On the robustness of the sign of nonadditivity index in a Choquet integral model

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Abstract

In the context of Multiple Criteria Decision Making, this paper studies the robustness of the sign of nonadditivity index for subset of criteria in a Choquet integral model. In the case where the set of alternatives is discrete, the use of the nonadditivity index proposed in the literature often leads to interpretations which are not always robust. Indeed, the sign of this nonadditivity index can depend on the arbitrary choice of a numerical representation in the set of all numerical representations compatible with the ordinal preferential information given by the Decision Maker. We characterize the ordinal preferential information for which the problem appears. We also propose a linear program allowing to test the non robustness of the sign of nonadditivity index for subset of criteria.

Keywords: Robustness, Nonadditivity index, Binary alternatives, Choquet integral model, Numerical representation.

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1. Introduction

The theory of value functions in Multiple Criteria Decision Making (MCDM), consists in assigning a real number to each alternative, so that the order on the alternatives induced by these real numbers is compatible with the ordinal preferential information of the Decision Maker (DM). Preferential independence is a necessary condition for these numbers to be obtained using an additive model[2]. But this property may not always be satisfied[13]. In this case it therefore becomes interesting to use a more general models such as the Choquet integral model. It was popularized by the work of Michel Grabisch[9,11] and is now considered as a central tool in MCDM when one wants to escape the preferential independence hypothesis[10,12,13]. The use of Choquet integral requires that criteria are "commensurate"[14].

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When a set of ordinal preferential information is not compatible with an additive model, it is common to interpret this situation by the existence of some interactions between criteria. It is well known that the interaction among criteria caused by the nonadditivity of a capacity can be measured by a cardinal probabilistic interaction index, in particular the Shapley interaction index[8,26]. More details in the literature on axiomatic properties of all kinds of cardinal probabilistic interaction indices can be found in[4,5,6,10]. Besides the Shapley interaction index[30] suggests to interpret this lack of compatibility with an additive model using the notion of nonadditivity, which is the subject of this paper.

After having modeled the preferences by a Choquet integral model, we are interested to the interactions between criteria. For a subset of criteria, when its Shapley interaction index is negative (resp. null, positive), it is usual to conclude that the interaction for this set of criteria is negative (resp. null, positive). However, it is possible to build another representation for which the conclusions are totally contradictory. Thus in[17], we showed that, in practice, when we elicit a capacity on the basis of preferential information, the interaction is not easy to interpret and requires caution. The objective of this present paper, is to extend this analysis to the nonadditivity index, which is used to take into account some interaction phenomenon.

In the case where all the alternatives are binary, we characterize the ordinal preferential information for which the use of the nonadditivity index proposed in the literature leads to interpretations that are not robust. Since, in practice, wag only binary alternatives is restrictive, so we propose a linear program, inspired by[22], allowing to test whether the interpretation of the nonadditivity index is ambiguous or not.

This paper is organized as follows. We recall in Section 2 some basic elements of Choquet integral model in MCDM. In Section 3, we give an example to motivate the introduction of the notion of necessary and possible nonadditivity. This notion is the subject of Section 4. In Section 5, we give our main results. Section 6 proposes a linear programming model allowing to test the existence of necessary and possible nonadditivity. A final section concludes.

2. Notation and preliminaries

2.1. The framework

Let X be a set of alternatives evaluated on a set of n criteria $N = \{1, 2, \ldots, n\}$ $(n \geq 2)$. For each subset $A \subseteq N$, throughout this paper we use the notation $A \subseteq_{\geq t} N$ (resp. $A \subseteq_{=t} N$) if $|A| \geq t$ (resp. $|A| = t$), where t is an integer in $\{1, 2, \ldots, n\}$. The set of all alternatives X is assumed to be a subset of a Cartesian product $X_1 \times \ldots \times X_n$, where X_i is the set of possible levels on criterion $i \in N$. The criteria are recoded numerically using, for all $i \in N$, a utility function u_i from X_i to $[0, +\infty[$. Using these functions, we assume that the various recoded criteria are commensurate and, hence, the application of the Choquet integral model is meaningful[14].

As in[21,23], we assume that the DM is able to identify on each criterion $i \in N$ two reference levels 0_i and 1_i:

- the level 0_i in X_i is considered as a neutral level and we set $u_i(0_i) = 0$,
- the level 1_i in X_i is considered as a good level and we set $u_i(1_i) = 1$.

For all $x = (x_1, \ldots, x_n) \in X$ and $S \subseteq N$, we will sometimes write $u(x)$ as a shorthand for $(u_1(x_1), \ldots, u_n(x_n))$ and we define the alternatives $a_S = (1_S, 0_{-S}) \in X$ such that $a_i = 1_i$ if $i \in S$ and $a_i = 0_i$ otherwise. We shall often work on the set $\mathcal{B}^g$ which we define as follows.

Definition 1. *The set of generalized binary alternatives is defined by:*

$$\mathcal{B}^g = \{a_S = (1_S, 0_{-S}) : S \subseteq N\}.$$

2.2. Choquet integral

The Choquet integral[3,8,10,13,19,24,29] is an aggregation function known in MCDM as a tool generalizing the weighted arithmetic mean. The Choquet integral uses the notion of capacity[3,25] defined as a function μ from the power set 2^N into $[0,1]$ such that:

- $\mu(\emptyset) = 0$,
- $\mu(N) = 1$,
- $\forall S, T \in 2^N, \big[S \subseteq T \Longrightarrow \mu(S) \leq \mu(T)\big]$ (monotonicity).

For an alternative $x = (x_1, \ldots, x_n) \in X$, the expression of the Choquet integral w.r.t. the capacity μ is given by:

$$\begin{aligned} C_\mu\big(u(x)\big) &= C_\mu\big(u_1(x_1), \ldots, u_n(x_n)\big) \\ &= \sum_{i=1}^{n} \Big[u_{\sigma(i)}\big(x_{\sigma(i)}\big) - u_{\sigma(i-1)}\big(x_{\sigma(i-1)}\big)\Big]\mu\big(N_{\sigma(i)}\big), \end{aligned}$$

where σ is a permutation on N such that: $N_{\sigma(i)} = \{\sigma(i), \ldots, \sigma(n)\}$, $u_{\sigma(0)}(x_{\sigma(0)}) = 0$ and $u_{\sigma(1)}(x_{\sigma(1)}) \leq u_{\sigma(2)}(x_{\sigma(2)}) \leq \ldots \leq u_{\sigma(n)}(x_{\sigma(n)})$.
We often suppose that the DM gives his preferences by comparing some elements of $\mathcal{B}^g$. We then obtain the binary relations P and I defined as follows.

