \succsim_i is complete.

Let us define the relation \succsim on X letting, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow x_i \succsim_i y_i, \text{ for all } i \in N.$$

Let $\mathcal{A}_* = \text{Min}(\mathcal{A}, \succsim)$ be the set of minimal elements in \mathcal{A} for \succsim .

The main result in Bouyssou et al. (2020) is as follows.

Proposition 3

Let $X = \prod_{i=1}^n X_i$ be a finite set and $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of X . The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E) iff it is linear. This representation can always be taken to be $\langle (\succsim_{i \in N}, \mathcal{F} = \{N\}, \mathcal{P} = \mathcal{A}_) \rangle$. Hence, Models (E) , (E^c) and (E^u) are equivalent.*

The logic of the proof of this result is simple. The relation \succsim is a partial order, being reflexive, antisymmetric and transitive. Because X is finite and \mathcal{A} is nonempty, we know that the set \mathcal{A}_* is nonempty. Our proof consists in taking $\mathcal{P} = \mathcal{A}_*$ and $\mathcal{F} = \{N\}$. All elements in \mathcal{A}_* are taken as profiles. But the cardinality of this set may be high (Bouyssou, Marchant, and Pirlot, 2019). Hence, it would be interesting to have a characterization of model (E) with a limited number of profiles.

3.2 A characterization of Models (E_1) , (E_1^c) and (E_1^u)

Models (E_1) , (E_1^c) and (E_1^u) have been studied in Bouyssou and Marchant (2007). Model (E_1) is the *noncompensatory sorting model with veto*. Model (E_1^c) is the *noncompensatory sorting model*. Model (E_1^u) is the particular case of the non-compensatory sorting model in which $\mathcal{F} = \{N\}$, called the *conjunctive sorting model*.

3.2.1 Model (E_1^c)

We say that a partition is strongly 2-graded on attribute $i \in N$ (condition i -s-2-graded) if

$$\left. \begin{array}{c} (x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (x_i, b_{-i}) \in \mathcal{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathcal{A}, \end{array} \right. \quad (i\text{-s-2-graded})$$

for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$.

Bouyssou and Marchant (2007) have shown that condition i -s-2-graded implies i -linearity and the fact that the relation \succsim_i has at most two equivalence classes (and, hence, two equivalence classes, since a relation \succsim_i with a single equivalence class means that this attribute is not influential). They also show how to factorize this condition so as to make i -linearity appear explicitly. We will not use this factorization in the present paper.

Bouyssou and Marchant (2007, Th. 21, p. 230) have shown the following.

Proposition 4

A partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in the noncompensatory sorting model (E_1^c) iff it is strongly 2-graded on all $i \in N$.

3.2.2 Model (E_1)

We say that the partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is strongly 3-graded with veto on attribute $i \in N$ (condition i -s-3v-graded) if, for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i}, c_i \in X_{-i}$,

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (z_i, c_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (x_i, b_{-i}) \in \mathcal{A} \\ \text{or} \\ (z_i, a_{-i}) \in \mathcal{A} \end{array} \right. \quad (i\text{-s-3v-graded})$$

for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i}, c_i \in X_{-i}$.

Bouyssou and Marchant (2007) have shown that condition $i\text{-s-3v-graded}$ implies $i\text{-linearity}$ and the fact that the relation \succsim_i has at most three equivalence classes. When the relation \succsim_i has exactly three equivalence classes, the last one acts as a veto, i.e., when an object has an evaluation in this equivalence class, it cannot belong to \mathcal{A} . They also show how to factorize this condition so as to make $i\text{-linearity}$ appear explicitly. This factorization will not be useful here.

The following definition, taken from Bouyssou and Marchant (2007), will be useful.

Definition 5

Let

$$Z_i = \{x_i \in X_i : (x_i, a_{-i}) \in \mathcal{U}, \text{ for all } a_{-i} \in X_{-i}\}.$$

Let $Y_i = X_i \setminus Z_i$. Consider the set $Y = \prod_{i=1}^n Y_i$ and let $\mathcal{A}' = \mathcal{A} \cap Y$ and $\mathcal{U}' = \mathcal{U} \cap Y$. \sqcup

Since $\langle \mathcal{A}, \mathcal{U} \rangle$ is a partition, we have $\mathcal{A} \neq \emptyset$ so that Y cannot be empty. It is easy to see that if condition $i\text{-s-3v-graded}$ holds for $\langle \mathcal{A}, \mathcal{U} \rangle$ on X , then condition $i\text{-s-2-graded}$ holds for the partition $\langle \mathcal{A}', \mathcal{U}' \rangle$ of Y .

Bouyssou and Marchant (2007, Th. 35, p. 237) have shown the following.

Proposition 6

A partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in the noncompensatory sorting model with veto (E_1) iff it is strongly 3v-graded on all $i \in N$.

3.2.3 Model (E_1^u)

We say $\langle \mathcal{A}, \mathcal{U} \rangle$ is $i\text{-strongly}^*\text{-conjunctive}$ (condition $i\text{-s}^*\text{-conj}$) if

$$\left. \begin{array}{l} (x_i, a_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, b_{-i}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (x_i, b_{-i}) \in \mathcal{A} \\ \text{and} \\ (y_i, a_{-i}) \in \mathcal{A} \end{array} \right.$$

for all $x_i, y_i, z_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$.

Bouyssou and Marchant (2007) have shown that this condition implies $i\text{-linear}$ and $i\text{-s-2-graded}$. They also show how to factorize this condition so as to make $i\text{-linearity}$ appear explicitly. This factorization will not be useful here.

Bouyssou and Marchant (2007, Prop. 32, p. 235) prove the following:

Proposition 7

A partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in the conjunctive sorting model (E_1^u) iff it satisfies i - s^* -conj, for all $i \in N$.

4 A characterization of Model $(E_{\leq 2}^u)$

Proposition 3 shows that, if k is taken large enough, a partition $\langle \mathcal{A}, \mathcal{U} \rangle$ that has a representation in model (E^c) or, more generally, in model (E) , also has a representation in model $(E_{\leq k}^u)$. This seems to call for the study of models $(E_{\leq k}^u)$ with small values of k . We start here with $k = 2$.

4.1 Observations and conventions

In Model (E_1^u) , all relations \succsim_i have at most two equivalence classes. The second equivalence class plays the role of a veto: an alternative that has an evaluation belonging to the second equivalence class on any attribute cannot belong to \mathcal{A} . In Model (E_1^u) , it is easy to see that it is not restrictive to take $S_i = \succsim_i$.

In Model (E_2^u) , all relations \succsim_i can have at most three equivalence classes: (i) strictly below the two profiles, (ii) above the lower profile and strictly below the upper profile (iii) above the upper profile. Moreover, when the relation \succsim_i has three distinct equivalence classes, the last class acts as a veto: an object that is below the two profiles on some attribute cannot belong to \mathcal{A} . In Model (E_2^u) , it is easy to see that it is not restrictive to take $S_i = \succsim_i$.

In Model (E_2^u) , it is easy to check that, on each attribute $i \in N$, the following three situations may happen:

1. the relation \succsim_i has exactly three distinct equivalence classes. This means that the two profiles have distinct (i.e., not belonging to the same equivalence class of \sim_i) values on attribute i and that something is below the lower profile.
2. the relation \succsim_i has exactly two distinct equivalence classes and the last class does not correspond to a veto. This means that the two profiles have distinct values on attribute i but that there is nothing below the lower profile.
3. the relation \succsim_i has exactly two distinct equivalence classes and the last class corresponds to a veto. This means that the two profiles have identical values on attribute i .

