A conjoint measurement view on fuzzy integrals

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(Based on joint work with Thierry Marchant & Marc Pirlot
Uses recent work by Greco, Matarazzo & Słowiński)

Example

Typical experiment in MCDM

Subjects (customers) express preferences for holiday packages

	cost	# of days	travel time	category of hotel	distance to beach	Wifi	cultural interest
A	200 €	15	12 h	***	45 km	Y	++
B	425 €	18	15 h	****	0 km	N	−−
C	150 €	4	7 h	**	250 km	N	+
D	300 €	5	10 h	***	5 km	Y	−

Data

$B \succ C \succ D \succ A$

B is Good, C and D are Acceptable, A is Unacceptable

Questions

Central questions

- can these preferences be represented using a given model aggregation model?
- how to infer the parameters of the aggregation model based on the data collected?

Importance of answering the central questions

- understanding and comparing aggregation models
- using a model for decision aiding purposes

Answer to central questions

Answer

- easy for a number of models
    - additive value function model
- not so easy for aggregation models based on fuzzy integrals (Sugeno, Choquet, or variants)

Prerequisite for fuzzy integral aggregation

- ordering levels on *distinct* attributes
    - is a 12 h travel time “better” than having a Wifi connection?
- question that is:
    - very specific to that type models
    - not a very natural
- answer is not easily inferred from data

What we are looking for?

Type of result wanted: Conjoint Measurement

"Let $\succsim$ be a binary relation on a set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$. This relation can be represented in the Sugeno (Choquet) integral model iff a number of conditions expressed in terms of $\succsim$ only are satisfied"

Important points

- no special structure for X besides being a Cartesian product
- conditions phrased in terms of $\succsim$ on X

Usefulness?

- theoretical interest
- comparing aggregation models based on fuzzy integrals with other models
- assessment techniques

Outline

1. Background on conjoint measurement
2. Setting
3. The noncompensatory sorting model
4. Sugeno integral

Conjoint measurement (Krantz et al., 1971, ch. 6–7, Wakker, 1989)

- $X \subseteq X_1 \times X_2 \times \cdots \times X_n$: set of objects evaluated on n attributes
- $\succsim$: *binary relation* on X
- **Aim**: Study under what conditions $\succsim$ can be represented in a given measurement model and the uniqueness of this representation

Remarks

- $\succsim$ is the *only* primitive of the model (observable, in principle)
- X_i have no special structure (numbers expressed in a well defined unit, levels on an ad hoc discrete scale, etc.)
- X_i and X_j may have a different structure
- we may suppose wlog that the sets X_i are pairwise disjoint

Additive value functions

Additive value functions

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) \geq \sum_{i=1}^{n} v_i(y_i)$$

$v_i : X_i \to \mathbb{R}$

Characterization

- when X is finite, necessary and sufficient conditions are known (Scott, 1964) (and may be tested using LP)
- when X is of arbitrary cardinality, sufficient conditions are known (Debreu, 1960, Luce & Tukey, 1964)

Additive value functions

Remarks

- central condition is *independence*: a common evaluation on some attribute does not affect preference

$$(x_{-i}, z_i) \succsim (y_{-i}, z_i) \Rightarrow (x_{-i}, w_i) \succsim (y_{-i}, w_i)$$

- Uniqueness: $(v_1, v_2, \ldots, v_n)$ define *interval scales with a common unit*

$$u_i = \alpha v_i + \beta_i \qquad \alpha > 0$$

- comparing values of v_i with values of v_j with $i \neq j$ leads to assertions that are *not* meaningful

Decomposable models

Decomposable model

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots, v_n(x_n)] \geq F[v_1(y_1), \ldots, v_n(y_n)]$$

$v_i : X_i \rightarrow \mathbb{R}$
$F : \mathbb{R}^n \rightarrow \mathbb{R}$, nondecreasing (increasing) in each argument

Characterization

- necessary and sufficient conditions are known (increasing: Krantz et al., 1971; nondecreasing: Bouyssou & Pirlot, 2004)
- central condition is *weak separability*: varying common evaluations on all but one attribute does not reverse strict preference

$$(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, w_{-i}) \succsim (y_i, w_{-i})$$

Decomposable models

Remarks

- uniqueness: quite weak
- comparing values of v_i with values of v_j with $i \neq j$ leads to assertions that are *not* meaningful

Motivations for fuzzy integrals in MCDM

Additive value functions

- additive model implies *independence*
  - no "interaction" between attributes, which may seem too restrictive