Definition 2. *An ordinal preferential information $\{P, I\}$ on $\mathcal{B}^g$ is given by:*

$P = \{(x, y) \in \mathcal{B}^g \times \mathcal{B}^g$: *DM strictly prefers* x *to* $y\}$,

$I = \{(x, y), (y, x) \in \mathcal{B}^g \times \mathcal{B}^g$: *DM is indifferent between* x *and* $y\}$.

The next definition makes explicit the compatibility of $\{P, I\}$ with a Choquet integral model.

Definition 3. *An ordinal preferential information* $\{P, I\}$ *on* X *is representable by a Choquet integral model if we can find a capacity* μ *such that: for all* $x, y \in X$*, we have,*

$$\begin{aligned} x\,P\,y &\Longrightarrow C_\mu\big(u(x)\big) > C_\mu\big(u(y)\big), \\ x\,I\,y &\Longrightarrow C_\mu\big(u(x)\big) = C_\mu\big(u(y)\big). \end{aligned}$$

The set of all capacities that can be used to represent the ordinal preferential information $\{P, I\}$ at hand will be denoted by $C_{\text{Pref}}(P, I)$. When there is no ambiguity on the underlying ordinal preferential information, we will simply write C_{Pref}.
As in [**? ?**], we add to this ordinal preferential information a binary relation M modeling the monotonicity relations between generalized binary alternatives, and allowing us to ensure the satisfaction of the monotonicity condition: $\big[S \subseteq T \implies \mu(S) \leq \mu(T)\big]$.

Definition 4. *For all* $a_S, a_T \in \mathcal{B}^g$*,* $a_S\,M\,a_T$ *if* $\big[not(a_S(P \cup I)a_T)$ *and* $S \supseteq T\big]$.

Remark 1. *For all* $S \subseteq N$*, we have* $C_\mu(u(a_S)) = \mu(S)$.

Remark 2. *For all* $S, T \subseteq N$*, we have* $a_S\,M\,a_T \Longrightarrow C_\mu(u(a_S)) \geq C_\mu(u(a_T))$.

In the sequel, we need the following two basic definitions in graph theory[20].

Definition 5. *There exists a strict path in* $(P \cup M \cup I)$*, from* x *to* y *if there exists the elements* $x_0, x_1, \ldots, x_r$ *of* $\mathcal{B}^g$ *such that* $x = x_0(P \cup M \cup I)x_1(P \cup M \cup I) \ldots (P \cup M \cup I)x_r = y$ *and for a least one* $i \in \{0, \ldots, r-1\}$*,* $x_i\,P\,x_{i+1}$*. In this case, we note* $x\,TC_P\,y$*.*
We speak of a strict cycle when $x = y$*.*

Definition 6. $x\,TC_{M\cup I}\,y$ *if there exists elements* $x_0, x_1, \ldots, x_r$ *of* $\mathcal{B}^g$ *such that* $x = x_0(M \cup I)x_1(M \cup I) \ldots (M \cup I)x_r = y$*. Hence,* $TC_{M\cup I}$ *is the transitive closure of the binary relation* $M \cup I$*.*

In the next subsection, we recall the definition of the nonadditivity index[30].

2.3. Nonadditivity index

Our work is based on the nonadditivity index, for which the definition and axiomatic properties can be found in[30].

Definition 7. *For all $A \subseteq_{\geq 2} N$, the nonadditivity index*[16,30,31] *w.r.t. a capacity μ is defined as follows:*

$$\eta_A^\mu = \frac{1}{2^{|A|-1}-1} \sum_{\substack{(B,\, A\setminus B)\\ \emptyset \subsetneq B \subsetneq A}} \Big(\mu(A) - \mu(B) - \mu(A \setminus B)\Big) \tag{1}$$

For all $A \subseteq_{\geq 2} N$, for each partition $(B, A \setminus B)$ of A with $\emptyset \subsetneq B \subsetneq A$, we compute the difference $\mu(A) - \big(\mu(B) + \mu(A \setminus B)\big)$. Thus η_A^μ corresponds to the arithmetic mean of these differences over all such partitions.

Remark 3. *We have $\eta_{ij}^\mu = \mu_{ij} - \mu_i - \mu_j$, therefore the nonadditivity index coincides with the Shapley interaction index I_{ij}^μ, for pairs $\{i, j\}$. We recall that, the Shapley interaction index*[26] *of $A \subseteq_{\geq 2} N$ is given by $I_A^\mu = \sum_{K \subseteq N \setminus A} \frac{(n-|K|-|A|)!|K|!}{(n-|A|+1)!} \sum_{L \subseteq A} (-1)^{|A|-|L|} \mu(K \cup L)$.*

The following remark gives two equivalent expressions of η_A^μ that can be found on pages 3 and 4 in[30].

Remark 4. *Given a capacity μ on N and $A \subseteq_{\geq 2} N$, Equation (1) is equivalent to each of Equations (2) and (3).*

$$\eta_A^\mu = \frac{1}{2^{|A|}-2} \sum_{\emptyset \subsetneq B \subsetneq A} \Big(\mu(A) - \mu(B) - \mu(A \setminus B)\Big) \tag{2}$$

$$\eta_A^\mu = \mu(A) - \frac{1}{2^{|A|-1}-1} \sum_{\emptyset \subsetneq B \subsetneq A} \mu(B) \tag{3}$$

In the next section, we give an example, motivating the introduction of the concept of necessary and possible nonadditivity.

3. A motivating example

This example is inspired by[13]. Four students are evaluated on three subjects Mathematics (M), Statistics (S) and Language skills (L). All marks are taken from the same scale, from 0 to 1. The evaluations of these students are given in the Table 1.