The treatment of the third case (in view of our convention regarding \sim_i , if p and q are the two profiles, the third case corresponds to an attribute for which $p_i = q_i$) is not difficult but it complicates things. Henceforth, we suppose that the third case does not happen. To avoid any misunderstanding, let us call (E_2^{u*}) the

particular case of Model (E_2^u) in which case 3 is excluded. Model $(E_{\leq 2}^{u*})$ is defined accordingly as either Model (E_1^u) or Model (E_2^{u*}) .

Using notation introduced in Definition 5, working with (E_2^{u*}) instead of (E_2^u) implies that all attributes are influential for the partition $\langle \mathcal{A}', \mathcal{U}' \rangle$ of Y .

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E_2^{u*}) with $\mathcal{P} = \{p, q\}$. This defines a partition of the set of all attributes into $N^p = \{i \in N : p_i \succ_i q_i\}$, the set of attribute for which p is above q and $N^q = \{i \in N : q_i \succ_i p_i\}$, the set of attributes for which q is above p . We have $N^p \neq \emptyset$, $N^q \neq \emptyset$, $N^p \cap N^q = \emptyset$ and $N^p \cup N^q = N$.

Remark 8

As already mentioned, our main motivation for studying Model (E_2^u) is linked to the fact that Model (E^u) is equivalent to Model (E) , when the number of profiles is not constrained. Besides this theoretical motivation, Model (E_2^u) may have an interest in itself because it relates to already familiar models.

For instance, a partition that can be represented in the noncompensatory sorting model (Model (E_1^c) , using the notation of this paper) studied in Bouyssou and Marchant (2007) can sometimes be represented in Model $(E_{\leq 2}^u)$.

In Model (E_1^c) , all relations \succsim_i have two distinct equivalence classes. Define $\mathcal{F}_* = \text{Min}(\mathcal{F}, \supseteq)$ as the set of all minimal winning coalitions w.r.t. set inclusion. When $|\mathcal{F}_*| = 2$, such a model can also be represented in Model $(E_{\leq 2}^u)$. A simple example of such a situation is the following. Take $N = \{1, 2, 3, 4\}$ and, for all $i \in N$, $X_i = \{0, 10\}$. Let $x \in \mathcal{A}$ iff $x_i = 10$ on, at least, either the first two or the last two attributes. This partition has a representation in Model (E_1) with the profile $p = (10, 10, 10, 10)$, $\mathcal{F}_* = \{\{1, 2\}, \{3, 4\}\}$, and $S_i = \geq$, for all $i \in N$. It can also be represented in Model $(E_{\leq 2}^u)$, keeping the same relations S_i , with $\mathcal{F} = \{N\}$ and the two profiles $(10, 10, 0, 0)$ and $(0, 0, 10, 10)$.

However, it is clear there are partitions that can be represented in Model (E_1^c) but that cannot be represented in Model $(E_{\leq 2}^u)$, see Example 35 below. This happens, in particular, when $|\mathcal{F}_*| > 2$.

Let us finally observe that there are some noncompensatory sorting models *with veto*, as defined in Bouyssou and Marchant (2007) (Model (E_1) , using the notation in the present paper) that can be represented in Model $(E_{\leq 2}^u)$. A simple example of such a situation is the following. Take $N = \{1, 2, 3, 4\}$ and, for all $i \in N$, $X_i = \{0, 5, 10\}$. Let $x \in \mathcal{A}$ iff $x_i \geq 10$ on either the first two or the last two attributes and, for all $i \in N$, $x_i \neq 0$. This partition has a representation in Model (E_1) with the profile $p = (10, 10, 10, 10)$, $\mathcal{F}_* = \{\{1, 2\}, \{3, 4\}\}$, $S_i = \geq$, for all $i \in N$, and $10 V_i 0$, for all $i \in N$. It can also be represented in Model $(E_{\leq 2}^{u*})$, keeping the same relations S_i and V_i , with $\mathcal{F} = \{N\}$ and the two profiles $(10, 10, 5, 5)$ and $(5, 5, 10, 10)$, •

4.2 Axioms

We introduce several conditions that are simple adaptations of already encountered conditions to the case of a nonempty subset $I \subseteq N$ of attributes, instead of just a single attribute $i \in N$.

4.2.1 I -linearity

We say that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -linearity (condition I -linear) if

$$\left. \begin{array}{c} (x_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (x_I, b_{-I}) \in \mathcal{A} \\ \text{or} \\ (y_I, a_{-I}) \in \mathcal{A} \end{array} \right. \quad (I\text{-linear})$$

for all $x_I, y_I \in X_I$ and all $a_{-I}, b_{-I} \in X_{-I}$.

This condition is familiar (e.g., Bouyssou and Marchant, 2009, 2010, Vind, 1991, 2003). Its interpretation is similar to that of condition i -linear. It ensures that on the Cartesian product of the sets X_i for the attributes i belonging to I , one can define a weak order that is compatible with the partition. The relation \succsim_I on X_I is defined letting

$$x_I \succsim_I y_I \Leftrightarrow [(y_I, a_{-I}) \in \mathcal{A} \Rightarrow (x_I, a_{-I}) \in \mathcal{A}],$$

for all $x_I, y_I \in X_I$ and all $a_{-I} \in X_{-I}$. We use \succ_I and \sim_I as is usual.

It is clear that the relation \succsim_I is always transitive but may not be complete. We leave to the reader the simple proof of the following lemma.

Lemma 9

A partition is I -linear iff \succsim_I is complete.

4.2.2 I -s-3v-graded

We say that $\langle \mathcal{A}, \mathcal{U} \rangle$ is strongly 3-graded with veto on I (condition I -s-3v-graded) if

$$\left. \begin{array}{c} (x_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{A} \\ \text{and} \\ (z_I, c_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (x_I, b_{-I}) \in \mathcal{A} \\ \text{or} \\ (z_I, a_{-I}) \in \mathcal{A} \end{array} \right. \quad (I\text{-s-3v-graded})$$

for all $x_I, y_I, z_I \in X_I$ and all $a_{-I}, b_{-I} \in X_{-I}$.

The interpretation of this condition is exactly similar to that of condition i -s-3v-graded. When I -s-3v-graded holds, the relation \succsim_I is a weak order that can have at most three equivalence classes. When it has three equivalence classes, the bottom class acts as a veto: an object that has on the attributes in I an evaluation belonging to this equivalence class cannot be in \mathcal{A} .

Remark 10

Following the analysis in Bouyssou and Marchant (2007), it is easy to factorize Condition I -s-3v-graded so as to make I -linear appear explicitly. This factorization will not be used in the present paper. •

We leave to the reader the simple proofs of the following two lemmas.

Lemma 11

Condition I -s-3v-graded implies Condition I -linear.

Lemma 12

Condition I -s-3v-graded is equivalent to saying that \succsim_I is a weak order having at most three equivalence classes and that, moreover, if \succsim_I has exactly three distinct equivalence classes and if x_I belongs to the last equivalence class of \succsim_I then $(x_I, a_{-I}) \in \mathcal{U}$, for all $a_{-I} \in X_{-I}$.

Let us also notice the following: when I and J partition N , their roles are symmetric in the expression of I -s-3v-graded.

