A conjoint measurement approach to the discrete Sugeno integral

A note on a result of Greco, Matarazzo and Słowiński

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Abstract

In a recent paper (*European Journal of Operational Research*, **158**, 271–292, 2004), S. Greco, B. Matarazzo and R. Słowiński have stated without proof a result characterizing binary relations on product sets that can be represented using a discrete Sugeno integral. To our knowledge, this is the first result about a fuzzy integral that applies to non-necessarily homogeneous product sets and only uses a binary relation on this set as a primitive. This is of direct interest to MCDM. The main purpose of this note is to propose a proof of this important result. Thereby, we study the connections between the discrete Sugeno integral and a non-numerical model called the noncompensatory model. We also show that the main condition used in the result of S. Greco, B. Matarazzo and R. Słowiński can be factorized in such a way that the discrete Sugeno integral model can be viewed as a particular case of a general decomposable representation.

Key words: MCDM, Sugeno integral, conjoint measurement.

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1 Introduction and motivation

In the area of decision-making under uncertainty, the use of fuzzy integrals, most notably the Choquet integral and its variants, has attracted much attention in recent years. It is a powerful and elegant way to extend the traditional model of (subjective) expected utility. Indeed, integrating with respect to a non-necessarily additive measure allows to weaken the independence hypotheses embodied in the additive representation of preferences underlying the expected utility model that have often been shown to be violated in experiments (see the pioneering experimental findings of Allais, 1953 and Ellsberg, 1961). Models based on Choquet integrals have been axiomatized in a variety of ways (see Gilboa, 1987, Schmeidler, 1989 or Wakker, 1989, Ch. 6. For related works in the area of decision-making under risk, see Quiggin, 1982 and Yaari, 1987). Recent reviews of this research trend can be found in Chateauneuf and Cohen (2000), Schmidt (2004), Starmer (2000) and Sugden (2004).

More recently, still in the area of decision-making under uncertainty, Dubois et al. (2000b) have suggested to replace the Choquet integral by a Sugeno integral, the latter being a kind of "ordinal counterpart" of the former, and provided an axiomatic analysis of this model (special cases of the Sugeno integral are analyzed in Dubois et al., 2001b. For a related analysis in the area of decision-making under risk, see Hougaard and Keiding, 1996). Dubois et al. (2001a) offer a nice survey of these developments.

Unsurprisingly, people working in the area of multiple criteria decision making (MCDM) have considered following a similar path to build models weakening the independence hypotheses embodied in the additive value function model that underlies most of existing MCDM techniques. The work of Grabisch (1995, 1996) has widely popularized the use of fuzzy integrals in MCDM. Since then, there has been many developments in this area. They are well surveyed in Grabisch and Roubens (2000) and Grabisch and Labreuche (2004) (an alternative approach to weaken the independence hypotheses of the traditional model that does not use fuzzy integrals is suggested in Gonzales and Perny, 2005).

It is well known that decision-making under uncertainty and MCDM are related areas. When there is only a finite number of states of nature, acts may indeed be viewed as elements of a homogeneous Cartesian product in which the underlying set is the set of all consequences (this is the approach advocated and developped in Wakker, 1989, Ch. 4). In the area of MCDM, a Cartesian product structure is also used to model alternatives. However, in MCDM the product set is generally not homogeneous: alternatives are evaluated on several attributes that do not have to be expressed on the same scale.

The recent development of the use of fuzzy integrals in the area of MCDM should not obscure the fact that there is a major difficulty involved in the transposition of techniques coming from decision-making under uncertainty to the area of MCDM. In the former area, any two consequences can easily be compared: considering constant acts gives a straightforward way to transfer a preference relation on the set of acts to the set of consequences. The situation is vastly different in the area of MCDM. The fact that the underlying product set is not homogeneous invalidates the idea to consider "constant acts". Therefore, there is no obvious way to compare consequences on different attributes. Yet, such comparisons are a prerequisite for the application of models based on fuzzy integrals.

Traditional conjoint measurement models (see, e.g., Krantz et al., 1971, Ch. 6 or Wakker, 1989, Ch. 3) lead to compare *preference differences* between consequences. It is indeed easy to give a meaning to a statement like "the preference difference between consequences x_i and y_i on attribute i is equal to the preference difference between consequences x_j and y_j on attribute j" (e.g., because they exactly compensate the same preference difference expressed on a third attribute). These models do *not* lead to comparing in terms of preference consequences expressed on distinct attributes. Indeed, in the additive value function model a statement like "x_i is better than x_j" is easily seen to be meaningless (this is reflected in the fact that, in this model, the origin of the value function on each attribute may be changed independently on each attribute).

In order to bypass this difficulty, most studies involving fuzzy integrals in the area of MCDM postulate that the attributes are somehow "commensurate", while the precise content of this hypothesis is difficult to analyze and test (see, e.g., Dubois et al., 2000a). Less frequently, researchers have tried to build attributes so that this commensurability hypothesis is adequate. This is the path followed in Grabisch et al. (2003) who use the MACBETH technique (see Bana e Costa and Vansnick, 1994, 1997, 1999) to build such scales. Such an analysis requires the assessment of a neutral level on each attribute that is supposed to be "equally attractive". In practice, the assessment of such levels does not seem to be an easy task. On a more theoretical level, the precise properties of these commensurate neutral levels are not easy to devise.

A major breakthrough for the application of fuzzy integrals in MCDM has recently been done in Greco et al. (2004) who give conditions characterizing binary relations on product sets that can be represented using a discrete Sugeno integral, using this binary relation as the only primitive. This is an important result that paves the way to a measurement-theoretic analysis of fuzzy integrals in the area of MCDM (Greco et al., 2004 also relate the discrete Sugeno integral model to models based on decision rules that they have advocated in Greco et al., 1999, 2001). It allows to analyze the discrete Sugeno integral model without any commensurateness hypothesis, which is of direct interest to MCDM.

Given the importance of the above result, it is a pity that Greco et al. (2004) offer no proof of it [1]. The purpose of this note is to propose such a proof, in the hope that this will contribute to popularize this result. In doing so, we will also study the relations between the discrete Sugeno integral model and a non-numerical model called the noncompensatory model that is inspired from the work of Bouyssou and Marchant (2006) in the area of sorting methods in MCDM. We will also show that the main condition used in the result in Greco et al. (2004) can be factorized in such a way that the discrete Sugeno integral model can be viewed as a particular case of a general decomposable representation.

This note is organized as follows. The result of Greco et al. (2004) is presented in Section 2. The following two sections present our proof: Section 3 is devoted to some intermediate results and Section 4 completes the proof. Section 5 presents examples showing that the conditions used in the main result are independent. Section 6 briefly concludes with the mention of some directions for future research.

2 The main result

2.1 Background on the discrete Sugeno integral

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_p) \in [0,1]^p$. Let $(\cdot)_\beta$ be a permutation on $P = \{1, 2, \ldots, p\}$ such that $\beta_{(1)_\beta} \leq \beta_{(2)_\beta} \leq \cdots \leq \beta_{(p)_\beta}$.