	1:Mathematics(M)	2:Language(L)	3: Statistics(S)
a	0.3	0.25	0.6
b	0.3	0.6	0.25
c	0.7	0.25	0.6
d	0.7	0.6	0.25

Table 1: Evaluations of the four students on the three criteria

To select the best students, the Dean of the faculty expresses his/her preferences where the notation $x\,P\,y$ means x is strictly preferred to y.
For a student bad in Mathematics, Statistics is more important that Language, so that

$$a\,P\,b. \tag{4}$$

For a student good in Mathematics, Language is more important that Statistics, so that

$$d\,P\,c. \tag{5}$$

It is not possible to model the two preferences $a\,P\,b$ and $d\,P\,c$ by an arithmetic mean model. Indeed let us denote by q_M, q_S and q_L the weights associated to Mathematics, Statistics and Language. We have:

$$a\,P\,b \Longrightarrow u_M(0.3)q_M + u_L(0.25)q_L + u_S(0.6)q_S > u_M(0.3)q_M + u_L(0.6)q_L + u_S(0.25)q_S.$$
$$d\,P\,c \Longrightarrow u_M(0.7)q_M + u_L(0.6)q_L + u_S(0.25)q_S > u_M(0.7)q_M + u_L(0.25)q_L + u_S(0.6)q_S.$$

Adding up the previous two inequalities leads to the contradiction $0 > 0$.

Let us assume that the scale of evaluation $[0,1]$ corresponds to the utility function associated to each subject, i.e., $u_M(0.3) = 0.3, u_M(0.7) = 0.7, u_L(0.25) = 0.25, u_L(0.6) = 0.6, u_S(0.25) = 0.25$ and $u_S(0.6) = 0.6$. In this case, the strict preferences $a\,P\,b$ and $d\,P\,c$, are now representable by a Choquet integral model w.r.t. any capacity given in Table 2. We choose six capacities compatible with these preferences (Cap. for short in Table 2) in order to illustrate the fact that the sign of nonadditivity index is strongly dependent upon the chosen capacity.

In this example, the interpretation of the nonadditivity index between criteria is not easy. For instance, the nonadditivity index between Language and Statistics, η^{μ}_{LS}, could be strictly positive (Cap. 1, Cap. 3) or null (Cap. 5), or strictly negative (Cap. 2, Cap. 4, Cap. 6). This conclusion is still valid concerning the nonadditivity index η^{μ}_{MS} between Mathematics and Statistics, the nonadditivity index η^{μ}_{ML} between Mathematics and Language, the nonadditivity index η^{μ}_{MLS} between Mathematics, Language and Statistics. Moreover, all nonadditivity indices are strictly positive w.r.t. Cap. 3 but strictly negative w.r.t. Cap. 2.

	Cap. 1	Cap. 2	Cap. 3	Cap. 4	Cap. 5	Cap. 6
μ_M	0	0.5	0	0.7	0.6	0.6
μ_L	0	0.4	0	0.3	0.2	0.1
μ_S	0.4	0.6	0.5	0.5	0.4	0.4
μ_{ML}	1	0.7	0.8	1	0.9	0.9
μ_{MS}	0.4	0.6	0.7	0.7	0.6	0.6
μ_{LS}	0.5	0.6	0.7	0.5	0.6	0.4
$C_\mu(a)$	0.39	0.46	0.43	0.43	0.4	0.4
$C_\mu(b)$	0.3	0.40	0.29	0.39	0.35	0.32
$C_\mu(c)$	0.39	0.51	0.49	0.56	0.52	0.5
$C_\mu(d)$	0.6	0.54	0.53	0.67	0.62	0.62
η^μ_{ML}	1	-0.2	0.8	$\boxed{0}$	0.1	0.2
η^μ_{MS}	$\boxed{0}$	-0.5	0.2	-0.3	-0.4	-0.4
η^μ_{LS}	0.1	-0.4	0.2	-0.3	$\boxed{0}$	-0.1
η^μ_{MLS}	0.23	-0.13	0.1	-0.23	-0.1	$\boxed{0}$

Table 2: A set of six capacities compatible with the preferences $a\,P\,b$ and $d\,P\,c$.

Depending on the choice of a capacity μ, the nonadditivity index can be null, or strictly positive, or strictly negative. This leads us to the definition of the notion of necessary and possible nonadditivity. This notion is similar at the notion of, necessary and possible interaction introduced in the case of a 2-additive Choquet integral model[22], and is somewhat similar to the notion of necessary and possible preference relations introduced in robust ordinal regression[7,15], replacing preferences with interaction indices.

4. Necessary and possible nonadditivity

The following definition of necessary and possible nonadditivity will be central in the rest of this text.

Definition 8. *Let $A \subseteq_{\geq 2} N$ and $\{P, I\}$ an ordinal preferential information. We say that:*

1. *There exists a possible positive (resp. null, negative) nonadditivity for A if there exists $\mu \in C_{Pref}$ such that $\eta^\mu_A > 0$ (resp. $\eta^\mu_A = 0$, $\eta^\mu_A < 0$),*

2. *There exists a necessary positive (resp. null, negative) nonadditivity for A if $\eta^\mu_A > 0$ (resp. $\eta^\mu_A = 0$, $\eta^\mu_A < 0$) for all $\mu \in C_{Pref}$.*

Remark 5. *Let $A \subseteq_{\geq 2} N$.*

- *If there exists a necessary positive (resp. null, negative) nonadditivity for A, then there exists a possible positive (resp. null, negative) nonadditivity for A.*

- *If there is no necessary positive (resp. null, negative) nonadditivity for A, then there exists a possible negative or null (resp. positive or negative, positive or null) nonadditivity for A.*

If we have a possible but not necessary nonadditivity, then the interpretation of the nonadditivity is difficult because it depends on the capacity chosen in C_{Pref}. Indeed, the interpretation of the nonadditivity only makes sense in the case of the necessary.

In[18], we treated the case where preferential information does not contains indifference. The next section treats the second case. Under some conditions, positive and negative nonadditivity are always possible.

5. Results when I is not empty

In the framework of generalized binary alternatives, we proved in[17], that, an ordinal preferential information $\{P, I\}$ on $\mathcal{B}^g$ is representable by a Choquet integral model if and only if the binary relation $(P \cup M \cup I)$ contains no strict cycle. In this section, we assume that this condition holds and the set of ordinal preferential information $\{P, I\}$ can contain an indifference. Given a subset $A \subseteq_{\geq 2} N$, Proposition 1 gives a sufficient condition on $\{P, I\}$ such that negative nonadditivity is always possible for A. Indeed, she shows that, if the DM is not indifferent between the worst alternative a_0 and another alternative, then negative nonadditivity is always possible for A.