Lemma 13

Let I, J be a partition of N . The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies Condition I -s-3v-graded iff it satisfies Condition J -s-3v-graded.

PROOF

Suppose that I -s-3v-graded is violated. Hence, we have, for some $x_I, y_I, z_I \in X_I$ and some $a_J, b_J, c_J \in X_J$, $(x_I, a_J) \in \mathcal{A}$, $(y_I, b_J) \in \mathcal{A}$, $(z_I, c_J) \in \mathcal{A}$, $(x_I, b_J) \in \mathcal{U}$, $(z_I, a_J) \in \mathcal{U}$. This implies $c_J \succ_J a_J \succ_J b_J$, while b_J is not a veto level. This is equivalent to saying that J -s-3v-graded is violated, as shown in Lemma 12. □

4.2.3 I -s-2-graded

We say that $\langle \mathcal{A}, \mathcal{U} \rangle$ is strongly 2-graded on I (condition I -s-2-graded) if

$$\left. \begin{array}{l} (x_I, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_I, b_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (x_I, b_{-I}) \in \mathcal{A} \\ \text{or} \\ (z_I, a_{-I}) \in \mathcal{A} \end{array} \right. \quad (I\text{-s-2-graded})$$

for all $x_I, y_I, z_I \in X_I$ and all $a_{-I}, b_{-I} \in X_{-I}$.

The interpretation of this condition is exactly similar to that of condition i -s-2-graded. When I -s-2-graded holds, the relation \succsim_I is a weak order that can have at most two equivalence classes.

Remark 14

Following the analysis in Bouyssou and Marchant (2007), it is easy to factorize Condition I -s-2-graded so as to make I -linear appear explicitly. This factorization will not be used in the present paper. •

We leave to the reader the simple proof of the following two lemmas.

Lemma 15

Condition I -s-2-graded implies Condition I -s-3v-graded and, hence, I -linear.

Lemma 16

A partition $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-2-graded iff \succsim_I is a weak order with at most two equivalence classes.

Remark 17

Let us notice here that Lemma 13 does not hold when I -s-3v-graded is replaced with I -s-2-graded. It is easy to build examples such that I -s-2-graded holds but J -s-2-graded fails while J -s-3v-graded holds. •

4.2.4 I -s-conj

We say that $\langle \mathcal{A}, \mathcal{U} \rangle$ is I -strongly-conjunctive (condition I -s-conj) if

$$\left. \begin{array}{c} (x_i, z_J, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (y_i, w_J, a_{-I}) \in \mathcal{A} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (y_i, z_J, a_{-I}) \in \mathcal{A} \\ \text{and} \\ (x_i, w_J, a_{-I}) \in \mathcal{A} \end{array} \right. \quad (I\text{-s-conj})$$

for all $i \in I$ (and defining $J = I \setminus \{i\}$), for all $x_i, y_i \in X_i$, all $z_J, w_J \in X_J$, and all $a_{-I} \in X_{-I}$.

Notice that Condition N -s-conj is nothing but condition i -s*-conj, for all $i \in N$, as defined above.

4.3 Main result

Theorem 18 will use conditions I -s-3v-graded, I -s-conj and J -s-conj for a given partition I, J of N . The logic of the proof will be as follows. First we show that veto effects on I only occur if there is a veto effect on $i \in I$. Restricting our attention to the case in which there are no veto effects, we show that the aggregation within the attributes in I is conjunctive. The same is true for the attributes in J . Hence, it remains to study how the aggregation of the attributes in I and J is performed. This is easily done.

Our main result is as follows.

Theorem 18

Let $X = \prod_{i=1}^n X_i$ be a finite set and $\langle \mathcal{A}, \mathcal{U} \rangle$ be a twofold partition of X . The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model $(E_{\leq 2}^{u})$ iff there is a partition I, J of N such that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-3v-graded, I -s-conj and J -s-conj.*

The proof appears below in 4.6 and 4.7, after some preparatory lemmas in 4.4 and 4.5. The independence of our conditions is discussed in 4.8. Notice that our conditions involve an existential statement (“there is a partition I, J of N such that”). Hence, they may not be easy to test. This is discussed below in 4.9.

4.4 Lemmas with I -s-3v-graded and I -s-conj

In all lemmas in 4.4 and 4.5, I is a proper nonempty subset of N . The common premise of all lemmas in 4.4 is that Conditions I -s-3v-graded and I -s-conj hold.

Lemma 19

If $i \in I$, the partition is i -linear.

PROOF

Take $J = I \setminus \{i\}$. Suppose that $(x_i, a_J, a_{-I}) \in \mathcal{A}$, $(y_i, b_J, b_{-I}) \in \mathcal{A}$, $(y_i, a_J, a_{-I}) \in \mathcal{U}$, $(x_i, b_J, b_{-I}) \in \mathcal{U}$, violating i -linearity. Since I -s-3v-graded implies I -linear, we have either $(x_i, a_J) \succsim_I (y_i, b_J)$ or $(y_i, b_J) \succsim_I (x_i, a_J)$.

In the first case, $(y_i, b_J, b_{-I}) \in \mathcal{A}$ implies $(x_i, a_J, b_{-I}) \in \mathcal{A}$. We now apply I -s-conj to the pair $(y_i, b_J, b_{-I}) \in \mathcal{A}$ and $(x_i, a_J, b_{-I}) \in \mathcal{A}$. This implies $(x_i, b_J, b_{-I}) \in \mathcal{A}$, a contradiction.

In the second case, $(x_i, a_J, a_{-I}) \in \mathcal{A}$ implies $(y_i, b_J, a_{-I}) \in \mathcal{A}$. We now apply I -s-conj to the pair $(y_i, b_J, a_{-I}) \in \mathcal{A}$ and $(x_i, a_J, a_{-I}) \in \mathcal{A}$. This implies $(y_i, a_J, a_{-I}) \in \mathcal{A}$, a contradiction. \square

Lemma 20

If $i \in I$, the partition satisfies i -s-3v-graded.

PROOF

Take $J = I \setminus \{i\}$. Suppose that $(x_i, a_J, a_{-I}) \in \mathcal{A}$, $(y_i, b_J, b_{-I}) \in \mathcal{A}$, $(z_i, c_J, c_{-I}) \in \mathcal{A}$, $(x_i, b_J, b_{-I}) \in \mathcal{U}$, and $(z_i, a_J, a_{-I}) \in \mathcal{U}$, violating i -s-3v-graded. Using I -s-3v-graded, $(x_i, a_J, a_{-I}) \in \mathcal{A}$, $(y_i, b_J, b_{-I}) \in \mathcal{A}$, $(z_i, c_J, c_{-I}) \in \mathcal{A}$, imply $(x_i, a_J, b_{-I}) \in \mathcal{A}$, or $(z_i, c_J, a_{-I}) \in \mathcal{A}$.

If $(x_i, a_J, b_{-I}) \in \mathcal{A}$, $(y_i, b_J, b_{-I}) \in \mathcal{A}$ together with I -s-conj imply $(x_i, b_J, b_{-I}) \in \mathcal{A}$, a contradiction. If $(z_i, c_J, a_{-I}) \in \mathcal{A}$, $(x_i, a_J, a_{-I}) \in \mathcal{A}$ together with I -s-conj imply $(z_i, a_J, a_{-I}) \in \mathcal{A}$, a contradiction. \square

Since I -s-3v-graded holds, we know that \succsim_I has at most three equivalence classes. Its last equivalence class may correspond to a veto situation. Since i -s-3v-graded holds, for all $i \in I$, the same is true for \succsim_i . Let us now show that the class of \succsim_I corresponding to a veto is exclusively composed of all elements c_I , for which we know that, for some $i \in I$, c_i corresponds to a veto.

