Biorders and bi-semiorders with frontiers

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October 2008
séminaire FNRS, Bruxelles

Introduction

Context of today's talk

- preference modelling for MCDA
- conjoint measurement

Conjoint measurement

- set of objects: $X = X_1 \times X_2 \times \cdots \times X_n$
- preference relation on X: $\succsim$
- study a number of models leading to a numerical representation of $\succsim$

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

- many alternative models

Ordered classification

New premises

- replace $\succsim$ with $\langle C^1, C^2, \ldots, C^r \rangle$
- $\langle C^1, C^2, \ldots, C^r \rangle$ is an ordered partition/covering of X
  - $x \in X$ is "good", x is "bad"
- objects in C^k are "more attractive" than objects in C^{k-1}
- objects in C^k are not necessarily "equally attractive"

Additive Model without frontier

$$x \in C^k \Leftrightarrow \sigma^{k-1} < \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k$$

Additive Model with frontier

$$x \in C^k \Leftrightarrow \sigma^{k-1} \leq \sum_{i=1}^{n} u_i(x_i) \leq \sigma^k$$

- $C^k \cap C^{k+1}$: thin frontier between categories C^k and C^{k+1}

Particular cases

Two attributes, two categories, no frontier

- $X = X_1 \times X_2$

$$\begin{aligned} x \in C^2 &\Leftrightarrow 0 < u_1(x_1) + u_2(x_2) \\ &\Leftrightarrow f(x_1) > g(x_2) \end{aligned}$$

- $x \in C^2 \Leftrightarrow x_1 \mathcal{T} x_2$
- $\mathcal{T}$ is a binary relation between X_1 and X_2
- $\mathcal{T}$ is a biorder (Ducamp & Falmagne, 1969)

Biorders

- Doignon, Ducamp & Falmagne (1984) give necessary and sufficient for the existence of a numerical representation of biorders

Particular cases

Two attributes, two categories with a frontier

$$(x_1, x_2) \in C^2 \setminus C^1 \Leftrightarrow 0 < u_1(x_1) + u_2(x_2) \Leftrightarrow x_1 \mathfrak{T} x_2$$
$$(x_1, x_2) \in C^2 \cap C^1 \Leftrightarrow 0 = u_1(x_1) + u_2(x_2) \Leftrightarrow x_1 \mathfrak{F} x_2$$

$$x_1 \mathfrak{T} x_2 \Leftrightarrow f(x_1) > g(x_2)$$
$$x_1 \mathfrak{F} x_2 \Leftrightarrow f(x_1) = g(x_2)$$

Main question

- generalize results on biorders to cope with a frontier

Outline

1. Definitions and notation
2. Biorders
3. Interval orders and semiorders
   - Interval orders
   - Semiorders
4. Biorders with frontier
   - Model
   - Interval order with frontier
   - Semiorder with frontier
5. Bi-semiorder
6. Bi-semiorder with frontiers
7. Discussion

Binary relations on a set

Relations on a set

- X is a set
- binary relation V on X is a subset of $X \times X$
- classic vocabulary and notation

Traces of a binary relation V

$$x \succsim_V^{\ell} y \Leftrightarrow [y \; V \; z \Rightarrow x \; V \; z]$$
$$x \succsim_V^{r} y \Leftrightarrow [z \; V \; x \Rightarrow z \; V \; y]$$
$$x \succsim_V y \Leftrightarrow [x \succsim_V^{\ell} y \text{ and } x \succsim_V^{r} y]$$

- $\succsim_V^{\ell}$, $\succsim_V^{r}$, and $\succsim_V$ are reflexive and transitive

Relations between two sets

Relations between two sets

- $A = \{a, b, \ldots\}$ and $Z = \{p, q, \ldots\}$ are two (wlog disjoint) sets
- a binary relation $\mathcal{V}$ between A and Z is a subset of $A \times Z$
- any binary relation on X may be viewed a a binary relation between X and a disjoint duplication of X
- $\mathcal{V}^{cd}$ as a relation between Z and A such that $p \; \mathcal{V}^{cd} \; a \Leftrightarrow Not[a \; \mathcal{V} \; p]$

Traces of $\mathcal{V}$

- trace of $\mathcal{V}$ on A

$$a \succsim_{\mathcal{V}}^{A} b \Leftrightarrow [b \; \mathcal{V} \; p \Rightarrow a \; \mathcal{V} \; p, \text{ for all } p \in Z]$$