Proposition 1. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ representable by a Choquet integral model and $A \subseteq_{\geq 2} N$. If for all $i \in A$, not$(a_0\, TC_{M\cup I} a_i)$, then there exists a capacity $\mu \in C_{\mathit{Pref}}$ such that $\eta^\mu_A < 0$.*

Proof. Let $A \subseteq_{\geq 2} N$, we assume that for all $i \in A$, not$(a_0\, TC_{M\cup I} a_i)$. Since $\{P, I\}$ is representable by a Choquet integral model, then $(P \cup M \cup I)$ contains no strict cycle, hence we can build a partition $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ of $\mathcal{B}^g$ using a suitable topological sorting on $(P \cup M \cup I)$ (see proof of Proposition 4 on[17]).
Let us define the capacity μ: $2^N \longrightarrow [0, 1]$ as follows:

$$\text{for all } S \subseteq N,\ \mu(S) = \begin{cases} 0, & \text{if } a_S \in \mathcal{B}_0 \\ \dfrac{\ell+1}{\ell+2}, & \text{if } a_S \in \mathcal{B}_\ell,\ \ell \in \{1, 2, \ldots, m-1\} \\ 1, & \text{if } a_S \in \mathcal{B}_m \end{cases}$$

Let $a_S, a_T \in \mathcal{B}^g$.

- If $a_S \, I \, a_T$, then $a_S, a_T \in \mathcal{B}_\ell$, thus $\mu(S) = \mu(T)$.

- If $a_S \, P \, a_T$, then there exists $r, q \in \{0, 1, \ldots, m\}$ such that $a_S \in \mathcal{B}_r$, $a_T \in \mathcal{B}_q$ since $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is a partition of $\mathcal{B}^g$. As $a_S \, P \, a_T$, then $r > q$. We have $C_\mu(u(a_S)) = \mu(S) = \dfrac{r+1}{r+2}$(if $1 \leq r \leq m-1$) or $\mu(S) = 1$ (if $r = m$), we then have $C_\mu(u(a_S)) \geq \dfrac{r+1}{r+2}$, since $1 \geq \dfrac{r+1}{r+2}$.

  - If $q = 0$, then $C_\mu(u(a_T)) = C_\mu(u(a_0)) = \mu(\emptyset) = 0 < \dfrac{r+1}{r+2} \leq C_\mu(u(a_S))$.
  - If $q \geq 1$, $C_\mu(u(a_T)) = \mu(T) = \dfrac{q+1}{q+2}$, since $1 \leq q \leq m-1$. But $r > q$ then $\dfrac{r+1}{r+2} > \dfrac{q+1}{q+2}$, since the sequence $(f_n)_{n\in\mathbb{N}}$ is strictly increasing, where $f_n = \dfrac{n+1}{n+2}$ for all $n \in \mathbb{N}$. Then $C_\mu(u(a_S)) > C_\mu(u(a_T))$.

  Hence, in both cases we have $C_\mu(u(a_S)) > C_\mu(u(a_T))$. We deduce that $\mu \in C_{\text{Pref}}$.

Let $\emptyset \subsetneq B \subsetneq A$, as $\forall i \in A$, not$(a_0 \, TC_{M\cup I} a_i)$, then $\forall i \in A$, we have $a_i \notin \mathcal{B}_0$, so $a_B \notin \mathcal{B}_0$. Hence $\mu(B) \geq \dfrac{\ell+1}{\ell+2}$ with $1 \leq \ell \leq m-1$. Thus $\mu(B) \geq \dfrac{2}{3} > \dfrac{1}{2}$, then $\sum_{\emptyset \subsetneq B \subsetneq A} \mu(B) > \dfrac{1}{2}(2^{|A|} - 2) = 2^{|A|-1} - 1$, i.e., $\dfrac{1}{2^{|A|-1} - 1} \sum_{\emptyset \subsetneq B \subsetneq A} \mu(B) > 1 \geq \mu(A)$. Thus $\eta^\mu_A < 0$. $\square$

Remark 6. *The sufficient condition of Proposition 1 is a necessary condition for* $A \subseteq_{=2} N$ *(see Proposition 2 below) but not necessary for* $A \subseteq_{\geq 3} N$*. Indeed, let us consider* $N = \{1, 2, 3\}$*,* $P = \{(a_{13}, a_2)\}$*,* $I = \{(a_0, a_1)\}$ *and* $A = N$*.* $\{P, I\}$ *is representable by the capacity given by the Table 3.*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0.5	1	1	1	0.5	1

Table 3: A capacity $\mu \in C_{\text{Pref}}$

We have $1 \in A$ *and* $a_0 TC_{M\cup I} a_1$*, but* $\eta^\mu_{123} = 1 - \dfrac{1}{3}(\mu_1 + \mu_2 + \mu_3 + \mu_{12} + \mu_{13} + \mu_{23}) = -\dfrac{1}{3} < 0$.

Given $\{i, j\} \subseteq N$, the following Proposition 2 gives a necessary and sufficient condition on $\{P, I\}$ such that negative nonadditivity that is always possible for $\{i, j\}$.

Proposition 2. *Let* $\{P, I\}$ *be an ordinal preferential information on* $\mathcal{B}^g$ *representable by a Choquet integral model and* $i, j \in N$*. There exists a capacity* $\mu \in C_{Pref}$ *such that* $\eta^\mu_{ij} < 0$ *if and only if not*$(a_0 \, TC_{M\cup I} a_i)$ *and not*$(a_0 \, TC_{M\cup I} a_j)$*.*

Proof. Let $i, j \in N$, we assume that $\{P, I\}$ is representable by a Choquet integral model.

Necessity. Assume that there exists a capacity $\mu \in C_{\text{Pref}}$ such that $\eta_{ij}^{\mu} < 0$. If $a_0 \, TC_{M\cup I} \, a_i$ or $a_0 \, TC_{M\cup I} \, a_j$, then $\eta_{ij}^{\mu} = \mu_{ij} - \mu_j \geq 0$ or $\eta_{ij}^{\mu} = \mu_{ij} - \mu_i \geq 0$ respectively. Therefore $\eta_{ij}^{\mu} \geq 0$ in the both cases. Contradiction since we assume that $\eta_{ij}^{\mu} < 0$.