Let $T_I = \{x_I \in X_I : (x_I, a_{-I}) \in \mathcal{U}, \text{ for all } a_{-I} \in X_{-I}\}$. Remember (see Definition 5) that we have $Z_i = \{x_i \in X_i : (x_i, a_{-i}) \in \mathcal{U}, \text{ for all } a_{-i} \in X_{-i}\}$.

Lemma 21

$x_I \in T_I \Leftrightarrow x_i \in Z_i$, for some $i \in I$.

PROOF

[\Leftarrow]. Take $J = I \setminus \{i\}$. It is clear that if $x_i \in Z_i$, then $(x_i, a_J) \in T_I$, for all $a_J \in X_J$.

[\Rightarrow]. Suppose that $x_I \in T_I$ and, for all $i \in I$, $x_i \notin Z_i$, i.e., $(x_i, a_J, a_{-I}) \in \mathcal{A}$, for some $a_J \in X_J, a_{-I} \in X_{-I}$. It is easy to check that repeated applications of I -s-conj and I -linear, which is implied by I -s-3v-graded, lead to the desired result.

Let us sketch the proof in the particular case in which $I = \{i, j, k\}$. By construction, we know that $(x_i, \alpha_j, \alpha_k, \alpha_{-I}) \in \mathcal{A}$, $(\beta_i, x_j, \beta_k, \beta_{-I}) \in \mathcal{A}$, $(\gamma_i, \gamma_j, x_k, \gamma_{-I}) \in \mathcal{A}$, for some $\beta_i, \gamma_i \in X_i$, some $\alpha_j, \gamma_j \in X_j$ and some $\alpha_k, \beta_k \in X_k$.

Because $(x_i, \alpha_j, \alpha_k, \alpha_{-I}) \in \mathcal{A}$ and $(\beta_i, x_j, \beta_k, \beta_{-I}) \in \mathcal{A}$, using I -linear, we obtain either $(x_i, \alpha_j, \alpha_k, \beta_{-I}) \in \mathcal{A}$ or $(\beta_i, x_j, \beta_k, \alpha_{-I}) \in \mathcal{A}$.

1. Suppose first that $(x_i, \alpha_j, \alpha_k, \beta_{-I}) \in \mathcal{A}$. Using I -s-conj and $(\beta_i, x_j, \beta_k, \beta_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, \alpha_k, \beta_{-I}) \in \mathcal{A}$. Using $(x_i, x_j, \alpha_k, \beta_{-I}) \in \mathcal{A}$ and $(\gamma_i, \gamma_j, x_k, \gamma_{-I}) \in \mathcal{A}$, I -linearity implies either $(x_i, x_j, \alpha_k, \gamma_{-I}) \in \mathcal{A}$ or $(\gamma_i, \gamma_j, x_k, \beta_{-I}) \in \mathcal{A}$.

If $(x_i, x_j, \alpha_k, \gamma_{-I}) \in \mathcal{A}$, using I -s-conj and $(\gamma_i, \gamma_j, x_k, \gamma_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, x_k, \gamma_{-I}) \in \mathcal{A}$, violating the fact that $x_I \in T_I$.

If $(\gamma_i, \gamma_j, x_k, \beta_{-I}) \in \mathcal{A}$, using I -s-conj and $(x_i, x_j, \alpha_k, \beta_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, x_k, \beta_{-I}) \in \mathcal{A}$, violating the fact that $x_I \in T_I$.

2. Suppose now that $(\beta_i, x_j, \beta_k, \alpha_{-I}) \in \mathcal{A}$. Using I -s-conj and $(x_i, \alpha_j, \alpha_k, \alpha_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, \beta_k, \alpha_{-I}) \in \mathcal{A}$. Using $(x_i, x_j, \beta_k, \alpha_{-I}) \in \mathcal{A}$ and $(\gamma_i, \gamma_j, x_k, \gamma_{-I}) \in \mathcal{A}$, I -linearity implies either $(x_i, x_j, \beta_k, \gamma_{-I}) \in \mathcal{A}$ or $(\gamma_i, \gamma_j, x_k, \alpha_{-I}) \in \mathcal{A}$.

If $(x_i, x_j, \beta_k, \gamma_{-I}) \in \mathcal{A}$, using I -s-conj and $(\gamma_i, \gamma_j, x_k, \gamma_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, x_k, \gamma_{-I}) \in \mathcal{A}$, violating the fact that $x_I \in T_I$.

If $(\gamma_i, \gamma_j, x_k, \alpha_{-I}) \in \mathcal{A}$, using I -s-conj and $(x_i, x_j, \beta_k, \alpha_{-I}) \in \mathcal{A}$, we obtain $(x_i, x_j, x_k, \alpha_{-I}) \in \mathcal{A}$, violating the fact that $x_I \in T_I$. \square

Remark 22

Let $S_I = X_I \setminus T_I$. It follows from the above lemma that, using notation introduced in Definition 5, we have:

$$S_I = \prod_{i \in I} Y_i.$$

It is easy to check that the relation \succsim_I on S_I is a weak order having at most two indifference classes. Indeed, we know that \succsim_I can have at most three equivalence

classes and, by construction, $x_I \in S_I$ implies that $(x_I, a_{-I}) \in \mathcal{A}$, for some $a_{-I} \in X_{-I}$, so that x_I cannot belong to the third equivalence class. •

Our next result generalizes Lemma 20 to subsets of attributes included in I .

Lemma 23

Let $J \subseteq I$. Condition J -s-3v-graded holds.

PROOF

Suppose that I -s-3v-graded and I -s-conj hold. Let J, K be a partition of I . Suppose that J -s-3v-graded is violated, so that $(x_J, a_K, a_{-I}) \in \mathcal{A}$, $(y_J, b_K, b_{-I}) \in \mathcal{A}$, and $(z_J, c_K, c_{-I}) \in \mathcal{A}$, while $(x_J, b_K, b_{-I}) \in \mathcal{U}$ and $(z_J, a_K, a_{-I}) \in \mathcal{U}$.

Using I -s-3v-graded, $(x_J, a_K, a_{-I}) \in \mathcal{A}$, $(y_J, b_K, b_{-I}) \in \mathcal{A}$, and $(z_J, c_K, c_{-I}) \in \mathcal{A}$, imply $(x_J, a_K, b_{-I}) \in \mathcal{A}$ or $(z_J, c_K, a_{-I}) \in \mathcal{A}$.

Suppose first that $(x_J, a_K, b_{-I}) \in \mathcal{A}$. Using $(y_J, b_K, b_{-I}) \in \mathcal{A}$ and $(x_J, a_K, b_{-I}) \in \mathcal{A}$, repeated uses of I -s-conj imply $(x_J, b_K, b_{-I}) \in \mathcal{A}$, a contradiction.