- trace of $\mathcal{V}$ on Z

$$p \succsim_{\mathcal{V}}^{Z} q \Leftrightarrow [a \; \mathcal{V} \; p \Rightarrow a \; \mathcal{V} \; q, \text{ for all } a \in A]$$

- $\succsim_{\mathcal{V}}^{A}$ and $\succsim_{\mathcal{V}}^{Z}$ are reflexive and transitive

Biorders

Ferrers Property

- $\mathcal{V}$ relation between A and Z
- $\mathcal{V}$ is said to be a *biorder* if it has the Ferrers property

$$\left.\begin{array}{c} a\ \mathcal{V}\ p \\ \text{and} \\ b\ \mathcal{V}\ q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a\ \mathcal{V}\ q \\ \text{or} \\ b\ \mathcal{V}\ p \end{array}\right.$$

Some elementary properties

- $\mathcal{V}$ is Ferrers iff $\succsim^A_{\mathcal{V}}$ is complete iff $\succsim^Z_{\mathcal{V}}$ is complete
- $\mathcal{V}$ is Ferrers iff $\mathcal{V}^{cd}$ is Ferrers
- traces generated by $\mathcal{V}$ and $\mathcal{V}^{cd}$ on A and Z are identical

Numerical representation of biorders

Theorem (Doignon at al. 1984)

Let A and Z be *finite or countably infinite* sets and $\mathcal{T}$ be a relation between A and Z. The following statements are equivalent:

1. $\mathcal{T}$ is Ferrers
2. there is a non-strict representation of $\mathcal{T}$
$$a\ \mathcal{T}\ p \Leftrightarrow f(a) \geq g(p)$$
3. there is a strict representation of $\mathcal{T}$
$$a\ \mathcal{T}\ p \Leftrightarrow f(a) > g(p)$$

The functions f and g can always be chosen in such a way that

$$a \succsim^A_{\mathcal{T}} b \Leftrightarrow f(a) \geq f(b)$$

$$p \succsim^Z_{\mathcal{T}} q \Leftrightarrow g(p) \geq g(q)$$

Strategy of proof

- build a relation Q on $A \cup Z$ such that the restriction of Q on $A \times Z$ is $\mathfrak{T}$

$$\alpha \; Q \; \beta \Leftrightarrow \begin{cases} \alpha \in A, \beta \in A, \text{ and } \alpha \succsim^A_{\mathfrak{T}} \beta, \\ \alpha \in Z, \beta \in Z, \text{ and } \alpha \succsim^Z_{\mathfrak{T}} \beta, \\ \alpha \in A, \beta \in Z, \text{ and } \alpha \; \mathfrak{T} \; \beta, \\ \alpha \in Z, \beta \in A, \text{ and } [\forall \gamma \in A, \delta \in Z, \gamma \; \mathfrak{T} \; \alpha \text{ and } \beta \; \mathfrak{T} \; \delta \Rightarrow \gamma \; \mathfrak{T} \; \delta] \end{cases}$$

- when $\mathfrak{T}$ is a biorder, Q is a weak order
- there is a real-valued function F on $A \cup Z$ such that

$$\alpha \; Q \; \beta \Leftrightarrow F(\alpha) \geq F(\beta)$$

- defining f (resp. g) as the restriction of F on A (resp. Z) leads to a non-strict representation

$$a \; \mathfrak{T} \; p \Leftrightarrow f(a) \geq g(p)$$

- to obtain a strict representation, use the same trick on $\mathfrak{T}^{cd}$

Strict numerical representation of biorders

General case

- order-denseness conditions have to be invoked
- the strict and non-strict representations are no more equivalent
- Doignon at al. (1984) have given the necessary order-denseness conditions in both cases

Theorem (Doignon at al., 1984)

Let $\mathfrak{T}$ be a binary relation between A and Z. The following statements are equivalent:

1. $\mathfrak{T}$ is Ferrers and there is a finite or countably infinite subset $\mathcal{B}^* \subseteq A$ such that, for all $a \in A$ and $p \in Z$

$$a \; \mathfrak{T} \; p \Rightarrow \left[a \succsim^A_{\mathfrak{T}} b^* \text{ and } b^* \; \mathfrak{T} \; p, \text{ for some } b^* \in \mathcal{B}^*\right]$$

2. there is a strict representation of $\mathfrak{T}$

The functions f and g can always be chosen in such a way that they represent the traces.