Sufficiency. According to the Proposition 1 it sufficient to consider $A = \{i, j\}$. □

Let $A \subseteq_{\geq 2} N$ and $\emptyset \neq B \subsetneq A$, suppose that DM is indifferent between alternatives a_A and $a_{A\setminus B}$. This would suggest that subset B is quite unimportant for the DM, so that a_B is indifferent with a_0. We translate this idea by the following Definition 9.

Definition 9. *Let $A \subseteq_{\geq 2} N$. We call Monotonicity of Ordinal Preferential Information for A, the following property (denoted $A -$ MOPI): for all $\emptyset \neq B \subsetneq A$,*

$$a_A \sim a_{A\setminus B} \implies \textit{not}(a_B \, TC_P \, a_0).$$

Given a subset $A \subseteq_{\geq 2} N$, the Proposition 3 gives a sufficient condition on $\{P, I\}$ such that null or positive nonadditivity that is always possible for A.

Proposition 3. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ representable by a Choquet integral model. Let $A \subseteq_{\geq 2} N$. If the A-MOPI property is satisfied, then there exists a capacity $\mu \in C_{Pref}$ such that $\eta_A^{\mu} \geq 0$.*

Proof. Since $\{P, I\}$ is representable by a Choquet integral model, then $(P \cup M \cup I)$ contains no strict cycle, hence we can build a partition $\{\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m\}$ of $\mathcal{B}^g$ using a suitable topological sorting on $(P \cup M \cup I)$ (see proof of Proposition 4 on[17]). Let us define the capacity μ: $2^N \longrightarrow [0, 1]$ as follows:

$$\text{for all } S \subseteq N,\ \mu(S) = \begin{cases} 0, & \text{if} \quad a_S \in \mathcal{B}_0 \\ \dfrac{(2n)^{\ell}}{(2n)^m}, & \text{if} \quad a_S \in \mathcal{B}_{\ell},\ \ell \in \{1, 2, \ldots, m\} \end{cases}$$

Let a_S, $a_T \in \mathcal{B}^g$.

- If $a_S \, I \, a_T$, then $a_S, a_T \in \mathcal{B}_q$, therefore $\mu(S) = \mu(T)$.
- If $a_S \, P \, a_T$, then $a_S \in \mathcal{B}_q$ and $a_T \in \mathcal{B}_r$ with $q > r$. Therefore $\mu(S) = \dfrac{(2n)^q}{(2n)^m}$ and $\mu(T) = 0$ (if $r = 0$) or $\mu(T) = \dfrac{(2n)^r}{(2n)^m}$ (if $r \geq 1$). But $\dfrac{(2n)^q}{(2n)^m} > \max\left(0, \dfrac{(2n)^r}{(2n)^m}\right)$ since $q > r \geq 0$, so $\mu(S) > \mu(T)$.

Hence, we have $\mu \in C_{\text{Pref}}$.

Let $A \subseteq_{\geq 2} N$, we consider the set $\Psi = \{\emptyset \neq B \subsetneq A : \text{ not}(a_A \sim a_B) \text{ and not}(a_A \sim a_{A\setminus B})\}$.

We can write:

$$
\begin{aligned}
(2^{|A|}-2)\eta^\mu_A &= \sum_{\emptyset\neq B\subsetneq A}\Big(\mu(A)-\mu(B)-\mu(A\setminus B)\Big)\\
&= \sum_{B\in\Psi}\Big(\mu(A)-\mu(B)-\mu(A\setminus B)\Big)+\sum_{B\notin\Psi}\Big(\mu(A)-\mu(B)-\mu(A\setminus B)\Big).
\end{aligned}
$$

Let $\emptyset\neq B\subsetneq A$. If $B\notin\Psi$, then $a_A\sim a_B$ or $a_A\sim a_{A\setminus B}$, therefore we have $\big(a_A\sim a_B$ and $\text{not}(a_{A\setminus B}\,TC_P\,a_0)\big)$ or $\big(a_A\sim a_{A\setminus B}$ and $\text{not}(a_B\,TC_P\,a_0)\big)$ since by hypothesis, the property A-MOPI is satisfied. Thus, $\big(\mu(A)=\mu(B)$ and $\mu(A\setminus B)=0\big)$ or $\big(\mu(A)=\mu(A\setminus B)$ and $\mu(B)=0\big)$ respectively, i.e., $\mu(A)-\mu(B)-\mu(A\setminus B)=0$ in the both cases, and we have $\displaystyle\sum_{B\notin\Psi}\Big(\mu(A)-\mu(B)-\mu(A\setminus B)\Big)=0$. Hence $(2^{|A|}-2)\eta^\mu_A=\displaystyle\sum_{B\in\Psi}\Big(\mu(A)-\mu(B)-\mu(A\setminus B)\Big)$.

- If $\Psi=\emptyset$, then $(2^{|A|}-2)\eta^\mu_A=0$, i.e., $\eta^\mu_A=0$.
- If $\Psi\neq\emptyset$, then for all $B\in\Psi$, we have $a_B\in\mathcal{B}_r$, $a_{A\setminus B}\in\mathcal{B}_s$ and $a_A\in\mathcal{B}_q$ with $q>r$ and $q>s$. Hence $\mu(B)\leq(2n)^r$, $\mu(A\setminus B)\leq(2n)^s$ and $\mu(A)=(2n)^q=(2n)(2n)^{q-1}>2(2n)^{q-1}=(2n)^{q-1}+(2n)^{q-1}\geq(2n)^r+(2n)^s$ since $q-1\geq r$ and $q-1\geq s$. Therefore $\mu(A)>(2n)^r+(2n)^s\geq\mu(B)+\mu(A\setminus B)$, i.e., $\mu(A)-\mu(B)-\mu(A\setminus B)>0$ for all $B\in\Psi$. Hence $\displaystyle\sum_{B\in\Psi}\big(\mu(A)-\mu(B)-\mu(A\setminus B)\big)>0$, so $\eta^\mu_A>0$.