Suppose now that $(z_J, c_K, a_{-I}) \in \mathcal{A}$. Using $(x_J, a_K, a_{-I}) \in \mathcal{A}$ and $(z_J, c_K, a_{-I}) \in \mathcal{A}$ repeated uses of I -s-conj imply $(z_J, a_K, a_{-I}) \in \mathcal{A}$, a contradiction. □

Remark 24

The above lemma implies that, for all $J \subseteq I$, Condition J -linear holds. Hence, under the hypothesis that I -s-3v-graded and I -s-conj hold, we know that for all partition J, K of I , J -linearity and K -linearity hold, so that both \succsim_J and \succsim_K are weak orders. It is easy to check that these three weak orders combine as expected, e.g.,

$$\begin{aligned} (x_K, x_J) \succsim_I y_I \text{ and } z_K \succsim_K x_K &\Rightarrow (z_K, x_J) \succsim_I y_I, \\ (x_K, x_J) \sim_I y_I \text{ and } z_K \sim_K x_K &\Rightarrow (z_K, x_J) \sim_I y_I, \\ (x_K, x_J) \succ_I y_I \text{ and } z_K \succsim_K x_K &\Rightarrow (z_K, x_J) \succ_I y_I. \end{aligned}$$

We will use such implications freely in what follows.

The proof of these implications is easy. For instance, suppose that $(x_K, x_J) \succsim_I y_I$, so that, for all $a_{-I} \in X_{-I}$, $(y_I, a_{-I}) \in \mathcal{A}$ implies $(x_J, x_K, a_{-I}) \in \mathcal{A}$. Since $z_K \succsim_K x_K$, $(x_K, x_J, a_{-I}) \in \mathcal{A}$ implies $(z_K, z_J, a_{-I}) \in \mathcal{A}$, so that $(z_K, x_J) \succsim_I y_I$. •

4.5 Lemmas with I -s-2-graded and I -s-conj

The common premise of all lemmas in 4.5 is that Conditions I -s-2-graded and I -s-conj hold.

Lemma 25

For all $J \subseteq I$, \succsim_J is a weak order that has at most two equivalence classes, so that J -s-2-graded holds.

PROOF

Suppose that I -s-2-graded and I -s-conj hold. Let J, K be a partition of I . Suppose that $x_J \succ_J y_J$ and $y_J \succ_J z_J$.

Hence, there are $a_K, b_K \in X_K$ and $a_{-I}, b_{-I} \in X_{-I}$ such that $(x_J, a_K, a_{-I}) \in \mathcal{A}$, $(y_J, a_K, a_{-I}) \in \mathcal{U}$, and $(y_J, b_K, b_{-I}) \in \mathcal{A}$, $(z_J, b_K, b_{-I}) \in \mathcal{U}$.

Because I -s-2-graded holds, we know that \lesssim_I has at most two equivalence classes. Hence, we know that both (x_J, a_K) and (y_J, b_K) belong to the first equivalence class of \lesssim_I . Hence, we have $(x_J, a_K) \sim_I (y_J, b_K)$, so that we have $(y_J, b_K, a_{-I}) \in \mathcal{A}$ and $(x_J, a_K, b_{-I}) \in \mathcal{A}$. Repeated uses of I -s-conj allows to permute a_K and b_K between the expressions $(y_J, b_K, a_{-I}) \in \mathcal{A}$ and $(x_J, a_K, b_{-I}) \in \mathcal{A}$. This leads to $(y_J, a_K, a_{-I}) \in \mathcal{A}$, a contradiction. \square

The next lemma is crucial. It shows how the information is aggregated with the attributes in I .

Lemma 26

Let $x_I \in X_I$. Then x_I belongs to the first equivalence class of \lesssim_I iff x_i belongs to the first equivalence class of \lesssim_i , for all $i \in I$.

PROOF

Proving that if x_i belongs to the first equivalence class of \lesssim_i , for all $i \in I$, then x_I belongs to the first equivalence class of \lesssim_I is easy. Indeed, with $i \in I$ and $J = I \setminus \{i\}$, if $x_i \lesssim_i y_i$ then we have $(x_i, z_J) \lesssim_I (y_i, z_J)$, for all $z_J \in X_J$. Let us prove the reverse implication.

Let $i \in I$ and $J = I \setminus \{i\}$. In the rest of this proof, we use the convention that on each $i \in N$, we have $a_i \succ_i b_i$. It is clear that (a_J, a_i) is a top element of \lesssim_I and that (b_J, b_i) is a bottom element of \lesssim_I . This follows from the fact that \lesssim_I has two equivalence classes and the monotonicity relations noted in Remark 24.

Suppose, in contradiction with the thesis, that $(a_J, a_i) \sim_I (a_J, b_i)$, for some $i \in I$ and letting $J = I \setminus \{i\}$.

Since $a_i \succ_i b_i$, we have $(x_J, a_i, x_{-I}) \in \mathcal{A}$ and $(x_J, b_i, x_{-I}) \in \mathcal{U}$, for some $x_J \in X_J$ and some $x_{-I} \in X_{-I}$. This implies $(x_J, a_i) \succ_I (x_J, b_i)$.

It is impossible that $x_J \sim_J a_J$ because we have supposed that $(a_J, a_i) \sim_I (a_J, b_i)$. Hence, we know that $x_J \sim_J b_J$, so that $(b_J, a_i, x_{-I}) \in \mathcal{A}$ and $(b_J, b_i, x_{-I}) \in \mathcal{U}$.

Because $a_J \succ_J b_J$, we know that $(a_J, y_i, y_{-I}) \in \mathcal{A}$ and $(b_J, y_i, y_{-I}) \in \mathcal{U}$, for some $y_J \in X_J$ and some $y_i \in X_i$. This implies that (a_J, y_i) is a top element of \lesssim_I , so that $(a_J, y_i) \sim_I (a_J, b_i)$. By hypothesis, (b_J, b_i) is a bottom element. Hence, we obtain $(a_J, b_i, y_{-I}) \in \mathcal{A}$ and $(b_J, b_i, y_{-I}) \in \mathcal{U}$.

Since we have $(a_J, b_i, y_{-I}) \in \mathcal{A}$ and $(b_J, a_i, x_{-I}) \in \mathcal{A}$, I -linearity implies $(b_J, a_i, y_{-I}) \in \mathcal{A}$ or $(a_J, b_i, x_{-I}) \in \mathcal{A}$.

It is impossible to have $(b_J, a_i, y_{-I}) \in \mathcal{A}$ since I -s-conj, together with $(a_J, b_i, y_{-I}) \in \mathcal{A}$, imply $(b_J, b_i, y_{-I}) \in \mathcal{A}$, a contradiction.

Similarly, it is impossible to have $(a_J, b_i, x_{-I}) \in \mathcal{A}$, since I -s-conj, together with $(b_J, a_i, x_{-I}) \in \mathcal{A}$ imply $(b_J, b_i, x_{-I}) \in \mathcal{A}$, a contradiction. \square

Using Lemma 26, we know that if I -s-2-graded and I -s-conj hold, \succsim_I has two equivalence classes, \succsim_i has two equivalence classes, for all $i \in I$, and that the aggregation of \succsim_i into \succsim_I is conjunctive.

4.6 Proof of Theorem 18: necessity

Lemma 27

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E_1^u) . Let I, J be any partition of N . Then $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies Conditions I -s-3v-graded, I -s-conj, and J -s-conj.

PROOF

By hypothesis, we know that $x \in \mathcal{A} \Leftrightarrow [x_i \succsim_i p_i, \text{ for all } i \in N]$.