A capacity on P is a function $\nu : 2^P \to [0,1]$ such that:

- $\nu(\varnothing) = 0$,
- $[A, B \in 2^P \text{ and } A \subseteq B] \Rightarrow \nu(A) \leq \nu(B)$.

The capacity ν is said to be normalized if, furthermore, $\nu(P) = 1$.

[1] To our knowledge, Greco, Matarazzo, and Słowiński have never presented or published their proof. It should be mentioned that a related result for the case of ordered categories is presented without proof in Słowiński et al. (2002). This result is a particular case of the one presented in Greco et al. (2004) for weak orders with a finite number of distinct equivalence classes. A complete and quite simple proof for this particular case was proposed in Bouyssou and Marchant (2006), using comments made on an early version of the latter paper by Greco, Matarazzo, and Słowiński.

The discrete Sugeno integral of the vector $(\beta_1, \beta_2, \ldots, \beta_p) \in [0,1]^p$ w.r.t. the normalized capacity ν is defined by:

$$S_\nu[\beta] = \bigvee_{i=1}^{p} \left[\beta_{(i)_\beta} \wedge \nu(A_{(i)_\beta})\right],$$

where $A_{(i)_\beta}$ is the element of 2^P equal to $\{(i)_\beta, (i+1)_\beta, \ldots, (p)_\beta\}$.

We refer the reader to Dubois et al. (2001a) and Marichal (2000a,b) for excellent surveys of the properties of the discrete Sugeno integral and its several possible equivalent definitions. Let us simply mention here that the reordering of the components of β in order to compute its Sugeno integral can be avoided noting that we may equivalently write:

$$S_\nu[\beta] = \bigvee_{T \subseteq P} \left[\nu(T) \wedge \left(\bigwedge_{i \in T} \beta_i\right)\right].$$

2.2 The model

Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. The relations $\succ$ and $\sim$ are defined as usual. We denote by X_{-i} the set $\prod_{j \in N \setminus \{i\}} X_j$. We abbreviate $Not[\, x \succsim y\,]$ as $x \not\succsim y$.

We say that $\succsim$ has a representation in the *discrete Sugeno integral model* if there are a normalized capacity μ on N and functions $u_i : X_i \to [0,1]$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow S_{\langle \mu, \boldsymbol{u} \rangle}(x) \geq S_{\langle \mu, \boldsymbol{u} \rangle}(y),$$

where $S_{\langle \mu, \boldsymbol{u} \rangle}(x) = S_\mu[(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))]$.

2.3 Axioms and result

A *weak order* is a complete and transitive binary relation. The set $Y \subseteq X$ is said to be dense in X for the weak order $\succsim$ if for all $x, y \in X$, $x \succ y$ implies $x \succsim z$ and $z \succsim y$, for some $z \in Y$. We say that the weak order $\succsim$ on X satisfies the *order-denseness condition* (condition *OD*) if there is a finite or countably infinite set $Y \subseteq X$ that is dense in X for $\succsim$. It is well-known (see Fishburn, 1970, p. 27 or Krantz et al., 1971, p. 40) that there is a real-valued function v on X such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow v(x) \geq v(y),$$

if and only if $\succsim$ is a weak order on X satisfying the order-denseness condition.

Remark 1
Let $\succsim$ be a weak order on X. It is clear that $\sim$ is an equivalence and that the elements of $X/\sim$ are linearly ordered. We often abuse terminology and speak of equivalence classes of $\succsim$ to mean the elements of $X/\sim$. When $X/\sim$ is finite, we speak of the first equivalence class of $\succsim$ to mean the elements of $X/\sim$ that precede all others in the induced linear order. •

The relation $\succsim$ on X is said to be strongly 2-graded on attribute $i \in N$ (condition 2^*-graded$_i$) if, for all $x, y, z, w \in X$ and all $a_i \in X_i$,

$$\left. \begin{array}{c} x \succsim z \\ \text{and} \\ y \succsim w \\ \text{and} \\ z \succsim w \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (a_i, x_{-i}) \succsim z \\ \text{or} \\ (x_i, y_{-i}) \succsim w, \end{array} \right. \qquad (2^*\text{-graded}_i)$$

where (a_i, x_{-i}) denotes the element of X obtained from $x \in X$ by replacing its ith coordinate by $a_i \in X_i$. The binary relation will be said to be *strongly 2-graded* (condition 2^*-graded) if it is strongly 2-graded on all attributes $i \in N$.

Consider the particular case of condition 2^*-graded$_i$ in which $z = w$. Suppose that $(x_i, y_{-i}) \not\succsim w$. Since $(y_i, y_{-i}) \succsim w$ and $(x_i, y_{-i}) \not\succsim w$, we know that the level x_i is worse than y_i (with respect to the alternative w). In this case, $(x_i, x_{-i}) \succsim w$ implies that $(a_i, x_{-i}) \succsim w$, for all $a_i \in X_i$. This means that, once we know that some level y_i is better than x_i, there does not exist any level in X_i that could be worse than x_i, so that if $(x_i, x_{-i}) \succsim w$ the same will be true replacing x_i by any element in X_i. This roughly implies that, for each $w \in X$, we can partition the elements of X_i into at most two categories of levels: the "satisfactory" ones and the "unsatisfactory" ones with respect to w. Condition 2^*-graded$_i$ implies these twofold partitions are not unrelated when considering distinct elements z and w in X. We have named this condition following Bouyssou and Marchant (2006).

Greco et al. (2004) state the following:

Theorem 2 (Greco et al., 2004, Th. 3, p. 284)
Let $\succsim$ be a binary relation on X. This relation has a representation in the discrete Sugeno integral model if and only if (iff) it is a weak order satisfying the order-denseness condition and being strongly 2-graded.

It is clear that if $\succsim$ has a representation in the discrete Sugeno integral model, then it must be a weak order satisfying *OD*. It is not difficult to show that it must also satisfy 2^*-graded. Indeed, suppose that condition 2^*-graded$_i$ is violated, so that, for some $x, y, z, w \in X$ and some $a_i \in X_i$, we have $x \succsim z$, $y \succsim w$, $z \succsim w$, $(a_i, x_{-i}) \not\succsim z$ and $(x_i, y_{-i}) \not\succsim w$. Using $y \succsim w$ and $(x_i, y_{-i}) \not\succsim w$, we obtain $u_i(x_i) < S_{\langle \mu, \boldsymbol{u} \rangle}(w)$. Because $z \succsim w$, we know that $S_{\langle \mu, \boldsymbol{u} \rangle}(z) \geq S_{\langle \mu, \boldsymbol{u} \rangle}(w)$, so that $S_{\langle \mu, \boldsymbol{u} \rangle}(z) > u_i(x_i)$. Since $x \succsim z$ and $S_{\langle \mu, \boldsymbol{u} \rangle}(z) > u_i(x_i)$, there is some $I \in 2^N$ such that $i \notin I$, $\mu(I) \geq S_{\langle \mu, \boldsymbol{u} \rangle}(z)$ and $u_j(x_j) \geq S_{\langle \mu, \boldsymbol{u} \rangle}(z)$, for all $j \in I$. This implies $S_{\langle \mu, \boldsymbol{u} \rangle}((a_i, x_{-i})) \geq S_{\langle \mu, \boldsymbol{u} \rangle}(z)$, so that $(a_i, x_{-i}) \succsim z$, a contradiction.