Decomposable model

- decomposable model implies *weak separability* but not independence
  - many degrees of freedom
  - assessment is problematic

Basic idea

- weak separability is hardly a restrictive hypothesis
- independence is a much more restrictive condition
- find models "in between"

Remarks

Aggregation using a Sugeno or a Choquet integral

- implies weak separability
- does not imply independence
  - "in between" the additive and the decomposable models
  - far more structured than the decomposable model

MCDM vs Decision making under uncertainty

Setting (with finite number of states)

- formally similar to MCDM with states of nature playing the role of attributes...
- ...with the additional *crucial* hypothesis that $X = Y^n$
- commensurability between consequences obtained in several sates of nature is "built in" as soon as there is a weak order on X

$$\alpha \overline{\succsim} \beta \Leftrightarrow (\alpha, \ldots, \alpha) \succsim (\beta, \ldots, \beta)$$

- the reordering of the states of nature needed to compute the fuzzy integrals causes no problem

Fuzzy integrals in Decision making under uncertainty

Results on Sugeno in the context of uncertainty

- characterization of relations representable in the Sugeno integral model: Hougaard & Keiding (1996), Dubois, Prade & Sabbadin (2001)

Results on Choquet in the context of uncertainty

- characterization of relations representable in the Choquet integral model: Schmeidler (1984, 1989), Wakker (1987, 1989)
- basic condition is a weakening of independence: *comonotonic independence*

Problem

- these results cannot be transposed to the case of MCDM

Previous works in MCDM

Theoretical study of fuzzy integrals in MCDM

1. supposing commensurability from the start: "aggregation operators" or "subjective evaluation" (Dubois & Prade, Grabisch, Marichal, Yager, . . .)
   - testing commensurability?
   - comparison with other models?
   - meaning of conditions (idempotence)?
2. building commensurate scales on each attribute (Grabisch & Labreuche)
   - requires a "neutral" level on each attribute
   - assessing the neutral level?

Exception

Greco, Matarazzo & Słowiński (2002, 2004)

Setting

Classical conjoint measurement setting

- $N = \{1, 2, \dots, n\}$: set of attributes
- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: set of alternatives
  - X_i are not supposed to have a special structure
- $\succsim$: binary relation on X

Our setting

- replace the binary relation $\succsim$ on X by a partition $\langle C^1, C^2, \dots, C^r \rangle$ of X
- worst category is C^1, best is C^r
  - $C^{\geq k} = \bigcup_{j=k}^{r} C^j$
  - $R = \{1, 2, \dots, r\}$

Decomposable models

Measurement model

$$x \in C^k \Leftrightarrow \sigma_k < F[v_1(x_1), v_2(x_2), \ldots, v_n(x_n)] < \sigma_{k+1}$$

$\sigma_1 < \sigma_2 < \cdots < \sigma_{r+1}$
$v_i : X_i \to \mathbb{R}$
$F : \mathbb{R}^n \to \mathbb{R}$, *increasing* in all its arguments

Remarks

- model proposed by Goldstein (1991) for two and three categories
- generalized to r categories by Greco, Matarazzo & Słowiński (2001, 2002)

Alternative equivalent formulation

$$x \in C^{\geq k} \Leftrightarrow F[v_1(x_1), v_2(x_2), \ldots, v_n(x_n)] > \sigma_k$$

Axiom and result

R-linearity

$\langle C^k \rangle_{k \in R}$ is R-linear on attribute $i \in N$

$$\left.\begin{array}{c}(x_i, a_{-i}) \in C^k \\ \text{and} \\ (y_i, b_{-i}) \in C^\ell\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \in C^{\geq k} \\ \text{or} \\ (x_i, b_{-i}) \in C^{\geq \ell}\end{array}\right.$$

Theorem (Goldstein, 1991, GMS 2001)

If X is finite or countably infinite, a partition $\langle C^k \rangle_{k \in R}$ can be represented in the decomposable model iff it is R-linear

Remarks

- can be generalized to sets of arbitrary cardinality
- increasing is equivalent to the nondecreasing

The noncompensatory sorting model

Ingredients

- sets $\mathscr{A}_i^r \subseteq \mathscr{A}_i^{r-1} \subseteq \cdots \subseteq \mathscr{A}_i^2 \subseteq X_i$
  - $\mathscr{A}_i^\ell$: set of satisfactory levels on attribute i for $C^{\geq \ell}$
- sets $\mathscr{F}^r \subseteq \mathscr{F}^{r-1} \subseteq \cdots \subseteq \mathscr{F}^2 \subseteq 2^N$ such that

$$[I \in \mathscr{F}^k \text{ and } I \subseteq J] \Rightarrow J \in \mathscr{F}^k$$

