# A general model of preference aggregation

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The classical way of modelling the preferences of a Decision Maker, consists in assuming the existence of a value function $u$ such that an alternative $a$ is at least as good as an alternative $b$ ($a \succeq b$) if and only if $u(a) \geq u(b)$. This leads to a model of preference in which $\succeq$ is complete and transitive. Using such a preference model to establish a recommendation in a decision-aid study is straightforward and the main task of the Analyst is to assess $u$. In a multicriteria/multiattribute (we will use these terms interchangeably in this paper) context, each alternative $a$ is usually seen as a vector $\overline{g}(a) = (g_1(a), \ldots, g_n(a))$ of evaluations of $a$ w.r.t $n$ points of view. Under some well-known conditions, $u$ can be obtained in an additive manner, i.e. there are functions $u_i$ such that

$$u(a) = \sum_{i=1}^{n} u_i \left( g_i(a) \right).$$

Modelling preferences therefore amounts to assess the partial value functions $u_i$; several techniques have been proposed to do so. It should be noticed that the additive model implies independence of each attribute, i.e. that the preference between alternatives which only differ on an attribute does not depend on their evaluations on the other attributes, and that individual (also called partial) preferences $\succeq_i$ deduced from $\succeq$ through independence are complete and transitive. In some situations, such a model might not appear to be appropriate, for instance because:

- indifference (seen as the symmetric part of $\succeq$) may not be transitive;
- $\succeq$ may not be a complete relation, i.e. some alternatives may be incomparable;
- compensation effects between criteria are more complex than with an additive model;
- criteria interact (there is no preference independence).

This calls for an extension of the additive utility framework allowing to better deal with some of these cases. Such an extension is also called for by a number of approaches developed since the early seventies. In those methods, the overall preference of $a$ over $b$ is usually determined by looking at the evaluation vectors $\overline{g}(a)$

and $\overline{g}(b)$ independently of the rest of the alternatives and treating the difference $g_i(a) - g_i(b)$ in rather an ordinal way by comparing the difference to a limited number of thresholds. This simple option usually leads to a global preference relation that is not a complete preorder (this being not unrelated to Arrow's theorem). This implies that the aggregation procedure results in structures from which it might not be easy to derive a recommendation (choice of an alternative, ranking of all alternatives). Therefore, elaborating a recommendation usually calls for the application of specific "exploitation techniques" . The perspective in which such methods were conceived is neither normative (what should the Decision Maker decide in order to be rational) nor descriptive (what are possibly the mechanisms at work in a Decision Maker's mind when he makes a decision); they claim to be constructive in the sense that, the resulting global preference is built or learned through a dialog between the Decision Maker and the Analyst based on supposedly intuitive concepts.

Among the methods alluded to, are the so-called *outranking* methods, where $\succeq$ is the outranking relation; the semantic content of a statement like "$a$ outranks $b$" has been expressed by B. Roy in the following manner:

> "An outranking relation is a binary relation $S$ defined in $A$ such that $a$ $S$ $b$ if, given what is known about the decision maker's preferences and given the quality of the valuations of the alternatives and the nature of the problem, there are enough arguments to decide that $a$ is at least as good as $b$, while there is no essential reason to refute that statement."

There has been relatively little interest in these methods outside Europe. There are several reasons to that. Two of them might be that

- they are not well founded from a formal point of view (no axiomatization);
- they may lead to preference structures from which it is not easy to derive a recommendation.

What we aim at doing in this paper is to show a sort of continuity between the dominant "value function" model and a number of pairwise comparisons approaches. This is done through exhibiting a very general model of preference aggregation and showing that a variety of methods fits into the model. Finally we are able to situate the different aggregation procedures as more or less compensatory, the utility approach being compensatory whereas outranking methods tend to be less compensatory.

The model studied in this paper is built on a product space $X = \prod_{i=1}^{n} X_i$, where $X_i$ can be viewed as the evaluations of a finite set $\mathcal{A}$ of alternatives with respect to criterion $i$. Denoting by $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, elements of $X$, the classical conjoint measurement model, alluded to above, reads;

$$x \succeq y \text{ iff } \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i),$$

where $u_i$ is a real valued function defined on $X_i$, for all $i = 1, \dots, n$.

The general model considered here is defined by

$$x \succeq y \text{ iff } F\left(\psi_i(u_i(x_i), u_i(y_i)), i = 1, \dots, n\right) \geq 0,$$

with $u_i$'s as above, $F : \mathcal{R}^n \longrightarrow \mathcal{R}$, a strictly increasing function and $\psi_i$: $\mathcal{R}^2 \longrightarrow \mathcal{R}$, nondecreasing in its first argument and nonincreasing in the second, for $i = 1, \dots, n$.

This model, though very general, shows fundamental features. A key concept emerging from it is a quaternary relation $\succeq_i^*$ encoding the comparison of pairs of levels on each criterion, i.e. the comparison of *differences* *of preference*; the relation $(x_i, y_i) \succeq_i^* (x'_i, y'_i)$ reads "the difference of preference between $x_i$ and $y_i$ is at least as large as that between $x'_i$ and $y'_i$".

It is easy to show that in our model, $\succeq_i^*$ is a complete preorder even if $\succeq$ is noncomplete and/or nontransitive. The number of equivalence classes of this relation may be considered as reflecting discrimination power in the perception of degrees of difference of preference. This point will be abundantly illustrated.

Another important characteristic of the model is that it implies that partial preference relations $\succeq_i$ on each criterion can be defined and are well behaved. Though our model does not necessarily imply independence of each attribute, it is not difficult to prove that (as soon as $\succeq$ is reflexive) the relations $\succeq_i$ are semiorders. Such an ordered structure appears a particularly desirable generalization of the usual complete preorder for at least two reasons:

- it encompasses the idea that there is a threshold under which differences of performance on a point of view are not perceived as implying definite preference; it thus allows to model preferences in which indifference is not transitive;

- it actually appears in one of the oldest and most famous family of methods based on pairwise comparisons and majority, the ELECTRE family.

Our main result is a characterization of the model in terms of two properties which we call (WC) and (WC') (WC stands for *Weak Cancellation*); (WC) is tightly linked with the fact that $\succeq_i^*$ is an ordering on the values taken by the alternatives on criterion $i$ while (WC') is connected to the fact that the relations $\succeq_i$) are semiorders.

In order to illustrate how this framework helps to contrast aggregation procedures,

- we will recall the aggregation mechanisms used in popular methods such as ELECTRE I, TACTIC and PROMETHEE;

- we show how they fit into the model;

- we interpret their differences in terms of the structure of equivalence classes of $\succeq_i^*$.

With the above model, we believe that we have defined a flexible aggregation scheme that admits a simple axiomatic foundation and encompasses many aggregation models; moreover, we believe that the comparison of preference differences is a key concept for analysing the similarities and dissimilarities of aggregation models A particularly appealing feature of our scheme is that it shows the "continuity" between full compensation and non-compensation.

The present paper emphasizes an interpretation of the technical results obtained by the authors. It opens some research perspectives both on axiomatic and experimental grounds. In the latter, particular models and conditions compatible with observed intuitive preferences could be searched for. On the theoretical side, it would be of interest

- to characterize special models where, e.g., $F$ is additive, the $\psi_i$'s are differences,...;

- to characterize, in a more precise manner, well-known aggregation procedures within our general framework;

- to examine in depth the interconnections of the complete preorder structures on preference differences and the semiorder structure $\succeq_i$ modelling individual preferences on each criterion.

- to further investigate valued preference relations in the framework of the valued version of our model.