Interval orders

Definition

- an interval order T is an *irreflexive* Ferrers relation on a set X

Interval order as biorders

- an interval order T may be viewed as a biorder between X and a disjoint duplication of X

Strict representation of interval orders on countable sets

- there is a strict representation of T as a biorder
$$x \; T \; y \Leftrightarrow u(x) > v(y)$$
- irreflexivity implies that $u(x) \leq v(x)$, for all $x \in X$

Non-strict representation of interval orders on countable sets

- there is a non-strict representation of T as a biorder
$$x \; T \; y \Leftrightarrow u(x) \geq v(y)$$
- irreflexivity implies that $u(x) < v(x)$, for all $x \in X$

Semiorders

Definition

- a *semiorder* T is a semitransitive interval order
$$\left.\begin{array}{c} x \; V \; y \\ \text{and} \\ y \; V \; z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \; V \; w \\ \text{or} \\ w \; V \; z \end{array}\right.$$
- the trace $\succsim_V$ of a relation V is complete iff V is Ferrers and semitransitive
- the left and right traces are not contradictory, i.e., it is never true that $x \succ^{\ell}_V y$ and $y \succ^{r}_V x$

Semiorders as biorders

- on at most countable sets this leads to representation *with no proper nesting* of semiorders (Aleskerov et al., 2007)
$$u(x) > u(y) \Rightarrow v(x) \geq v(y)$$
- the general case is dealt with using the order-denseness condition presented above

Three models for semiorders

Representations with no proper nesting

$$x \; T \; y \Leftrightarrow u(x) > v(y)$$
$$u(x) > u(y) \Rightarrow v(x) \geq v(y)$$
$$u(x) \leq v(x)$$

Representations with no nesting

$$x \; T \; y \Leftrightarrow u(x) > v(y)$$
$$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$$
$$u(x) \leq v(x)$$

Constant threshold representations

$$x \; T \; y \Leftrightarrow u(x) > u(y) + 1$$

Sample result on semiorders

Results (Fishburn, 1985)

1. representations with no nesting are equivalent to representations with no proper nesting
   - no proper nesting: u represents $\succsim_*^\ell$, v represents $\succsim_*^r$
   - no nesting: u and v represent $\succsim_*$
2. on finite sets, constant threshold representations are equivalent to representations with no nesting

Biorders with frontier

Definition

- two disjoint relations $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z
- $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$
- $a \; \mathcal{N} \; p \Leftrightarrow Not[a \; \mathcal{R} \; p]$

Numerical representation with frontier

$$a \; \mathcal{T} \; p \Leftrightarrow f(a) > g(p)$$
$$a \; \mathcal{F} \; p \Leftrightarrow f(a) = g(p)$$

Traces

- $\succsim^A_{\mathcal{T}}$ (resp. $\succsim^Z_{\mathcal{T}}$) is the trace of $\mathcal{T}$ on A (resp. on Z)
- $\succsim^A_{\mathcal{R}}$ (resp. $\succsim^Z_{\mathcal{R}}$) is the trace of $\mathcal{R}$ on A (resp. on Z)
- $\succsim^A_{\star} = \succsim^A_{\mathcal{T}} \cap \succsim^A_{\mathcal{R}}$
- $\succsim^Z_{\star} = \succsim^Z_{\mathcal{T}} \cap \succsim^Z_{\mathcal{R}}$

Biorders with frontier

Necessary conditions

- $\mathcal{T}$ is a biorder
- $\mathcal{R}$ is a biorder
- traces of $\mathcal{T}$ and $\mathcal{R}$ must be compatible ($\succsim^A_{\star}$ and $\succsim^Z_{\star}$ are complete)
  - specific conditions
- $\mathcal{F}$ is “thin”

Thinness

- *thinness* for $\mathcal{F}$ holds on A if

$$\left.\begin{array}{c} a \; \mathcal{F} \; p \\ \text{and} \\ b \; \mathcal{F} \; p \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a \; \mathcal{F} \; q \Leftrightarrow b \; \mathcal{F} \; q \\ \text{and} \\ a \; \mathcal{T} \; q \Leftrightarrow b \; \mathcal{T} \; q \end{array}\right\} \Leftrightarrow a \sim^A_{\star} b$$

- *thinness* for $\mathcal{F}$ holds on Z if

$$\left.\begin{array}{c} a \; \mathcal{F} \; p \\ \text{and} \\ a \; \mathcal{F} \; q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b \; \mathcal{F} \; p \Leftrightarrow b \; \mathcal{F} \; q \\ \text{and} \\ b \; \mathcal{T} \; p \Leftrightarrow b \; \mathcal{T} \; q \end{array}\right\} \Leftrightarrow p \sim^Z_{\star} q$$