  - $\mathscr{F}^\ell$: set of "sufficiently important" coalition of attributes for $C^{\geq \ell}$

Model

$$x \in C^{\geq k} \Leftrightarrow \{i \in N : x_i \in \mathscr{A}_i^k\} \in \mathscr{F}^k$$

Interpretation

- $x \in C^{\geq k}$ iff x is "satisfactory" ($x_i \in \mathscr{A}_i^k$) at level k on a subset of attributes that is "sufficiently important" ($\in \mathscr{F}^k$) at level k

Remarks

The noncompensatory sorting model

- has a simple interpretation (Fishburn, 1978)
- has a definite ordinal character
- has intimate connections with some well-known MCDM techniques (ELECTRE TRI, Wei, 1992, Roy & Bouyssou, 1993)

Observations

Remarks

- if $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model then it is R-linear
  - the noncompensatory sorting model is a particular case of the decomposable model

Specific feature

- for the twofold partition $\langle C^{\geq k}, C^{<k} \rangle$ each x_i is either satisfactory or unsatisfactory
  - dichotomic character of the model

Axiom

R-2-gradedness

$$\left.\begin{array}{c}(x_i, a_{-i}) \in C^{\geq k} \\ \text{and} \\ (y_i, a_{-i}) \in C^{\geq k} \\ \text{and} \\ (y_i, b_{-i}) \in C^{\geq \ell}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, b_{-i}) \in C^{\geq \ell} \\ \text{or} \\ (z_i, a_{-i}) \in C^{\geq k}\end{array}\right.$$

for all $k, \ell \in R^+$ with $\ell \leq k$ (ℓ worse then k)

Observation

- if a partition $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model then it is R-2-graded

Result

Theorem (Bouyssou & Marchant, 2006)

A partition $\langle C^k \rangle_{k \in R}$ of X has a representation in the noncompensatory sorting model iff it is R-linear and R-2-graded

Remark

- R-linearity and R-2-gradedness are independent conditions
- uniqueness is well understood
- no condition on the cardinality of X

Definition

Partition representable using a Sugeno integral

- a non-negative real valued function f on $\bigcup_{i=1}^{n} X_i$
- a normalized capacity μ on 2^N
- $r-1$ real numbers such that $0 < \rho_2 < \rho_3 < \ldots < \rho_r$

such that, for all $x \in X$,

$$x \in C^{\geq k} \Leftrightarrow \mathcal{S}_{\langle \mu, f \rangle}(x) = \bigvee_{I \subseteq N} \left[\mu(I) \wedge \left(\bigwedge_{i \in I} [f(x_i)] \right) \right] > \rho_k$$

Interpretation

- natural extension of Sugeno to the context of ordered partitions

Observations

Remark

- if a partition has a representation in the noncompensatory sorting model, it can be represented using a Sugeno integral

skip proof

Proof

$\langle \mathscr{F}^k, \langle \mathscr{A}_i^k \rangle_{i \in N} \rangle$: representation of $\langle C^k \rangle_{k \in R}$ in the noncompensatory sorting model
$0 < \lambda_1 < \rho_2 < \lambda_2 < \rho_3 < \dots \lambda_{r-1} < \rho_r < \lambda_r$

$$\begin{cases} f(x_i) = \lambda_r & \text{if } x_i \in \mathscr{A}_i^r \\ f(x_i) = \lambda_{r-1} & \text{if } x_i \in \mathscr{A}_i^{r-1} \setminus \mathscr{A}_i^r \\ \dots & \\ f(x_i) = \lambda_2 & \text{if } x_i \in \mathscr{A}_i^2 \setminus \mathscr{A}_i^3 \\ f(x_i) = \lambda_1 & \text{otherwise} \end{cases} \qquad \begin{cases} \mu(A) = \lambda_r & \text{if } A \in \mathscr{F}^r \\ \mu(A) = \lambda_{r-1} & \text{if } A \in \mathscr{F}^{r-1} \setminus \mathscr{F}^r \\ \dots & \\ \mu(A) = \lambda_2 & \text{if } A \in \mathscr{F}^2 \setminus \mathscr{F}^3 \\ \mu(A) = \lambda_1 & \text{otherwise} \end{cases}$$

$$x \in C^k \Leftrightarrow \mathcal{S}_{\langle \mu, f \rangle}(x) = \lambda_k$$
$$x \in C^{\geq k} \Leftrightarrow \mathcal{S}_{\langle \mu, f \rangle}(x) > \rho_k$$

Observations

Remark

- if a partition has a representation using Sugeno, it is R-linear and R-2-graded
  [routine check]

Main Result

Theorem

A partition $\langle C^k \rangle_{k \in R}$ of a set X has a representation in the noncompensatory sorting model iff it has a representation using a Sugeno integral (iff it is R-linear and R-2-graded)

Remarks

- gives a new and simple interpretation of the Sugeno integral
- answers our initial question for ordered partitions

What about commensurability?