In the both case, we deduce that $\eta^\mu_A\geq 0$. □

Remark 7. *The sufficient condition of Proposition 3 is a necessary condition for $A\subseteq_{=2}N$ (see Theorem 4 in[22]) but not necessary for $A\subseteq_{\geq 3}N$. Indeed, if we consider $N=\{1,2,3\}$, $P=\{(a_{23},a_1),\,(a_3,a_0)\}$, $I=\{(a_{12},a_{123})\}$ and $A=N$. The ordinal preferential information $\{P,I\}$ is representable by the Choquet integral model w.r.t the capacity given in the Table 4.*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0	0.5	1	0.5	0.5	1

Table 4: A capacity $\mu\in C_{\text{Pref}}$

We have $a_{12}\sim a_{123}$ and $a_3\,TC_P\,a_0$ so the $\{1,2,3\}-$MOPI property is not satisfied, while $\eta^\mu_{123}=1-\dfrac{1}{3}(\mu_1+\mu_2+\mu_3+\mu_{12}+\mu_{13}+\mu_{23})=\dfrac{1}{6}\geq 0.$

Given a subset $A\subseteq_{\geq 2}N$, we are interested in the class of alternatives for which the DM is neutral for all the criteria not belonging to A. The alternative a_A is the best of them. Proposition 4 shows that, if the DM is not indifferent between this best alternative a_A and another alternative a_B (with $B\subsetneq A$), then positive nonadditivity is always possible for A.

Proposition 4. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ representable by a Choquet integral model and $A \subseteq_{\geq 2} N$. If for all $i \in A$, not$(a_{A\setminus\{i\}}\, TC_{M\cup I}\, a_A)$, then there exists a capacity $\mu \in C_{Pref}$ such that $\eta^{\mu}_{A} > 0$.*

Proof. Let $A \subseteq_{\geq 2} N$. We assume that for all $i \in A$, not$(a_{A\setminus\{i\}}\, TC_{M\cup I}\, a_A)$. We define the set Ψ as in the proof of Proposition 3. Since, for all $i \in A$, not$(a_{A\setminus\{i\}}\, TC_{M\cup I}\, a_A)$, then for all $\emptyset \subsetneq B \subsetneq A$, we have not$(a_B \sim a_A)$ and not$(a_{A\setminus B} \sim a_A)$. Therefore each subset $\emptyset \neq B \subsetneq A$ is an element of Ψ, hence $\Psi \neq \emptyset$. According to the proof of Proposition 3, we can build $\mu \in C_{\text{Pref}}$ such that $\eta^{\mu}_{A} > 0$. □

Remark 8. *The sufficient condition of Proposition 4 is a necessary condition for $A \subseteq_{=2} N$ (see Proposition 5 below) but not necessary for $A \subseteq_{\geq 3} N$. Indeed, let us consider $N = \{1, 2, 3\}$, $P = \{(a_{12}, a_3)\}$, $I = \{(a_{13}, a_{123})\}$ and $A = N$. $\{P, I\}$ is representable by the capacity given by the Table 5.*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0	0	1	1	0	1

Table 5: A capacity $\mu \in C_{\text{Pref}}$

We have $2 \in A$ *and* $a_{13} TC_{M\cup I} a_{123}$*, but* $\eta^{\mu}_{123} = 1 - \frac{1}{3}(\mu_1 + \mu_2 + \mu_3 + \mu_{12} + \mu_{13} + \mu_{23}) = \frac{1}{3} > 0$.

Given a pair of criteria $\{i, j\}$, Proposition 5 gives a necessary and sufficient condition on $\{P, I\}$ such that positive nonadditivity that is always possible for $\{i, j\}$.

Proposition 5. *Let $\{P, I\}$ be an ordinal preferential information on $\mathcal{B}^g$ representable by a Choquet integral model and $i, j \in N$. There exists a capacity $\mu \in C_{Pref}$ such that $\eta^{\mu}_{ij} > 0$ if and only if [not$(a_i\, TC_{M\cup I}\, a_{ij})$ and not$(a_j\, TC_{M\cup I}\, a_{ij})$].*

Proof. Let $i, j \in N$.

Necessity. Assume that there exists a capacity $\mu \in C_{\text{Pref}}$ such that $\eta^{\mu}_{ij} > 0$. If $a_i\, TC_{M\cup I}\, a_{ij}$ or $a_j\, TC_{M\cup I}\, a_{ij}$, then $\eta^{\mu}_{ij} = -\mu_j \leq 0$ or $\eta^{\mu}_{ij} = -\mu_i \leq 0$ respectively. Therefore $\eta^{\mu}_{ij} \leq 0$ in the both cases. Contradiction since we assume that $\eta^{\mu}_{ij} > 0$.

Sufficiency. According to the Proposition 4, it is sufficient to consider $A = \{i, j\}$. □

The results of this section show that, under some conditions, it is possible to represent the preferences of DM, in such a way as to have indices of nonadditivity with non-constant sign. Therefore, the interpretation of nonadditivity between criteria requires caution.

All previous results are based on the set of generalized binary alternatives $\mathcal{B}^g$. As this set is restrictive in practice, we propose in the next section, a process based on linear programming, allowing us to test the existence of a necessary positive (resp. negative) nonadditivity for a subset of criteria $A \subseteq_{\geq 2} N$.

6. A LP model testing for necessary nonadditivity

In[17] we proposed outside the framework of generalized binary alternatives, a linear program allowing to test the existence of some necessary interactions when $I = \emptyset$. We show how to test the existence necessary positive or negative nonadditivity on the basis of information given on a subset of X that is not necessarily $\mathcal{B}^g$, when $I = \emptyset$.
We assume that the DM provides at least one strict preference, ($i.e., P \neq \emptyset$) and an indifference I relations on a subset of X. Our approach consists in testing first, the compatibility of this ordinal preferential information with a Choquet integral model, and then, in the second step, the existence of a necessary positive or negative nonadditivity for a subset $A \subseteq_{\geq 2} N$.