Remarks

- if $\mathcal{T}$ is a biorder, $\mathcal{R}$ is a biorder, and $\mathcal{F}$ is thin on both A and Z then both $\succsim_\star^A$ and $\succsim_\star^Z$ are complete
- the following four conditions are independent: $\mathcal{T}$ is a biorder, $\mathcal{R}$ is a biorder, thinness for $\mathcal{F}$ holds on A, thinness for $\mathcal{F}$ holds on Z
- $\mathcal{F}$ is strictly monotonic wrt to $\succsim_\star^A$ and $\succsim_\star^Z$

$$[a\ \mathcal{F}\ p \text{ and } b \succ_\star^A a] \Rightarrow b\ \mathcal{T}\ p$$
$$[a\ \mathcal{F}\ p \text{ and } p \succ_\star^Z q] \Rightarrow a\ \mathcal{T}\ q$$
$$[a\ \mathcal{F}\ p \text{ and } a \succ_\star^A c] \Rightarrow c\ \mathcal{N}\ p$$
$$[a\ \mathcal{F}\ p \text{ and } r \succ_\star^Z p] \Rightarrow a\ \mathcal{N}\ r$$

Numerical representation of biorders with frontier

Proposition (B & M, 2008)

Let A and Z be *finite or countably infinite* sets and let $\mathcal{T}$ and $\mathcal{F}$ be a pair of disjoint relations between A and Z.
The following statements are equivalent:

1. there is a representation of $\mathcal{T}$ and $\mathcal{F}$ as a biorder with frontier
2. $\mathcal{T}$ is a biorder, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ is a biorder and thinness holds on A and Z

The functions f and g can always be chosen in such a way that

$$a \succsim_\star^A b \Leftrightarrow f(a) \geq f(b)$$
$$p \succsim_\star^Z q \Leftrightarrow g(p) \geq g(q)$$

Idea of proof

The binary relation $\mathscr{L}$ on $A \cup Z$ defined letting

$$\alpha \mathrel{\mathscr{L}} \beta \Leftrightarrow \begin{cases} \alpha \in A, \beta \in A, \text{ and } \alpha \succsim_\star^A \beta \\ \alpha \in Z, \beta \in Z, \text{ and } \alpha \succsim_\star^Z \beta \\ \alpha \in A, \beta \in Z, \text{ and } \alpha \mathrel{\mathcal{R}} \beta \\ \alpha \in Z, \beta \in A, \text{ and } Not[\beta \mathrel{\mathcal{T}} \alpha] \end{cases}$$

is a weak order when $\mathcal{T}$ is a biorder, $\mathcal{R}$ is a biorder, and $\mathcal{F}$ is thin on both A and Z

$$\alpha \mathrel{\mathscr{L}} \beta \Leftrightarrow F(\alpha) \geq F(\beta)$$

$$a \mathrel{\mathcal{T}} p \Leftrightarrow [a \mathrel{\mathscr{L}} p \text{ and } Not[p \mathrel{\mathscr{L}} a]] \Rightarrow F(a) > F(p)$$
$$a \mathrel{\mathcal{F}} p \Leftrightarrow [a \mathrel{\mathscr{L}} p \text{ and } p \mathrel{\mathscr{L}} a] \Rightarrow F(a) = F(p)$$
$$Not[a \mathrel{\mathcal{R}} b] \Leftrightarrow [Not[a \mathrel{\mathscr{L}} p] \text{ and } p \mathrel{\mathscr{L}} a] \Rightarrow F(a) < F(p)$$

The general case

A subset $\mathcal{A}^* \subseteq A$ is dense for the pair $\mathcal{T}$ and $\mathcal{F}$ if, for all $a \in A$ and all $p \in Z$,

$$a \mathrel{\mathcal{T}} p \Rightarrow [a \succsim_\star^A a^* \text{ and } a^* \mathrel{\mathcal{T}} p]$$
$$a \mathrel{\mathcal{N}} p \Rightarrow [a^* \mathrel{\mathcal{N}} p \text{ and } a^* \succsim_\star^A a]$$

for some $a^* \in \mathcal{A}^*$

Proposition (B & M, 2008)

The following statements are equivalent:

