A choice procedure $\mathcal{S}$ satisfies β^+ if $[A \subseteq B$ and $A \cap \mathcal{S}(B,\pi) \neq \varnothing] \Rightarrow \mathcal{S}(A,\pi) \subseteq \mathcal{S}(B,\pi)$, for all $\pi \in \mathcal{G}(X)$ and all $A, B \in \mathcal{P}(X)$.

Proposition 1 ($\boldsymbol{\beta^+}$ and weak monotonicity)
If $\mathcal{S}$ is monotonic and satisfies β^+ then $\succsim_{\mathcal{S}}$ is weakly monotonic.

Proof
We use the following:

Claim (Perny (1995))
If $\mathcal{S}$ is monotonic and satisfies β^+ then, for all $a, b \in X$ and all $\pi \in \mathcal{G}(X)$, $a \succ_{\mathcal{S}} (\pi) b \Rightarrow a \succsim_{\mathcal{S}} (\pi^{a\uparrow}) b$.

Proof
In violation of the claim suppose that $a \succ_{\mathcal{S}} (\pi) b$ and $b \succ_{\mathcal{S}} (\pi^{a\uparrow}) a$. Since $b \succ_{\mathcal{S}} (\pi^{a\uparrow}) a$, there is $A \subseteq X$ such that $a, b \in A$, $b \in \mathcal{S}(A, \pi^{a\uparrow})$ and $a \notin \mathcal{S}(A, \pi^{a\uparrow})$. Using β^+, we must have $\{b\} = \mathcal{S}(\{a,b\}, \pi^{a\uparrow})$. Since $\mathcal{S}$ is monotonic, this implies $\{b\} = \mathcal{S}(\{a,b\}, \pi)$. Using β^+, this implies that for all $B \subseteq X$, $a, b \in B \Rightarrow b \in \mathcal{S}(B, \pi^{x\uparrow})$. Therefore, it cannot be true that $a \succ_{\mathcal{S}} (\pi) b$. $\square$

In view of the above claim, it remains to prove that $a \sim_{\mathcal{S}} (\pi) b \Rightarrow a \succsim_{\mathcal{S}} (\pi^{a\uparrow}) b$. Let $R(a,\pi) = \{x \in X : a \succsim_{\mathcal{S}} (\pi) x\}$, $P(a,\pi) = \{x \in X : a \succ_{\mathcal{S}} (\pi) x\}$, and $I(a,\pi) = \{x \in X : a \sim_{\mathcal{S}} (\pi) x\}$.

Let $c \in I(a,\pi)$ be such that $c \succsim_{\mathcal{S}} (\pi^{a\uparrow}) x$, for all $x \in I(a,\pi)$. This implies $c \succsim_{\mathcal{S}} (\pi^{a\uparrow}) a$. If $c \sim_{\mathcal{S}} (\pi^{a\uparrow}) a$, there is nothing to prove. Suppose therefore that $c \succ_{\mathcal{S}} (\pi^{a\uparrow}) a$, so that $R(a, \pi^{a\uparrow}) \subseteq R(c, \pi^{a\uparrow})$

We have, by construction, $I(a,\pi) \subseteq R(c, \pi^{a\uparrow})$. Using the above claim, we know that $P(a,\pi) \subseteq R(a, \pi^{a\uparrow})$. Hence, we know that $I(a,\pi) \cup P(a,\pi) = R(a,\pi) \subseteq R(c, \pi^{a\uparrow})$.

By definition, we know that $a \in \mathcal{S}(R(a,\pi), \pi)$. Since π is monotonic, this implies $a \in \mathcal{S}(R(a,\pi), \pi^{a\uparrow})$. Hence, we have $R(a,\pi) \subseteq R(c, \pi^{a\uparrow})$, $c \in R(a,\pi)$ and $c \in \mathcal{S}(R(c, \pi[a]), \pi^{a\uparrow})$. Using β^+ this implies $a \in \mathcal{S}(R(a,\pi), \pi^{a\uparrow})$, a contradiction. $\square$

References

Perny, P. (1995), Monotonie des méthodes de rangement par choix répétés. Communication to the *Groupe de Contact FNRS sur les Procédures de Choix dans les méthodes d'aide à la décision*, January, Université de Liège, Liège, Belgium.