6.1. The process

Step 1. At this step, we test the compatibility of $\{P, I\}$ with a Choquet integral. This is similar to what is done in[1,22]. The following linear program LP_1 models each preference of P by introducing a non negative slack variable ε in the corresponding constraint (Equation $(1a)$). The Equation $(1c)$ (resp. $(1d)$) ensures the normalization (resp. monotonicity) of capacity μ. The objective function Z_1 maximizes the non negative variable ε introduced in $(1a)$.

$$
\begin{array}{lr}
\text{Maximize } Z_1 = \varepsilon & LP_1 \\
\text{Subject to} & \\
C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \;\; \forall x, y \in X \text{ such that } x\ P\ y & (1a) \\
C_\mu(u(x)) - C_\mu(u(y)) = 0 \;\; \forall x, y \in X \text{ such that } x\ I\ y & (1b) \\
\mu(N) = 1 & (1c) \\
\mu(S \cup \{i\}) \geq \mu(S) \;\; \forall S \subsetneq N,\ \forall i \in N \setminus S. & (1d) \\
\varepsilon \geq 0 &
\end{array}
$$

We have one of the following two cases:

1. If the linear program LP_1 is not feasible or feasible with an optimal solution $Z_1^* = 0$, then there is no Choquet integral model compatible with $\{P, I\}$.

2. If the linear program LP_1 is feasible with an optimal solution $Z_1^* > 0$, then ordinal information $\{P, I\}$ is representable by a Choquet integral model.

Step 2. At this step, we suppose that the preference information $\{P, I\}$ is representable by a Choquet integral model, i.e., $Z_1^* > 0$. In order to know if the nonadditivity index for subset of criteria A is necessarily negative, at LP_1, we add the constraint $(1e)$ and we obtain the following linear program denoted by LP_{NN}^A.

$$
\begin{aligned}
&\text{Maximize } Z_2 = \varepsilon && LP_{NN}^A\\
&\text{Subject to}\\
&C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \;\; \forall x, y \in X \text{ such that } x\; P\; y && (1a)\\
&C_\mu(u(x)) - C_\mu(u(y)) = 0 \;\; \forall x, y \in X \text{ such that } x\; I\; y && (1b)\\
&\mu(N) = 1 && (1c)\\
&\mu(S \cup \{i\}) \geq \mu(S) \;\; \forall S \subsetneq N,\; \forall i \in N \setminus S && (1d)\\
&\varepsilon \geq 0\\
&\eta_A^\mu \geq 0. && (1e)
\end{aligned}
$$

To know if the nonadditivity index for subset of criteria A is necessarily positive, we change the constraint $(1e)$ by $\eta_A^\mu \leq 0$ and we obtain the linear program denoted LP_{NP}^A. After a resolution of the linear programs, we have one of the following three possible conclusions:

1. If LP_{NN}^A (resp. LP_{NP}^A) is not feasible, then there is a necessary negative (resp. positive) nonadditivity index for the subset A. Indeed, as the program LP_1 is feasible with an optimal solution $Z_1^* > 0$, the contradiction about the representation of $\{P, I\}$ only comes from the introduction of the constraint $\eta_A^\mu \geq 0$ (resp. $\eta_A^\mu \leq 0$).

2. If LP_{NN}^A (resp. LP_{NP}^A) is feasible and the optimal solution $Z_2^* = 0$, then the constraint $C_\mu(u(x)) - C_\mu(u(y)) \geq \varepsilon \;\; \forall x, y \in X$ such that $x\, P\, y$ is satisfied with $\varepsilon = 0$, i.e., it is not possible to model strict preference by adding the constraint $\eta_A^\mu \geq 0$ (resp. $\eta_A^\mu \leq 0$) in LP_{NN}^A (resp. LP_{NP}^A). Therefore, we can conclude that there is a necessary negative (resp. positive) nonadditivity index for A.

3. If LP_{NN}^A (resp. LP_{NP}^A) is feasible and the optimal solution $Z_2^* > 0$, then there is no necessary negative (resp. positive) nonadditivity index for A.

Note that this process can be done in three steps[22]. This way of doing it saves from having to specify an arbitrary parameter ε. But here we chose to do it in two steps.

For each of the previous linear programs, we have $n(2^{n-1}-1)$ monotonicity constraints. Furthermore, the Table 6 gives an idea of the decision variables and Table 7 gives an idea of number of variables and number of monotonicity constraints.

	Decision variables
LP_1	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$
LP_{NN}^A	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$
LP_{NP}^A	ε, $\mu(S)$ $(\emptyset \subsetneq S \subsetneq N)$

Table 6: Decision variables

	Number of variables $\mu(S)$	Number of constraints of monotonicity
$n = 3$	6	9
$n = 4$	14	28
$n = 5$	30	75
$n = 6$	62	186
$n = 7$	126	441
$n = 8$	254	1 016
$n = 9$	510	2 295
$n = 10$	1 022	5 110
$n = 11$	2 046	11 253
$n = 12$	4 094	24 564

Table 7: Number of variables $\mu(S)$ and number of monotonicity constraints with $3 \leq n \leq 12$

In practice, the number of criteria generally does not exceed 12. Thus, with a standard LP solver, we are able to deal with these linear programs.

6.2. Example

In this section, we illustrate our decision procedure with an example given by Brice Mayag in[21]. Six young artists without a producer take part in a high-audience singing competition program, where a winner will see his/her work produced by a famous record company. Each candidate performs, in front of a jury his own song. The jury is subdivided into three groups: a group of choreography professionals, another of professional singers and vocals, and the last group is formed by professional musicians. The following three criteria are used to classify candidates.

1. Choreography: the choreography chosen by the candidate during his performance. Evaluations are given as a number of vertical bars |. There are four members of the sub jury and each marks from 0 to 5 bars. The best candidate in choreography will be the one who will collect the greatest number of bars.

2. Singing: the quality of the song performed, taking into account the voice of the performer. The Singing sub jury evaluates the candidates in a classic way, assigning them marks between 0 and 20. The best candidate in singing will be the one who will collect the greatest number of marks.

3. Music: the quality of the music used to accompany the chosen song. The ability to play musical instruments is also taken into account at this level. The evaluations are given between 0 and 100. The best candidate in music will be the one who will collect the greatest number of marks.

The evaluations obtained by the candidates are given in the Table 8.

Candidates	1: Choreography	2: Singing	3: Music
a: Anita	||||| ||	17	70
b: Bertrand	||||| ||||	17	60
c: Carine	||||| ||	8	70
d: Didier	||||| ||||	8	60
e: Elise	||||| ||||| |	10	45
f: Fabien	||||| ||||| ||	10	45

Table 8: Evaluation matrix

In this example we have $N = \{1,\ 2,\ 3\}$ and $X = \{a, b, c, d, e, f\}$. To choose the winner, the jury establishes the following two rules:

- When two candidates have good marks in singing and in music, the jury will prefer the one who has a better evaluation in choreography, even if it means being less good in singing or in music. Therefore he strictly prefers b to a.
- When two candidates have a poor performance in singing, the jury will prefer the one who has the best rating in music. Therefore he strictly prefers c to d.