- derived commensurability

$$x_i \precsim x_j \Leftrightarrow x_i \in \mathscr{A}_i^k \text{ and } x_j \in \mathscr{A}_j^\ell \text{ with } \ell \leq k \text{ (}\ell\text{ worse than }k\text{)}$$

Observations

Remarks

- if $\langle C^k \rangle_{k \in R}$ has a representation in the noncompensatory sorting model ...
- ...it has a representation using a Sugeno integral model...
- ...such that $\forall x \in C^k$, $\mathcal{S}_{\langle \mu, f \rangle}(x) = \lambda_k$

Consequence

- rewording the axioms for the noncompensatory sorting model in terms of the weak order induced by $\langle C^k \rangle_{k \in R}$ ...
- ...gives conditions for the representation of a weak order using a Sugeno integral

jump to conclusion if late

R-linearity and weak separability

Reformulation of R-linearity

A partition $\langle C^k \rangle_{k \in R}$ is R-linear iff its associated weak order $\succsim$ is *weakly separable*

$$(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, w_{-i}) \succsim (y_i, w_{-i})$$

Theorem (Bouyssou & Pirlot, 2004)

Let $\succsim$ be a binary relation on a finite or countably infinite set X. Then $\succsim$ has a representation in the decomposable model

$$x \succsim y \Leftrightarrow F[v_1(x_1), \ldots, v_n(x_n)] \geq F[v_1(y_1), \ldots, v_n(y_n)]$$

with F being nondecreasing in each argument iff

- $\succsim$ *is a weak order*
- $\succsim$ *is weakly separable*

R-2-graded

Reformulation R-2-gradedness

A partition $\langle C^k \rangle_{k \in R}$ is R-2-graded iff its associated weak order satisfies condition D (Dichotomy)

$$\left.\begin{array}{c} (x_i, a_{-i}) \succsim t \\ \text{and} \\ (y_i, a_{-i}) \succsim t \\ \text{and} \\ (y_i, b_{-i}) \succsim w \\ \text{and} \\ t \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (x_i, b_{-i}) \succsim w \\ \text{or} \\ (z_i, a_{-i}) \succsim t \end{array}\right.$$

Result

Theorem

Let $\succsim$ be a weak order on X with a finite number of equivalence classes. Then $\succsim$ can be be represented in the Sugeno integral model iff

- *it is weakly separable*
- *it satisfies D*

Remarks

Remarks

- can be extended to arbitrary weak orders
- characterization of the Sugeno integral model *within* the decomposable model
- condition D is expressed using $\succsim$
  - we can use data collected in our experiment to test it
- new interpretation of Sugeno integral using the noncompensatory sorting model

For the record...

Who should be credited?

- GMS (2002) characterize the Sugeno integral model for ordered partitions (using slightly different axioms than ours: only one axiom; no proof given)
- GMS (2004) give axioms for the Sugeno integral model for weak orders (using slightly different axioms than here: only one axiom; no proof given)

- Bouyssou & Marchant (2006) characterize the noncompensatory sorting model being unaware of GMS (2002)
- GMS reacting on Bouyssou & Marchant (2006) brought GMS (2002) to our attention and observed that the axioms were similar

...hence, the results presented here

Summary

Main points

- conjoint measurement analysis of fuzzy integrals is
  - useful
  - enlightening
- complete analysis for the Sugeno integral model

Extensions

- GMS have many reformulations of the decomposable and the Sugeno integral models in terms of decision rules
- Bouyssou & Marchant (2006) study many variants of the noncompensatory sorting model
  - inclusion of discordance effects
  - characterization of ELECTRE TRI (Wei, 1992, Roy & Bouyssou, 1993)

Future Research

Open problems

- what about particular cases of the Sugeno integral?
  - GMS (2004) have results (ordered weighted minimum and maximum)
- what about the Choquet integral?
  - no results to date
  - likely to be difficult
  - likely to be rewarding

BibliographyTools

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