The rest of this note is mainly devoted to a proof of the converse assertion.

3 Preliminary results

3.1 Factorization of 2^*-graded$_i$

Let us first show how condition 2^*-graded$_i$ can be factorized using two conditions.

Let $\succsim$ be a binary relation on X. We say that $\succsim$ satisfies $AC1_i$ if, for all $x, y, z, w \in X$,

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i, x_{-i}) \succsim y, \\ \text{or} \\ (x_i, z_{-i}) \succsim w. \end{array}\right. \qquad (AC1_i)$$

We say that $\succsim$ satisfies $AC1$ if it satisfies $AC1_i$ for all $i \in N$. Condition $AC1$ was proposed and studied in Bouyssou and Pirlot (2004). It plays a central rôle in the characterization of binary relations (that may be incomplete or intransitive) admitting a decomposable representation of the type:

$$x \succsim y \Leftrightarrow G[u_1(x_1), \ldots, u_n(x_n), u_1(y_1), \ldots, u_n(y_n)] \geq 0,$$

with G being nondecreasing (resp. nonincreasing) in its first (resp. last) n arguments (see Bouyssou and Pirlot, 2004, Theorem 2). We refer to Bouyssou and Pirlot (2004) for a detailed interpretation of this condition. Let us simply mention here that condition $AC1_i$, independently of any transitivity or completeness properties of $\succsim$, allows to order the elements of X_i in such a way that this ordering is compatible with $\succsim$ (see Lemma 5 below).

We say that $\succsim$ is 2-graded on attribute $i \in N$ (condition 2-graded$_i$) if, for all $x, y, z, w \in X$ and all $a_i \in X_i$,

$$\left.\begin{array}{c} x \succsim z \\ \text{and} \\ (y_i, x_{-i}) \succsim z \\ \text{and} \\ y \succsim w \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a_i, x_{-i}) \succsim z \\ \text{or} \\ (x_i, y_{-i}) \succsim w. \end{array}\right. \qquad (\text{2-graded}_i)$$

We say that $\succsim$ is *2-graded* (condition 2-graded) if it is 2-graded on all attributes $i \in N$. Condition 2-graded weakens condition 2^*-graded adjoining it the additional premise $(y_i, x_{-i}) \succsim z$. It has a similar interpretation. We have:

Lemma 3
Let $\succsim$ be a weak order on X. Then $\succsim$ satisfies $AC1_i$ and 2-graded$_i$ iff it satisfies 2^*-graded$_i$.

PROOF
[$AC1_i$ & 2-graded$_i$ $\Rightarrow$ 2^*-graded$_i$]. Suppose that $x \succsim z$, $y \succsim w$ $z \succsim w$. Using $AC1_i$, $x \succsim z$ and $y \succsim w$ implies either $(y_i, x_{-i}) \succsim z$ or $(x_i, y_{-i}) \succsim w$. In the latter case, one of the two conclusions of 2^*-graded$_i$ holds. In the former case, we have $x \succsim z$, $(y_i, x_{-i}) \succsim z$, $y \succsim w$ and $z \succsim w$, so that 2-graded$_i$ implies either $(a_i, x_{-i}) \succsim z$, for all $a_i \in X_i$ or $(x_i, y_{-i}) \succsim w$, which is the desired conclusion.

[2^*-graded$_i$ $\Rightarrow$ $AC1_i$ & 2-graded$_i$]. It is clear that 2^*-graded$_i$ implies 2-graded$_i$ since 2-graded$_i$ is obtained from 2^*-graded$_i$ by adding to it an additional premise. Suppose that $x \succsim y$ and $z \succsim w$. Since $\succsim$ is complete, we have either $y \succsim w$ or $w \succsim y$. If $y \succsim w$, we have $x \succsim y$, $z \succsim w$ and $y \succsim w$, so that 2^*-graded$_i$ implies $(x_i, z_{-i}) \succsim w$ or $(a_i, x_{-i}) \succsim y$, for all $a_i \in X_i$. Taking $a_i = z_i$ shows that $AC1_i$ holds in this case. The proof is similar if it is supposed that $w \succsim y$. □

Remark 4
When $\succsim$ is a weak order, condition $AC1_i$ is equivalent to supposing that, for all $x_i, y_i \in X_i$ and all $z_{-i}, w_{-i} \in X_{-i}$ $(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, w_{-i}) \succsim (y_i, w_{-i})$, i.e., that attribute i is weakly separable, using the terminology of Bouyssou and Pirlot (2004).

Indeed suppose that $\succsim$ satisfies $AC1_i$ and is such that attribute i is not weakly separable. Therefore there are $x_i, y_i \in X_i$ and $z_{-i}, w_{-i} \in X_{-i}$ such that $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(y_i, w_{-i}) \succ (x_i, w_{-i})$. Since $\succsim$ is reflexive, we have $(x_i, z_{-i}) \succsim (x_i, z_{-i})$ and $(y_i, w_{-i}) \succsim (y_i, w_{-i})$. Using $AC1_i$, we have either $y_i \succsim_i x_i$ or $x_i \succsim_i y_i$, so that either $(y_i, z_{-i}) \succsim (x_i, z_{-i})$ or $(x_i, w_{-i}) \succsim (y_i, w_{-i})$, a contradiction.

Conversely, suppose that $\succsim$ is complete and transitive and that attribute i is weakly separable. Suppose that $AC1_i$ is violated so that, since $\succsim$ is complete, $(x_i, x_{-i}) \succsim y$, $(z_i, z_{-i}) \succsim w$, $y \succ (z_i, x_{-i})$ and $w \succ (x_i, z_{-i})$, for some $x, y, z, w \in X$. Since $\succsim$ is a weak order, we obtain $(x_i, x_{-i}) \succ (z_i, x_{-i})$ and $(z_i, z_{-i}) \succ (x_i, z_{-i})$, which violates the weak separability of attribute i.

We say that a weak order $\succsim$ is *weakly separable* if, for all $i \in N$, it is weakly separable for attribute i.

Hence, combining Lemma 3 with Theorem 2 shows that a relation has a representation in the discrete Sugeno integral model iff it is a weakly separable weak order satisfying *OD* and 2-graded.