Suppose that $(x_I, a_{-I}) \in \mathcal{A}$, $(y_I, b_{-I}) \in \mathcal{A}$, $(z_I, c_{-I}) \in \mathcal{A}$, while $(x_I, b_{-I}) \in \mathcal{U}$ and $(z_I, a_{-I}) \in \mathcal{U}$. By construction, $(x_I, b_{-I}) \in \mathcal{U}$ implies that either $p_i \succ_i x_i$, for some $i \in I$ or $p_j \succ_j b_j$, for some $j \notin I$. This implies either $(x_I, a_{-I}) \in \mathcal{U}$ or $(y_I, b_{-I}) \in \mathcal{U}$, a contradiction. Hence, I -s-3v-graded holds.

Suppose now that $(x_i, z_J, a_{-I}) \in \mathcal{A}$, $(y_i, w_J, a_{-I}) \in \mathcal{A}$, and $(y_i, z_J, a_{-I}) \in \mathcal{U}$ (the case in which $(x_i, w_J, a_{-I}) \in \mathcal{U}$ is dealt with similarly). Now, $(x_i, z_J, a_{-I}) \in \mathcal{A}$, and $(y_i, z_J, a_{-I}) \in \mathcal{U}$ imply that $x_i \succsim_i p_i$, while $p_i \succ_i y_i$. But this contradicts $(y_i, w_J, a_{-I}) \in \mathcal{A}$. Hence, I -s-conj holds. A similar proof shows that J -s-conj holds. \square

The following lemma makes use of the sets N^p and N^q partitioning N that were defined at the end of Section 4.1.

Lemma 28

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E_2^{u*}) . Let I, J be the partition of N induced by the two profiles p and q , i.e., $I = N^p$ and $J = N^q$. Then $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-3v-graded, I -s-conj, and J -s-conj.

PROOF

Suppose that $(x_I, a_J) \in \mathcal{A}$, $(y_I, b_J) \in \mathcal{A}$, $(z_I, c_J) \in \mathcal{A}$, while $(x_I, b_J) \in \mathcal{U}$ and $(z_I, a_J) \in \mathcal{U}$. By hypothesis, we know that $x_i \succsim_i q_i$, $y_i \succsim_i q_i$ and $z_i \succsim_i q_i$, for all $i \in I$. Similarly, we know that $a_j \succsim_j p_j$, $b_j \succsim_j p_j$ and $c_j \succsim_j p_j$, for all $j \in J$.

If $a_j \succ_j q_j$, for all $j \in J$, this would imply $(z_I, a_J) \in \mathcal{A}$, a contradiction. Hence, it must be true that $x_i \succsim_i p_i$, for all $i \in I$. This contradicts the fact that $(x_I, b_J) \in \mathcal{U}$. Hence, $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-3v-graded.

Let $K = I \setminus \{i\}$ and suppose that $(x_i, z_K, a_J) \in \mathcal{A}$, $(y_i, w_K, a_J) \in \mathcal{A}$, together with either $(y_i, z_K, a_J) \in \mathcal{U}$, or $(x_i, w_K, a_J) \in \mathcal{U}$. By hypothesis, we know that

$x_i \succsim_i q_i$ and $y_i \succsim_i q_i$. Moreover, $z_k \succsim_k q_k$ and $w_k \succsim_k q_k$, for all $k \in K$. Similarly, we know that $a_j \succsim_j p_j$, for all $j \in J$.

We have either $(y_i, z_K, a_J) \in \mathcal{U}$, or $(x_i, w_K, a_J) \in \mathcal{U}$. Suppose that $(y_i, z_K, a_J) \in \mathcal{U}$, the other case being similar. It is clearly impossible that $a_j \succsim_j q_j$, for all $j \in J$. Hence, we must have: $x_i \succsim_i p_i$, $y_i \succsim_i p_i$, $z_k \succsim_k p_k$ and $w_k \succsim_k p_k$, for all $k \in K$. This implies $(y_i, z_K, a_J) \in \mathcal{A}$ and $(x_i, w_K, a_J) \in \mathcal{A}$, a contradiction. Hence, $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-conj. A similar proof shows that J -s-conj holds. \square

4.7 Proof of Theorem 18: sufficiency

Suppose that there is a partition I, J of N such that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-3v-graded, I -s-conj and J -s-conj. Using Lemma 13, we know that J -s-3v-graded holds.

As above, define $Z_i = \{x_i \in X_i : (x_i, a_{-i}) \in \mathcal{U}, \text{ for all } a_{-i} \in X_{-i}\}$, $Y_i = X_i \setminus Z_i$, $T_I = \{x_I \in X_I : (x_I, a_{-I}) \in \mathcal{U}, \text{ for all } a_{-I} \in X_{-I}\}$ and $T_J = \{x_J \in X_J : (x_J, a_{-J}) \in \mathcal{U}, \text{ for all } a_{-J} \in X_{-J}\}$. $S_I = X_I \setminus T_I$ and $S_J = X_J \setminus T_J$.

We know from Lemma 21 that $S_I = \prod_{i \in I} Y_i$ and $S_J = \prod_{j \in J} Y_j$. Consider the set $Y = \prod_{i=1}^n Y_i$ and let $\mathcal{A}' = \mathcal{A} \cap Y$ and $\mathcal{U}' = \mathcal{U} \cap Y$.

By construction, Condition I -s-2-graded and J -s-2-graded hold for $\langle \mathcal{A}', \mathcal{U}' \rangle$. Let us show that $\langle \mathcal{A}', \mathcal{U}' \rangle$ has a representation in Model $(E_{\leq 2}^{u*})$, which will complete the proof.

Using Lemma 26, we know that both \succsim_I and \succsim_J are weak orders having at most two equivalence classes. Moreover, we know that x_I belongs to the first equivalence class of \succsim_I iff x_i belongs to the first equivalence class of \succsim_i , for all $i \in I$, a similar conclusion holding for J . Our conventions imply that all attributes are influential for $\langle \mathcal{A}', \mathcal{U}' \rangle$.

We use the convention that on each $i \in N$, we have $a_i \succ_i b_i$.

There are two types of elements in X_I and two types of elements in X_J . We know (see Lemma 26) that a top element of \succsim_I (resp. \succsim_J) consists exclusively of top elements of \succsim_i , for all $i \in I$ (resp. $j \in J$). Hence, there are four cases to consider, using the convention that a_I (resp. b_I) is a top element of \succsim_I , a similar convention holding for J .

1. $(a_I, a_J) \in \mathcal{A}'$, $(a_I, b_J) \in \mathcal{A}'$, $(b_I, a_J) \in \mathcal{A}'$, $(b_I, b_J) \in \mathcal{U}'$. This is exactly Model (E_2^{u*}) , with p (resp. q) consisting of any element in the first (resp. second) equivalence class of \succsim_i , for $i \in I$ and the second (resp. first) equivalence class of \succsim_j , for $j \in J$.
2. $(a_I, a_J) \in \mathcal{A}'$, $(a_I, b_J) \in \mathcal{U}'$, $(b_I, a_J) \in \mathcal{U}'$, $(b_I, b_J) \in \mathcal{U}'$. This is exactly Model (E_1^u) , with p consisting of any element in the first equivalence class of each \succsim_i .

3. $(a_I, a_J) \in \mathcal{A}'$, $(a_I, b_J) \in \mathcal{A}'$, $(b_I, a_J) \in \mathcal{U}'$, $(b_I, b_J) \in \mathcal{U}'$. The attributes in J are not influential for $\langle \mathcal{A}', \mathcal{U}' \rangle$, which is impossible.
4. $(a_I, a_J) \in \mathcal{A}'$, $(a_I, b_J) \in \mathcal{U}'$, $(b_I, a_J) \in \mathcal{A}'$, $(b_I, b_J) \in \mathcal{U}'$. The attributes in I are not influential for $\langle \mathcal{A}', \mathcal{U}' \rangle$, which is impossible.