Besides, the jury finding the evaluations of candidates e and f very similar, considers them indifferent. The ordinal preferential information on X provided by the jury will therefore consist of the four following binaries relations $P = \{(b, a), (c, d)\}$, $I = \{(e, f), (f, e)\}$
To apply the Choquet integral to our example, we need to define commensurable scales. We assume that the construction of the utility functions u_1, u_2 and u_3 is done simply by normalization the evaluations according to each criterion by reducing them to marks between 0 and 20. Thus, the evaluations of the choreography criterion will be done by counting just the number of bars obtained by each candidate. On the music criterion, the evaluations will be normalized by a division by 5. We then obtain the Table 9.

	1	2	3
$u_i(a)$	7	17	14
$u_i(b)$	9	17	12
$u_i(c)$	7	8	14
$u_i(d)$	9	8	12
$u_i(e)$	11	10	9
$u_i(f)$	12	10	9

Table 9: Utility function u_i scaled between 0 and 20

Step 1. The linear program corresponding to the test of the existence of a capacity μ compatible with $\{P, I\}$ is the following.

Maximize $Z_1 = \varepsilon$ $\qquad LP_1$
Subject to
$C_\mu(u(b)) - C_\mu(u(a)) \geq \varepsilon$
$C_\mu(u(c)) - C_\mu(u(d)) \geq \varepsilon$
$C_\mu(u(e)) - C_\mu(u(f)) + \alpha^+_{ef} - \alpha^-_{ef} = 0$
$C_\mu(u(a)) = 7 + 7\mu_{23} + 3\mu_2$
$C_\mu(u(b)) = 9 + 3\mu_{23} + 5\mu_2$
$C_\mu(u(c)) = 7 + \mu_{23} + 6\mu_3$
$C_\mu(u(d)) = 8 + \mu_{13} + 3\mu_3$
$C_\mu(u(e)) = 9 + \mu_{12} + \mu_1$
$C_\mu(u(f)) = 9 + \mu_{12} + 2\mu_1$
$\mu_1 \geq 0$; $\mu_2 \geq 0$; $\mu_3 \geq 0$
$\mu_{12} \geq \mu_1$; $\mu_{12} \geq \mu_2$; $\mu_{13} \geq \mu_1$; $\mu_{13} \geq \mu_3$; $\mu_{23} \geq \mu_2$; $\mu_{23} \geq \mu_3$
$\mu_{123} \geq \mu_{12}$; $\mu_{123} \geq \mu_{13}$; $\mu_{123} \geq \mu_{23}$
$\mu_{123} = 1$
$\varepsilon \geq 0$.
The linear program LP_1 is feasible with an optimal solution $Z^*_1 = 0.8 > 0$, then we can conclude that, $\{P, I\}$ is representable by a Choquet integral model.

Step 2. In order to know if the nonadditivity for $\{1, 2, 3\}$ is necessarily negative (resp. positive). We obtain the LP^{123}_{NN} (resp. LP^{123}_{NP}) by adding at the previous linear program LP_1 the constraints $\eta^\mu_{123} \geq 0$ (resp. $\eta^\mu_{123} \leq 0$) with $\eta^\mu_{123} = 1 - \frac{1}{3}(\mu_{12}+\mu_{13}+\mu_{23}+\mu_1+\mu_2+\mu_3)$.

- The linear program LP^{123}_{NN} is feasible with an optimal solution $Z^*_2 = 0.8 > 0$. Then the nonadditivity for {Choreography, Singing, Music} is not necessarily negative. Moreover, the results obtained by solving LP^{123}_{NN} are given by the Tables 10 and 11 (with $\eta^\mu_{123} = 0.3 > 0$).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0.4	0.4	0.5	0.4	0.4	1

Table 10: A capacity $\mu \in C_{\text{Pref}}$

x	a	b	c	d	e	f
$C_\mu(u(x))$	11	12.2	9.8	9.6	9.5	9.5

Table 11: Choquet integral corresponding at the capacity μ of Table 10

- The linear program LP_{NP}^{123} is feasible with an optimal solution $Z_2^* = 0.8 > 0$. Then the nonadditivity for {Choreography, Singing, Music} is not necessarily positive. Moreover, the results obtained by solving LP_{NP}^{123} are given by the Tables 12 and 13 (with $\eta_{123}^{\mu} = -0.2 < 0$).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\mu(S)$	0	0.7	0.7	0.8	0.7	0.7	1

Table 12: A capacity $\mu \in C_{\text{Pref}}$

x	a	b	c	d	e	f
$C_\mu(u(x))$	14	14.6	11.9	10.8	9.8	9.8

Table 13: Choquet integral corresponding at the capacity μ of Table 12

7. Conclusion

In the Choquet integral model, the capacity elicited to represent the preferences of the decision maker is not likely to be unique. This contrasts with the "continuous case" studied in[27,28]. This non-uniqueness complicates the interpretation of the nonadditivity index. Indeed, we give some examples in which the sign of the nonadditivity index depends upon the arbitrary choice of a capacity within the set of all capacities compatible with the preferences that were obtained. We define the concept of necessary and possible nonadditivity. This concept is similar at necessary and possible interaction, introduced in[22] in the case of 2-additive capacities. Necessary nonadditivity is the only nonadditivity that can safely be interpreted since its sign does not vary within the set of all compatible capacities. We have given conditions under which preferences on binary alternatives can be represented using a capacity in a Choquet integral model. We do the same by adding the extra conditions so that one of the representative capacities induces strictly positive nonadditivity indices for all groups of criteria, and another representative capacities induces strictly negative nonadditivity indices for all groups of criteria. These results show that, in practice, when we elicit a capacity on the basis of preferential information, it is not easy to interpret what we find using the nonadditivity index. Therefore, the interpretation of nonadditivity between criteria requires caution.

Our results leave some important questions open. The first one would be to develop tools allowing to analyze "necessary nonadditivity" for a large class of aggregation models, including the Choquet integral model. The second would be to study of aggregation models using bipolar scales[10].

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