Bouyssou and Pirlot (2004, Propositions 8 and B.3) have shown that, for weak orders satisfying *OD*, weak separability is a necessary and sufficient condition to obtain a general decomposable representation in which, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow F[u_1(x_1), \ldots, u_n(x_n)] \geq F[u_1(y_1), \ldots, u_n(y_n)],$$

with F being nondecreasing in all its arguments (see also Greco et al., 2004, Theorem 1). Hence, condition 2-graded is exactly what must be added to go from this general decomposable representation to a representation in the discrete Sugeno integral model. •

3.2 Traces

Consider an attribute $i \in N$. We define the *left marginal trace* on attribute $i \in N$ letting, for all $x_i, y_i \in X_i$, all $a_{-i} \in X_{-i}$ and all $z \in X$,

$$x_i \succsim_i y_i \Leftrightarrow [(y_i, a_{-i}) \succsim z \Rightarrow (x_i, a_{-i}) \succsim z].$$

Similarly, given $a \in X$, we define the left marginal trace on attribute $i \in N$ with respect to $a \in X$, letting, for all $x_i, y_i \in X_i$ and all $z_{-i} \in X_{-i}$,

$$x_i \succsim_i^a y_i \Leftrightarrow [(y_i, z_{-i}) \succsim a \Rightarrow (x_i, z_{-i}) \succsim a].$$

The symmetric and asymmetric parts of $\succsim_i$ (resp. $\succsim_i^a$) are denoted $\sim_i$ and $\succ_i$ (resp. $\sim_i^a$ and $\succ_i^a$). It is clear that $\succsim_i$ and $\succsim_i^a$ are always reflexive and transitive. They may be incomplete however.

We note a few useful obvious connections between $\succsim_i^a$, $\succsim_i$ and $\succsim$ in the following lemma.

Lemma 5
We have, for all $i \in N$, all $z, w \in X$ and all $x_i, y_i \in X_i$,

1. $x_i \succsim_i y_i \Leftrightarrow [x_i \succsim_i^a y_i,$ *for all* $a \in X]$,

2. $[z \succsim w, x_i \succsim_i z_i] \Rightarrow (x_i, z_{-i}) \succsim w$.

3. *Furthermore, if* $\succsim$ *is reflexive then,* $[z_j \sim_j w_j,$ *for all* $j \in N] \Rightarrow z \sim w$.

4. *The relation* $\succsim_i$ *is complete iff* $AC1_i$ *holds.*

Proof
Parts 1 and 2 easily follow from the definitions. Part 3 follows from Part 2 and the fact that $w \succsim w$. It is obvious that negating the completeness of $\succsim_i$ is equivalent to negating $AC1_i$. □

The following lemma makes precise the structure of the relations $\succsim_i^a$ when $\succsim$ is a weak order satisfying $AC1_i$ and 2-graded$_i$.

Lemma 6
Let $\succsim$ *be a weak order on* X *satisfying* $AC1_i$ *and* 2-graded$_i$. *Then*

1. $\succsim_i^a$ *is complete for all* $a \in X$,

2. $x_i \succ_i^a y_i \Rightarrow [x_i \succsim_i^b y_i$ *for all* $b \in X]$,

3. $\succsim_i^a$ *has at most two distinct equivalence classes, for all* $a \in X$,

4. $[x_i \sim_i^a z_i$ *and* $x_i \succ_i^a y_i] \Rightarrow x_i \sim_i^b z_i$, *for all* $b \in X$ *such that* $a \succsim b$.

5. *If* $a \succsim b$ *and both* $\succsim_i^a$ *and* $\succsim_i^b$ *are nontrivial then the first equivalence class of* $\succsim_i^a$ *is included in the first equivalence class of* $\succsim_i^b$.

Proof
Parts 1 and 2 follow from Lemma 5 since $AC1_i$ implies that $\succsim_i$ is complete.

Part 3. Suppose that $\succsim_i^a$ has at least three distinct equivalence classes. This implies that $(x_i, c_{-i}) \succsim a$, $(y_i, c_{-i}) \not\succsim a$, $(y_i, d_{-i}) \succsim a$ and $(z_i, d_{-i}) \not\succsim a$, for some $x_i, y_i, z_i \in X_i$, some $c_{-i}, d_{-i} \in X_{-i}$ and some $a \in X$. Using $AC1_i$, $(x_i, c_{-i}) \succsim a$, $(y_i, d_{-i}) \succsim a$ and $(y_i, c_{-i}) \not\succsim a$ imply $(x_i, d_{-i}) \succsim a$. Using 2-graded$_i$, $(y_i, d_{-i}) \succsim a$, $(x_i, d_{-i}) \succsim a$, $(x_i, c_{-i}) \succsim a$ and $a \succsim a$ imply $(y_i, c_{-i}) \succsim a$ or $(z_i, d_{-i}) \succsim a$, a contradiction.

Part 4. Suppose that $x_i \sim_i^a z_i$, $x_i \succ_i^a y_i$, $a \succsim b$ and $x_i \succ_i^b z_i$ (the proof for the case $z_i \succ_i^b x_i$ being similar). By construction, we have $(x_i, w_{-i}) \succsim b$, $(z_i, w_{-i}) \not\succsim b$, $(x_i, t_{-i}) \succsim a$ and $(y_i, t_{-i}) \not\succsim a$. Since $x_i \sim_i^a z_i$, we must have $(z_i, t_{-i}) \succsim a$. Using $AC1_i$, $(x_i, w_{-i}) \succsim b$, $(z_i, t_{-i}) \succsim a$ and $(z_i, w_{-i}) \not\succsim b$ imply

$(x_i, t_{-i}) \succsim a$. Using 2-graded$_i$, $(z_i, t_{-i}) \succsim a$, $(x_i, t_{-i}) \succsim a$, $(x_i, w_{-i}) \succsim b$ and $a \succsim b$ imply $(z_i, w_{-i}) \succsim b$ or $(y_i, t_{-i}) \succsim a$, a contradiction.

Part 5. Suppose that $a \succsim b$, $x_i \succ_i^a y_i$ and $z_i \succ_i^b x_i$. Using Part 2, we know that $z_i \succsim_i^a x_i$. Because we know from Part 3 that $\succsim_i^a$ has at most two equivalence classes, we must have $z_i \sim_i^a x_i$. Using Part 4, $a \succsim b$, $z_i \sim_i^a x_i$ and $x_i \succ_i^a y_i$ imply $z_i \sim_i^b x_i$, a contradiction. □

Let $\succsim$ be a weak order on X satisfying $AC1_i$ and 2-graded$_i$. Let $i \in N$. For all $a \in X$, we know that either $\succsim_i^a$ is trivial or $\succsim_i^a$ has two distinct equivalence classes. Define $B_i^a \subset X_i$ as the empty set in the first case and as the elements in the first equivalence class in the second case. Define C_i^a letting:

$$C_i^a = \bigcup_{\{x \in X : x \succsim a\}} B_i^x.$$

The following lemma studies the properties of the sets C_i^a.

Lemma 7
Let $\succsim$ be a weak order on X satisfying $AC1$ and 2-graded. For all $x, y, z, w \in X$ and all $i \in N$,

1. $z \succsim w \Rightarrow C_i^z \subseteq C_i^w$,
2. $\{j \in N : y_j \in C_j^z\} \subseteq \{j \in N : x_j \in C_j^z\} \Rightarrow [x_i \succsim_i^z y_i$ *for all* $i \in N]$,
3. $C_i^x \subsetneq X_i$.