This completes the proof. \square

4.8 Independence of the axioms: examples

The following examples show that the three conditions used in Theorem 18 are independent.

Example 29

Let $N = \{1, 2, 3, 4\}$ and $X_i = \{a, b\}$, for all $i \in N$. Consider the partition given below, abusing notation in an obvious way:

$$\begin{array}{llll} aaaa \in \mathcal{A}, & abaa \in \mathcal{A}, & baaa \in \mathcal{A}, & bbaa \in \mathcal{A}, \\ aaab \in \mathcal{A}, & abab \in \mathcal{A}, & baab \in \mathcal{U}, & bbab \in \mathcal{U}, \\ aaba \in \mathcal{A}, & abba \in \mathcal{U}, & baba \in \mathcal{U}, & bbba \in \mathcal{U}, \\ aabb \in \mathcal{A}, & abbb \in \mathcal{U}, & babb \in \mathcal{U}, & bbbb \in \mathcal{U}. \end{array}$$

It is easy to check that both $\{1, 2\}$ -s-conj and $\{3, 4\}$ -s-conj are satisfied. $\{1, 2\}$ -s-3v-graded (and, consequently, $\{3, 4\}$ -s-3v-graded) is violated since we have, using obvious notation, $aa \succ_{12} ab \succ_{12} [ba \sim_{12} bb]$, while $(b, b, a, a) \in \mathcal{A}$ (see Lemma 12).

Notice that this example also shows that we cannot replace in our characterization I -s-3v-graded by i -s-3v-graded, for all $i \in I$. Indeed, in the above example, i -s-3v-graded trivially holds, for all $i \in N$ (since each X_i has only 2 elements). \diamond

The next example shows that I -s-3v-graded, J -s-3v-graded and J -s-conj do not imply I -s-conj. A similar example shows that the same conclusion holds, permuting the role of I and J .

Example 30

Let $N = \{1, 2, 3, 4, 5, 6\}$ and $X_i = \{a, b\}$, for all $i \in N$. Let $I = \{1, 2, 3\}$ and $J = \{4, 5, 6\}$. The partition $\langle \mathcal{A}, \mathcal{U} \rangle$ is such that $x \in \mathcal{A}$ iff $x_j = a$, for all $j \in J$ or $|\{i \in I; x_i = a\}| \geq 2$.

It is clear that J -s-3v-graded holds: either an element of X_J is a top element of \succsim_J (being a top element of \succsim_j , for all $j \in J$) or not. The same is true for I -s-3v-graded: either an element of X_I is a top element of \succsim_I (being a top element of \succsim_i , for at least two elements $i \in I$) or not. Condition J -s-conj clearly holds. Condition I -s-conj is violated since $(a, a, b, b, b, b) \in \mathcal{A}$, $(a, b, a, b, b, b) \in \mathcal{A}$, while $(a, b, b, b, b, b) \in \mathcal{U}$. \diamond

4.9 Testing the conditions

The conditions used in Theorem 18 involve an existential clause: “there is a partition I, J of N such that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-3v-graded, I -s-conj and J -s-conj”. Testing such conditions may not be easy. The following results aim at simplifying this test.

Lemma 31

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies I -s-conj and let $J = I \setminus \{i\}$. Then $\langle \mathcal{A}, \mathcal{U} \rangle$ satisfies J -s-conj.

PROOF

J -s-conj applies to all pairs of alternatives that have common evaluations on $N \setminus J$. But since $J \subsetneq I$, having common evaluation on $N \setminus J$ implies having common evaluations on $N \setminus I$. Hence I -s-conj implies J -s-conj. \square

Lemma 32

Condition $\{i\}$ -s-conj is trivial.

PROOF

Condition $\{i\}$ -s-conj says that $(x_i, a_{-i}) \in \mathcal{A}$ and $(y_i, a_{-i}) \in \mathcal{A}$ imply $(x_i, a_{-i}) \in \mathcal{A}$ and $(y_i, a_{-i}) \in \mathcal{A}$. \square

Lemma 33

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E_2^{u}) with the two profiles p and q . Let $I = N^p$ and $J = N^q$. Let $j \in J$. Then $\langle \mathcal{A}, \mathcal{U} \rangle$ does not satisfy $I \cup \{j\}$ -s-conj.*

PROOF

In this proof, we use the convention that on each $i \in N$, we have $a_i \succ_i b_i \succ_i c_i$. Consider the following two alternatives: (a_I, b_j, b_K) and (b_I, a_j, a_K) , with $K = J \setminus \{j\}$. These two alternatives are in \mathcal{A} . Indeed, by construction, (a_I, b_j, b_K) dominates the profile p and (b_I, a_j, a_K) dominates the profile q . However, it is clear that $(b_I, b_j, a_K) \in \mathcal{U}$, since it does not dominate p and q . Hence, condition $I \cup \{j\}$ -s-conj is violated. \square

Lemma 34

Suppose that $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model (E_2^{u}) with the two profiles p and q . Let $I = N^p$ and $J = N^q$. Let $i \in I$ and $j \in J$. Then $\langle \mathcal{A}, \mathcal{U} \rangle$ does not satisfy $\{i, j\}$ -s-conj.*

PROOF

In this proof, we use the convention that on each $i \in N$, we have $a_i \succ_i b_i \succ_i c_i$. Consider the following two alternatives: (a_K, a_i, b_j, a_L) and (a_K, b_i, a_j, a_L) , with $K = I \setminus \{i\}$ and $L = J \setminus \{j\}$. While both alternatives are in \mathcal{A} , it is clear that $(a_K, b_j, b_i, a_L) \in \mathcal{U}$, so that $\{i, j\}$ -s-conj is violated. \square

Hence the test of I -s-conj could proceed as follows in a greedy-like way. Choose a pair $\{i, j\}$ of attributes such that $\{i, j\}$ -s-conj holds. If such a pair does not exist, Model $(E_{\leq 2}^{u*})$ does not hold. If such a pair exists, try to iteratively enlarge it until this is no longer possible. This gives rise to a maximal set I such that I -s-conj holds. If $I = N$, then Model (E_1^u) applies (remember that N -s-conj implies i -s*-conj, for all $i \in N$, see Proposition 7). Otherwise, test if $(N \setminus I)$ -s-conj holds. If yes both conditions I -s-conj and J -s-conj are satisfied. Otherwise, Model $(E_{\leq 2}^{u*})$ does not hold (see Lemma 28).

A similar reasoning does not apply starting with the test of I -s-3v-graded. Hence, after having applied the above procedure, we test if I -s-3v-graded applies with the set I that we have obtained above. If not, Model $(E_{\leq 2}^{u*})$ does not hold. If yes, then the conditions of Theorem 18 are satisfied and Model $(E_{\leq 2}^{u*})$ holds.

5 Discussion

Using classical tools from conjoint measurement, we have proposed a characterization of Model $(E_{\leq 2}^u)$. For the time being, we have no useful results for $(E_{\leq k}^u)$, for $k = 3, 4, \dots$. Similarly, we have no interesting results for $(E_{\leq k}^c)$ and $(E_{\leq k}^{u*})$, for $k = 2, 3, 4, \dots$. Hence, we are still far from our initial objective of understanding how the increase in the number of profiles leads to an expressiveness gain.