Proof
Part 1. We have $x_i \in C_i^z$ iff $x_i \in B_i^a$, for some $a \succsim z$. Because $z \succsim w$ and $\succsim$ is a weak order, we have $a \succsim z$. Hence, $x_i \in B_i^a$, for some $a \succsim w$, so that $x_i \in C_i^w$.

Part 2. If $\succsim_i^z$ is trivial, we have by definition $x_i \sim_i^z y_i$. If $\succsim_i^z$ is not trivial, it follows from Part 5 of Lemma 6 that C_i^z is equal to the first equivalence class of $\succsim_i^z$. If $y_i \in C_i^z$, we have $x_i \in C_i^z$, so that $x_i \sim_i^z y_i$. If $y_i \notin C_i^z$, then we have $z_i \succsim_i^z y_i$, for all $z_i \in X_i$.

Part 3. By construction, B_i^y is strictly included in X_i. As the set C_i^x is obtained by taking the union of sets B_i^y, the conclusion follows. □

Lemma 8
Let $\succsim$ be a weak order on X satisfying $AC1_i$ and 2-graded$_i$. Define, for all $x \in X$, the set $G^x \subseteq 2^N$ letting $I \in G^x$ whenever we have $\{i \in N : z_i \in C_i^x\} \subseteq I$, for some $z \in X$ such that $z \succsim x$. We have, for all $x, y \in X$,

1. $x \succsim y \Leftrightarrow \{i \in N : x_i \in C_i^y\} \in G^y$,

2. $[I \in G^x$ *and* $I \subseteq J] \Rightarrow J \in G^x$,

3. $x \succsim y \Rightarrow G^x \subseteq G^y$.

Proof

Part 1. By construction, if $x \succsim y$ then $\{i \in N : x_i \in C_i^y\} \in G^y$. Let us show that the reverse implication is true. Suppose that $\{i \in N : x_i \in C_i^y\} \in G^y$. This implies that $\{i \in N : z_i \in C_i^y\} \subseteq \{i \in N : x_i \in C_i^y\}$, for some $z \in X$ such that $z \succsim y$. Using Part 2 of Lemma 7, $\{i \in N : z_i \in C_i^y\} \subseteq \{i \in N : x_i \in C_i^y\}$ implies $x_i \succsim_i^y z_i$, for all $i \in N$. Hence, $z \succsim y$ implies $x \succsim y$.

Part 2 follows from the definition of the sets G^x.

Part 3. Suppose that $x \succsim y$ and let $I \in G^x$. Let us show that we must have $I \in G^y$. By construction, $I \in G^x$ implies that $\{i \in N : z_i \in C_i^x\} \subseteq I$, for some $z \in X$ such that $z \succsim x$. Consider the alternative $w \in X$ defined in the following way.

- If $z_i \in C_i^x$, let $w_i = z_i$. We have $w_i \in C_i^x$. Using Part 1 of Lemma 7, we know that this implies $w_i \in C_i^y$.
- If $z_i \notin C_i^x$. Using Part 3 of Lemma 7, we know that $C_i^y \subsetneq X_i$. We take w_i to be any element in $X_i \setminus C_i^y$. Because, we know that $C_i^x \subseteq C_i^y$, we have $w_i \notin C_i^x$.

By construction we have, for all $i \in N$, $z_i \in C_i^x \Leftrightarrow w_i \in C_i^x \Leftrightarrow w_i \in C_i^y$. Hence, we have $\{i \in N : z_i \in C_i^x\} = \{i \in N : w_i \in C_i^x\} = \{i \in N : w_i \in C_i^y\}$. The first equality implies $w \succsim x$. Using the fact that $\succsim$ is a weak order, we obtain $w \succsim y$. Hence, we have $\{i \in N : w_i \in C_i^y\} \subseteq I$ and $w \succsim y$. This implies $I \in G^y$. □

3.3 The noncompensatory model for weak orders

The following model is used as an intermediary step in the construction of the discrete Sugeno integral model. It may be viewed as a kind of "non-numerical version" of the discrete Sugeno integral model.

Definition 9

A weak order $\succsim$ on X has a representation in the noncompensatory model *if for all $x \in X$, there are sets*

1. $A_i^x \subseteq X_i$, *for all* $i \in N$,

2. $F^x \subseteq 2^N$ *such that*

$$[I \in F^x \text{ and } I \subseteq J \in 2^N] \Rightarrow J \in F^x, \quad (1)$$

that are such that, for all $x, y \in X$,

$$x \succsim y \Rightarrow \begin{cases} A_i^x \subseteq A_i^y \\ \text{and} \\ F^x \subseteq F^y \end{cases} \tag{2}$$

and

$$x \succsim y \Leftrightarrow \{i \in N : x_i \in A_i^y\} \in F^y. \tag{3}$$

We often write $A(x, y)$ instead of $\{i \in N : x_i \in A_i^y\}$.

The noncompensatory model for weak orders [2] is inspired from the work of Bouyssou and Marchant (2006) in the area of sorting model in MCDM. The results in Bouyssou and Marchant (2006) may be viewed as dealing with the noncompensatory model for weak orders that have a finite number of equivalent classes (this is in Bouyssou and Marchant (2006) phrased in the language of "ordered categories").

The noncompensatory model can be interpreted as follows. For each $x \in X$ we isolate on each attribute a subset $A_i^x \subseteq X_i$ containing the levels on attribute i that are satisfactory for x. In order for an alternative to be at least as good as x, it must have evaluations that are satisfactory for x on a subset of attributes belonging to F^x. The subsets of attributes belonging to F^x are interpreted as subsets that are "sufficiently important" to warrant preference on x.

With this interpretation in mind, the constraint (2) means that if x is at least as good as y then every level that is satisfactory for x must be satisfactory for y. Furthermore, subsets of attributes that are "sufficiently important" to warrant preference on x must also be "sufficiently important" to warrant preference on y. Given the above interpretation of F^x, the constraint (1) simply says that any superset of a set that is "sufficiently important" to warrant preference on x must have the same property.

Suppose that $x \not\succsim y$ and that $x_i \in A_i^y$, for some $i \in N$. In the noncompensatory model, we have $(z_i, x_{-i}) \not\succsim y$, for all $z_i \in X_i$. It is therefore impossible, starting from x, to obtain an alternative that would be at least as good as y by modifying the evaluation of x on the ith attribute. In other terms, the fact that $A(x, y) \notin F^y$ cannot be compensated by improving the evaluation of x on an attribute in $A(x, y)$. Hence, our name for this model.

[2] The noncompensatory model for weak orders must not be confused with "noncompensatory preferences" as introduced in Fishburn (1976). Noncompensatory preferences in the sense of Fishburn (1976) are preferences that result form an "ordinal aggregation" in the context of MCDM that is quite close from the type of aggregation studied in social choice theory in the vein of Arrow (1963). For a recent analysis of such preferences, see Bouyssou and Pirlot (2005).

A weak order having a representation in the noncompensatory model must satisfy $AC1$ and 2-graded. We have:

Lemma 10
If weak order $\succsim$ on X has a representation in the noncompensatory model, then it satisfies $AC1$ and 2-graded.

Proof
$[AC1_i]$. Suppose that $x \succsim y$, $z \succsim w$, $(z_i, x_{-i}) \not\succsim y$ and $(x_i, z_{-i}) \not\succsim w$. It is easy to see that $x \succsim y$ and $(z_i, x_{-i}) \not\succsim y$ imply $x_i \in A_i^y$ and $z_i \notin A_i^y$. Similarly, $z \succsim w$ and $(x_i, z_{-i}) \not\succsim w$ imply $z_i \in A_i^w$ and $x_i \notin A_i^w$. Because $\succsim$ is complete, we have either $y \succsim w$ or $w \succsim y$. Hence, we have either $A_i^y \subseteq A_i^w$ or $A_i^w \subseteq A_i^y$, a contradiction.

[2-graded$_i$]. Suppose that 2-graded$_i$ is violated, so that, for some $x, y, z, w \in X$ and some $a_i \in X_i$, $(x_i, x_{-i}) \succsim z$, $(y_i, x_{-i}) \succsim z$, $(y_i, y_{-i}) \succsim w$, $z \succsim w$, $(a_i, x_{-i}) \not\succsim z$ and $(x_i, y_{-i}) \not\succsim w$. Using the definition of the noncompensatory model, $(y_i, y_{-i}) \succsim w$ and $(x_i, y_{-i}) \not\succsim w$ imply $y_i \in A_i^w$ and $x_i \notin A_i^w$. Similarly, $(x_i, x_{-i}) \succsim z$ and $(a_i, x_{-i}) \not\succsim z$ imply $x_i \in A_i^z$ and $a_i \notin A_i^z$. Since $z \succsim w$, we have $A_i^z \subseteq A_i^w$, a contradiction. □

The main result of this section says that, for weak orders, the noncompensatory model is fully characterized by the conjunction of $AC1$ and 2-graded. Notice that we may equivalently replace the conjunction of $AC1$ and 2-graded either by condition 2^*-graded or by the conjunction of weak separability and 2-graded.

Proposition 11
If a weak order on X satisfies $AC1$ and 2-graded then it has a representation in the noncompensatory model.

Proof
Define $A_i^x = C_i^x$ and $F^x = G^x$. The proof follows from combining Lemmas 7 and 8. □

4 Completion of the proof

The main result in this section says that if a weak order has a representation in the noncompensatory model and has a numerical representation, then it has a representation in the discrete Sugeno integral model.

Proposition 12
Let $\succsim$ be a weak order on X. Suppose that $\succsim$ can be represented in the noncompensatory model and that there is a real function v on X such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow v(x) \geq v(y). \tag{4}$$

Then $\succsim$ has a representation in the discrete Sugeno integral model.

Proof

Let $\succsim$ be a weak order representable in the noncompensatory model and such that there is a real-valued function v satisfying (4). We may assume w.l.o.g. that, for all $x \in X$, $v(x) \in [0, 1]$. Furthermore, if there are minimal elements in X for $\succsim$, we may assume w.l.o.g. that v gives the value 0 to these elements. We consider now any such function v.

For all $i \in N$, define u_i letting, for all $x_i \in X_i$,

$$u_i(x_i) = \begin{cases} \sup\limits_{\{w \in X : x_i \in A_i^w\}} v(w) & \text{if } \exists w : x_i \in A_i^w, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Define μ on 2^N letting, for all $I \in 2^N$,

$$\mu(I) = \begin{cases} \sup\limits_{\{w \in X : I \in F^w\}} v(w) & \text{if } \exists w : I \in F^w, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Since $I \in F^w$ and $J \supseteq I$ entails $J \in F^w$, we have that $\mu(J) \geq \mu(I)$. Hence, μ is a nondecreasing set function.

Let us show that $\mu(\varnothing) = 0$. If there is no $w \in X$ such that $\varnothing \in F^w$, then we have, by construction, $\mu(\varnothing) = 0$. Suppose that $X_\varnothing = \{w \in X : \varnothing \in F^w\} \neq \varnothing$. From the definition of the noncompensatory model, it follows that, for all $x \in X$ and all $w \in X_\varnothing$, we have $x \succsim w$. Hence, for all $w \in X_\varnothing$, w is minimal for $\succsim$. We therefore have $v(w) = 0$, for all $w \in X_\varnothing$ and, hence, $\mu(\varnothing) = 0$. This shows that μ defined by (6) is a capacity on 2^N. It is not necessarily normalized, i.e., we may not have that $\mu(N) = 1$.

Independently of the normalization of μ, we can compute, for all $x \in X$, $S_{\mu,\mathbf{u}}(x)$ letting:

$$S_{\langle \mu, \boldsymbol{u} \rangle}(x) = \bigvee_{I \subseteq N} \left[\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right) \right]. \tag{7}$$

It is clear that, for all $y \in X$, $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \in [0, 1]$. Let us show that, for all $y \in X$, $S_{\langle \mu, \boldsymbol{u} \rangle}(y) = v(y)$, which will complete the proof if μ happens to be normalized.

Let $x, y \in X$ be such that $x \succsim y$. This implies $A(x, y) = \{i \in N : x_i \in A_i^y\} \in F^y$. Hence, for all $i \in A(x, y)$, $y \in \{w \in X : x_i \in A_i^w\}$, so that $u_i(x_i) \geq v(y)$. Similarly, $y \in \{w \in X : A(x, y) \in F^w\}$, so that

$\mu(A(x,y)) \geq v(y)$. Hence, for $I = A(x,y)$, we have

$$\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right) \geq v(y).$$

In view of (7), this implies $S_{\langle \mu, \boldsymbol{u} \rangle}(x) \geq v(y)$. Since $\succsim$ is reflexive, this shows that, for all $y \in X$, $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \geq v(y)$.

We now prove that, for all $y \in X$, $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \leq v(y)$. If y is maximal for $\succsim$ (i.e., $y \succsim x$, for all $x \in X$), we have $v(y) \geq v(x)$, for all $x \in X$. The definition of u_i and μ obviously implies that they cannot exceed the maximal value of v on X. Hence, in this case, we have $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \leq v(y)$.

Suppose henceforth that $y \in X$ is not maximal for $\succsim$, so that $x \succ y$, for some $x \in X$. This implies that $A(y,x) = \{i \in N : y_i \in A_i^x\} \notin F^x$. Define $A_y = \bigcup_{z \succ y} A(y,z)$. Because $A(y,z) \subseteq N$, N is a finite set, and $z' \succsim z$ implies $A(y,z') \subseteq A(y,z)$, there is an element $z_0 \in X$ with $z_0 \succ y$ that is such that $A(y,z_0) = A_y$ and $A(y,z) = A_y$, for all $z \in X$ such that $z_0 \succsim z \succ y$.

We claim the following:

Claim 1: for all $j \notin A_y$, $u_j(y_j) \leq v(y)$,

Claim 2: for all $I \subseteq A_y$, $\mu(I) \leq v(y)$.