Comparing Theorem 18 with Propositions 3 and 7 shows that the case of at most two limiting profiles is more complex than the case of many limiting profiles (Proposition 3) and the case of one profile (Proposition 7). Moreover³, it seems difficult to extend our results to cover the case of Model $(E_{\leq 2}^c)$. Indeed, as shown below, the conditions we use seem to be quite specific to Model $(E_{\leq 2}^{u*})$ (or $(E_{\leq 2}^u)$).

Example 35

The example has $n = 4$ and $X_1 = X_2 = X_3 = X_4 = \{0, 5, 10\}$. The partition is built in Model $(E_{\leq 2}^c)$ with the following two profiles $(10, 10, 5, 5)$ and $(5, 5, 10, 10)$ together with $\mathcal{F} = \{\{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.

It is simple to check that on all attributes, we have $10 \succ_i 5 \succ_i 0$. For instance,

³Although this case has little practical importance (see Fernández et al., 2017, p. 216, 2nd col., beginning of Sect. 2.1 or Roy, 1996, p. 235), more powerful results can be obtained when $n = 2$. It is easy to devise a condition equivalent to requiring, together with i -linear, that \succsim_i has at most $\ell \geq 2$ equivalence classes. Imposing that each relation \succsim_i has at most $\ell + 1$ equivalence classes is necessary if $\langle \mathcal{A}, \mathcal{U} \rangle$ has a representation in Model $(E_{\leq \ell}^c)$. When $n = 2$, this requirement is not only necessary but also sufficient to guarantee the existence of a representation in Model $(E_{\leq \ell}^c)$. Since this case is of little importance, we leave the easy proof of this fact to the interested reader. Simple examples show that the result does not generalize to the case $n \geq 3$.

we have with $i = 1$,

$$\begin{aligned} (10, 0, 5, 10) &\in \mathcal{A}, & (5, 0, 5, 10) &\in \mathcal{U}, \\ (5, 0, 10, 10) &\in \mathcal{A}, & (0, 0, 10, 10) &\in \mathcal{U}, \end{aligned}$$

The two profiles partition the set of attributes into two subsets $I = \{1, 2\}$ and $J = \{3, 4\}$.

Observe that i -s-3v-graded does not hold since 0 is not a veto level. Indeed, we have, e.g., $(0, 5, 10, 10) \in \mathcal{A}$, $(10, 0, 5, 10) \in \mathcal{A}$, $(10, 5, 0, 10) \in \mathcal{A}$, and $(10, 5, 10, 0) \in \mathcal{A}$.

On $I = \{1, 2\}$, the partition is not I -s-3v-graded. Indeed, we have, using obvious notation,

$$(10, 10) \succ_{12} [(10, 5), (5, 10)] \succ_{12} (5, 5) \succ_{12} [(5, 0), (0, 5)] \succ_{12} (0, 0).$$

Indeed, we have

$$\begin{aligned} (10, 10, 5, 0) &\in \mathcal{A}, & (10, 5, 5, 0) &\in \mathcal{U}, \\ (10, 5, 5, 5) &\in \mathcal{A}, & (5, 5, 5, 5) &\in \mathcal{U}, \\ (5, 5, 10, 0) &\in \mathcal{A}, & (5, 0, 10, 0) &\in \mathcal{U}, \\ (5, 0, 10, 10) &\in \mathcal{A}, & (0, 0, 10, 10) &\in \mathcal{U}. \end{aligned}$$

It is simple to find similar examples for J .

Finally, it is clear that I -s-conj does not hold. We have, e.g., $(10, 0, 5, 10) \in \mathcal{A}$ and $(0, 10, 5, 10) \in \mathcal{A}$, while $(0, 0, 5, 10) \in \mathcal{U}$. It is simple to find similar examples for J .

This shows that the axioms used for the characterization of Model $(E_{\leq 2}^{u*})$ do not hold with $(E_{\leq 2}^c)$, i.e., when we have two profiles but $\mathcal{F} \neq \{N\}$. \diamond

Hence, the study of the particular cases of Models (E) , (E^c) and (E^u) with constraints on the number of limiting profiles is not an easy task. Our analysis of the case of two limiting profiles coupled with unanimity shows that it is unlikely that a purely axiomatic investigation will allow us to obtain clear answers to this question. Hence, this is also a plea to combine axiomatic work with other types of work, e.g., based on computer simulation.

References

- D. Bouyssou and T. Marchant. An axiomatic approach to noncompensatory sorting methods in MCDM, I: The case of two categories. *European Journal of Operational Research*, 178(1):217–245, 2007.
- D. Bouyssou and T. Marchant. Ordered categories and additive conjoint measurement on connected sets. *Journal of Mathematical Psychology*, 53(2):92–105, 2009.

- D. Bouyssou and T. Marchant. Additive conjoint measurement with ordered categories. *European Journal of Operational Research*, 203(1):195–204, 2010. doi: 10.1016/j.ejor.2009.07.032.
- D. Bouyssou, T. Marchant, and M. Pirlot. The size of the largest antichains in products of linear orders. *arXiv e-prints*, art. arXiv:1903.07569, Mar 2019.
- D. Bouyssou, T. Marchant, and M. Pirlot. A theoretical look at ELECTRE TRI-nB. Working Paper, 2020.
- E. Fernández, J. R. Figueira, J. Navarro, and B. Roy. ELECTRE TRI-nB: A new multiple criteria ordinal classification method. *European Journal of Operational Research*, 263(1):214–224, 2017. doi: 10.1016/j.ejor.2017.04.048.
- E. Fernández, J. R. Figueira, and J. Navarro. An indirect elicitation method for the parameters of the ELECTRE TRI-nB model using genetic algorithms. *Applied Soft Computing Journal*, 77:723–733, 2019. doi: 10.1016/j.asoc.2019.01.050.
- W. M. Goldstein. Decomposable threshold models. *Journal of Mathematical Psychology*, 35(1):64–79, 1991.
- S. Greco, B. Matarazzo, and R. Słowiński. Conjoint measurement and rough set approach for multicriteria sorting problems in presence of ordinal criteria. In A. Colorni, M. Paruccini, and B. Roy, editors, *A-MCD-A, Aide Multicritère à la Décision / Multiple Criteria Decision Aid*, pages 117–144. European Commission, Joint Research Centre, Luxembourg, 2001.
- B. Roy. *Multicriteria methodology for decision aiding*. Kluwer, Dordrecht, 1996. Original version in French: “*Méthodologie multicritère d’aide à la décision*”, Economica, Paris, 1985.
- B. Roy and D. Bouyssou. *Aide multicritère à la décision : méthodes et cas*. Economica, Paris, 1993. ISBN 2-7178-2473-1.
- R. Słowiński, S. Greco, and B. Matarazzo. Axiomatization of utility, outranking and decision-rule preference models for multiple-criteria classification problems under partial inconsistency with the dominance principle. *Control and Cybernetics*, 31(4):1005–1035, 2002.
- K. Vind. Independent preferences. *Journal of Mathematical Economics*, 20(1):119–135, 1991.
- K. Vind. *Independence, Additivity, Uncertainty*. Springer-Verlag, Berlin, 2003.
- W. Yu. *Aide multicritère à la décision dans le cadre de la problématique du tri : concepts, méthodes et applications*. Thèse de doctorat, Université Paris Dauphine, Paris, France, 1992. (in French).