Proof of Claim 1. Let $j \notin A_y$, so that $y_j \notin A_j^{z_0}$. If the set $\{w \in X : y_j \in A_j^w\}$ is empty, we have $u_j(y_j) = 0$ and the claim trivially holds. Otherwise, let $w \in X$ such that $y_j \in A_j^w$. If $w \succ z_0$, we have $A_j^w \subseteq A_j^{z_0}$, so that $y_j \in A_j^w$ implies $y_j \in A_j^{z_0}$, a contradiction. If $z_0 \succsim w \succ y$, we know that $A(y,w) = A(y,z_0)$. This is contradictory since $y_j \in A_j^w$ and $y_j \notin A_j^{z_0}$. Hence, when $j \notin A_y$, we must have $y \succsim w$, for all $w \in X$ such that $y_j \in A_j^w$. This implies that $u_j(y_j) = \sup_{\{w \in X : y_j \in A_j^w\}} v(w) \leq v(y)$, for all $j \notin A_y$.

Proof of Claim 2. Let $I \subseteq A_y$. If the set $\{w \in X : I \in F^w\}$ is empty, we have $\mu(I) = 0$ and the claim follows. Otherwise, let $w \in X$ such that $I \in F^w$. Suppose that $w \succ z_0$. This implies $F^w \subseteq F^{z_0}$, so that $I \in F^{z_0}$. Because $I \subseteq A_y$, we obtain $A_y \in F^{z_0}$. This is contradictory since $z_0 \succ y$ implies that $A_y = A(y,z_0) \notin F^{z_0}$. Suppose now that $z_0 \succsim w \succ y$. We have $A(y,w) = A_y \notin F^w$. But, since $I \in F^w$ and $I \subseteq A_y$, we obtain $A_y \in F^w$, a contradiction. Hence, for all $w \in X$ such that $I \in F^w$, we have $y \succsim w$. This implies $\mu(I) = \sup_{\{w \in X : I \in F^w\}} v(w) \leq v(y)$.

Using Claims 1 and 2, we establish that $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \leq v(y)$ for any $y \in X$ that is not maximal. Let $I \subseteq N$. We distinguish two cases in order to compute

$$\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right).$$

1. If I is not included in A_y, we know that there is $j \in I$ such that $j \notin A_y$. Hence, using Claim 1, $u_j(y_j) \leq v(y)$ so that $\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(y_i)\right) \leq v(y)$.

2. If I is included in A_y, using Claim 2, we have $\mu(I) \leq v(y)$. Hence, we know that $\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(y_i)\right) \leq v(y)$.

Hence, for all $I \subseteq N$, we have $\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(y_i)\right) \leq v(y)$, so that $S_{\langle \mu, \boldsymbol{u} \rangle}(y) \leq v(y)$. This proves that, for all $y \in X$, $S_{\langle \mu, \boldsymbol{u} \rangle}(y) = v(y)$.

It remains to show that we may always build a representation in the discrete Sugeno integral model using a *normalized* capacity, i.e., a capacity ν such that $\nu(N) = 1$.

Using the above construction, the value of $\mu(N)$ is obtained using (6). We have $\mu(N) = \sup_{w \in X} v(w)$, since for all $w \in X$, $N \in F^w$. If the weak order $\succsim$ is not trivial, we have $\mu(N) > 0$. In order to obtain a representation leading to a normalized capacity, it suffices to apply the above construction to the function u obtained by dividing v by $\mu(N)$. If the weak order $\succsim$ is trivial, it is easy to see that it has a representation in the noncompensatory model such that, for all $x \in X$ and all $i \in N$, $A_i^x = X_i$ and $F^x = \{N\}$. Defining, for all $i \in N$ and all $x_i \in X_i$, $u_i(x_i) = 1$, $\mu(N) = 1$ and $\mu(A) = 0$, for all $A \subsetneq N$, leads to a representation of this trivial weak order in the discrete Sugeno integral model. □

The sufficiency proof of Theorem 2 follows from combining Lemma 3 with Propositions 11 and 12. This amounts to characterizing the discrete Sugeno integral model by the conjunction of any of the following three equivalent sets of conditions:

- completeness, transitivity, *OD*, *AC*1 and 2-graded,
- completeness, transitivity, *OD*, weak separability and 2-graded,
- completeness, transitivity, *OD* and 2*-graded.

The examples in the following section show no condition in the first set is redundant.

Remark 13
Consider a nontrivial weak order $\succsim$ on X that satisfies the hypotheses of Proposition 12. The proof of this proposition establishes that *any* function $v : X \to [0,1]$ satisfying (4) and giving a value 0 to the minimal elements in X for $\succsim$ (if any) can be used to define a representation in the Sugeno integral model. The functions u_i and the (non necessarily normalized) capacity μ used in this representation can be defined on the basis of v using (5) and (6). Furthermore, as shown in this proof, (5) and (6) can be viewed as *inversion*

formulas for the discrete Sugeno integral model in the following sense. If we know the value of $S_{\langle \mu, \boldsymbol{u} \rangle}(x)$, for all $x \in X$, without knowing the functions μ and u_i, it is possible to use (5) and (6) to build functions u_j and a capacity μ that allow to reconstruct all these values using the discrete Sugeno integral formula (7). •

5 Independence of conditions

Proposition 14
Let $\succsim$ be a binary relation on X. The following conditions are independent:

1. *$\succsim$ is complete,*
2. *$\succsim$ is transitive,*
3. *$\succsim$ satisfies $AC1$,*
4. *$\succsim$ is 2-graded.*

Proof
We provide the required four examples.

Example 15
Let $X = \{x_1, y_1\} \times \{x_2, y_2\}$. Let $\succsim$ be identical to the weak order

$$(y_1, y_2) \succ [(x_1, y_2), (y_1, x_2)] \succ (x_1, x_2),$$ ◇

except that we have removed two arcs from $\succsim$, so as to have $(x_1, y_2) \not\succsim (y_1, x_2)$ and $(y_1, x_2) \not\succsim (x_1, y_2)$. It is clear that $\succsim$ is transitive but is not complete. Since X_1 and X_2 have only two elements, condition 2-graded trivially holds. It is not difficult to check that we have $y_1 \succ_1 x_1$ and $y_2 \succ_2 x_2$, so that $AC1$ holds.

Example 16
Let $X = \{x_1, y_1\} \times \{x_2, y_2\}$. Let $\succsim$ be identical to the trivial weak order except that we have removed one arc from $\succsim$, so as to have $(x_1, x_2) \not\succsim (y_1, y_2)$. It is not difficult to see that the resulting relation is complete but not transitive (it is a semi-order). Since X_1 and X_2 have only two elements, condition 2-graded trivially holds. It is not difficult to check that we have $y_1 \succ_1 x_1$ and $y_2 \succ_2 x_2$, so that $AC1$ holds.

Example 17
$X = \{x_1, y_1, z_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$. Let $\succsim$ be the weak order such that:

$$[(x_1, x_2, x_3), (y_1, x_2, x_3)]$$
$$\succ$$
$$[(x_1, x_2, y_3), (x_1, y_2, x_3), (y_1, x_2, y_3), (y_1, y_2, x_3),$$
$$(y_1, y_2, y_3), (z_1, x_2, x_3), (z_1, x_2, y_3), (z_1, y_2, x_3)]$$
$$\succ$$
$$[(z_1, y_2, y_3), (x_1, y_2, y_3)].$$

We have $y_1 \succ_1 x_1 \succ_1 z_1$, $x_2 \succ_2 y_2$ and $x_3 \succ_3 y_3$, which shows that $AC1$ holds. Conditions 2-graded$_2$ and 2-graded$_3$ are trivially satisfied. Condition 2-graded$_1$ is violated since $(x_1, x_2, x_3) \succsim (y_1, x_2, x_3)$, $(y_1, x_2, x_3) \succsim (y_1, x_2, x_3)$, $(y_1, y_2, y_3) \succsim (x_1, x_2, y_3)$ and $(y_1, x_2, x_3) \succsim (x_1, x_2, y_3)$ but $(z_1, x_2, x_3) \not\succsim (y_1, x_2, x_3)$ and $(x_1, y_2, y_3) \not\succsim (x_1, x_2, y_3)$. $\diamond$

Example 18
Let $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$. Let $\succsim$ be the weak order such that:

$$[(x_1, x_2, x_3), (x_1, y_2, x_3), (y_1, y_2, x_3)]$$
$$\succ$$
$$[(y_1, y_2, y_3), (y_1, x_2, x_3)]$$
$$\succ$$
$$[(x_1, x_2, y_3), (x_1, y_2, y_3), (y_1, x_2, y_3)].$$

Condition 2-graded trivially holds. We have $y_2 \succ_2 x_2$ and $x_3 \succ_3 y_3$, so that conditions $AC1_2$ and $AC1_3$ hold. Since $(x_1, x_2, x_3) \succsim (y_1, y_2, x_3)$ and $(y_1, y_2, y_3) \succsim (y_1, x_2, x_3)$ but $(y_1, x_2, x_3) \not\succsim (y_1, y_2, x_3)$ and $(x_1, y_2, y_3) \not\succsim (y_1, x_2, x_3)$, condition $AC1_1$ is violated. $\diamond$

$\square$

Remark 19
It is easy to check that the weak order in Example 18 satisfies the following condition

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim y \end{array}\right\} \Rightarrow \left\{\begin{array}{l} (z_i, x_{-i}) \succsim y, \\ \text{or} \\ (x_i, z_{-i}) \succsim y, \end{array}\right.$$

for all $x, y, z \in X$. This condition is a weakening of $AC1_i$ obtained by requiring that $y = w$ in the expression of $AC1_i$ (it is equivalent to requiring that all relations $\succsim_i^a$ are complete). It is therefore not possible to weaken $AC1_i$ in this way.

Similarly, it is easy to check that the weak order in Example 17 satisfies the weakening of 2-graded$_i$ obtained by requiring that $z = w$ in the expression of 2-graded$_i$ (and, hence, removing the last redundant premise), i.e., for all $x, y, z \in X$ and all $a_i \in X_i$,

$$\left.\begin{array}{c} x \succsim z \\ \text{and} \\ (y_i, x_{-i}) \succsim z \\ \text{and} \\ y \succsim z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a_i, x_{-i}) \succsim z \\ \text{or} \\ (x_i, y_{-i}) \succsim z, \end{array}\right.$$

Hence, condition 2-graded$_i$ cannot be weakened in this way. •

Finally, as shown by the following example, there are weak orders satisfying $AC1$ and 2-graded but violating OD.

Example 20
Let $X = 2^{\mathbb{R}} \times \{0, 1\}$. We consider the weak order on X such that $(x_1, x_2) \succsim (y_1, y_2)$ if $[x_2 = 1]$ or $[x_2 = 0,\ y_2 = 0$ and $x_1 \geq^* y_1]$, where $\geq^*$ is any linear order on $2^{\mathbb{R}}$. It is easy to see that $\succsim$ is a weak order. It violates OD since the restriction of $\succsim$ to $2^{\mathbb{R}} \times \{0\}$ is isomorphic to $\geq^*$ on $2^{\mathbb{R}}$ and $\geq^*$ violates OD. The relation $\succsim$ has a representation in the noncompensatory model. Indeed, for all $x = (x_1, 1)$, take $A_1^x = \varnothing$, $A_2^x = \{1\}$ and $F^x = \{\{2\}, \{1, 2\}\}$. For all $x = (x_1, 0)$, take $A_1^x = \{y_1 \in 2^{\mathbb{R}} : y_1 \geq^* x_1\}$, $A_2^x = \{1\}$ and $F^x = \{\{1\}, \{2\}, \{1, 2\}\}$. It is easy to check that this defines a representation of the weak order $\succsim$ in the noncompensatory model. Using Lemma 10, this implies that $\succsim$ satisfies $AC1$ and 2-graded. ◇

6 Discussion

This note has proposed a proof of Greco et al. (2004, Theorem 3), in the hope that this will contribute to popularize this useful result. By the same token, we have analyzed the relations between the discrete Sugeno integral model and the noncompensatory model as well as proposed a factorization of the main condition used in Greco et al. (2004, Theorem 3). Many questions are nevertheless left open. Let us briefly mention here what seem to us the most important ones.

The result in Greco et al. (2004) is a first step in the systematic study of models using fuzzy integrals in MCDM. A first and major open problem is to derive a similar result for the discrete Choquet integral. This appears very difficult and we have no satisfactory answer at this time. A second open problem is to use the above result as a building block to study particular cases of the discrete Sugeno integral. This was started in Greco et al. (2004) who showed how to characterize ordered weighted minimum and maximum. There are nevertheless many other particular cases of the discrete Sugeno integral that would be worth investigating. A third problem is to investigate assessment protocols of the various parameters of the discrete Sugeno integral model using the above result and conditions. This will clearly require to investigate the uniqueness properties of a representation in the discrete Sugeno integral model. This will allow to understand better the type of commensurateness that is implied by the noncompensatory model for weak orders and the discrete Sugeno integral model[3]. Finally, it should be mentioned that we have mainly used here the noncompensatory model for weak orders as a tool for obtaining a proof of the result of Greco et al. (2004). The noncompensatory model that we introduced can be extended in many possible directions. This will be the subject of a subsequent paper.

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[3] The noncompensatory model for weak orders allows, indirectly, to compare in terms of preference levels on distinct attributes. Indeed, the level $x_i \in X_i$ can be considered as "better" than the level $x_j \in X_j$ if we have $x_i \in A_i^z$ and $x_j \notin A_j^z$, for some $z \in X$. In other terms, x_i is better than x_j if, for some $z \in X$, some $y_i \in X_i$, some $y_j \in X_j$, some $a_{-i} \in X_{-i}$ and some $b_{-j} \in X_{-j}$, we have $(x_i, a_{-i}) \succsim z$, $(y_i, a_{-i}) \not\succsim z$, $(y_j, b_{-j}) \succsim z$ and $(x_j, b_{-j}) \not\succsim z$. The constraints of the noncompensatory model ensure that this will never lead to contradictory information of the type "x_i is better than x_j" and "x_j is better than x_i". They also imply that this relation is transitive.

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