1. there is a representation of $\mathcal{T}$ and $\mathcal{F}$ as a biorder with frontier
2. $\mathcal{T}$ is a biorder, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ is a biorder, thinness holds on A and Z, and there is a finite or countably infinite set $\mathcal{A}^* \subseteq A$ that is dense for the pair $\langle \mathcal{T}, \mathcal{F} \rangle$

The functions f and g can always be chosen in such a way that, for all $a, b \in A$ and $p, q \in Z$,

$$a \succsim_\star^A b \Leftrightarrow f(a) \geq f(b)$$
$$p \succsim_\star^Z q \Leftrightarrow g(p) \geq g(q)$$

Interval order with frontier

Definition

- let T and F be two disjoint relations on X
- let $R = T \cup F$ and $I = R^{sc}$ (symmetric complement of R)

$$x\ T\ y \Leftrightarrow u(x) > v(y)$$
$$x\ F\ y \Leftrightarrow u(x) = v(y)$$
$$u(x) < v(x)$$

Remark

- results for interval orders with frontier are obvious corollaries of results on biorders with frontier

Necessary conditions

- T is an interval order
- R is an interval order

Traces

- traces of T: $\succsim^{\ell}_T$ and $\succsim^{r}_T$
- traces of R: $\succsim^{\ell}_R$ and $\succsim^{r}_R$
- intersection of traces:

$$\succsim^{\ell}_* = \succsim^{\ell}_T \cap \succsim^{\ell}_R$$
$$\succsim^{r}_* = \succsim^{r}_T \cap \succsim^{r}_R$$

- we have to ensure that $\succsim^{\ell}_*$ and $\succsim^{r}_*$ are complete

Necessary conditions

Thinness

- F is *upper thin* if

$$\left.\begin{array}{c} x\ F\ z \\ \text{and} \\ y\ F\ z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x\ F\ w \Leftrightarrow y\ F\ w \\ \text{and} \\ x\ T\ w \Leftrightarrow y\ T\ w \end{array}\right\} \Leftrightarrow x \sim_*^\ell y$$

- F is *lower thin* if

$$\left.\begin{array}{c} z\ F\ x \\ \text{and} \\ z\ F\ y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} w\ F\ x \Leftrightarrow w\ F\ y \\ \text{and} \\ w\ T\ x \Leftrightarrow w\ T\ y \end{array}\right\} \Leftrightarrow x \sim_*^r y$$

Results

Proposition (B & M, 2008)

Let T and F be two disjoint relations on a *finite or countably infinite* set X. The following statements are equivalent:

1. the pair of relations T and F has a numerical representation as an interval order with frontier
2. T is an interval order, R is an interval order and F is upper and lower thin

We can always choose u and v in such a way that

$$x \succsim_*^\ell y \Leftrightarrow u(x) \geq u(y)$$
$$x \succsim_*^r y \Leftrightarrow v(x) \geq v(y)$$

General case

- A subset $\mathcal{X}^* \subseteq X$ is dense for the pair T and F if, for all $x, y \in X$,

$$x\ T\ y \Rightarrow [x \succsim_*^\ell x^* \text{ and } x^*\ T\ y]$$

$$x\ R^c\ y \Rightarrow [x^*\ R^c\ y \text{ and } x^* \succsim_*^\ell x]$$

for some $x^* \in \mathcal{X}^*$

Semiorder with frontier

Representations with no proper nesting

- let T and F be two disjoint relations on X
- let $R = T \cup F$ and $I = R^{sc}$ (symmetric complement of R)

$$x \; T \; y \Leftrightarrow u(x) > v(y)$$
$$x \; F \; y \Leftrightarrow u(x) = v(y)$$
$$u(x) < v(x)$$
$$u(x) > u(y) \Rightarrow v(x) \geq v(y)$$

Necessary conditions

Necessary conditions

- $\langle T, F, I \rangle$ is a pseudo-order with a thin relation F
  - T is a semiorder
  - R is a semiorder
  - consistency conditions

Consistency conditions

$$TFI \subseteq T$$
$$IFT \subseteq T$$
$$FIT \subseteq T$$
$$TIF \subseteq T$$

Representations with no proper nesting

Proposition (B & M, 2008)

Let T and F be two disjoint relations on a finite or countably infinite X. The following statements are equivalent:

1. the pair of relations has a representation with no proper nesting
2. T is a semiorder, R is a semiorder, $TFI \subseteq T$, $IFT \subseteq T$, F is upper thin, and F is lower thin

We can always choose u and v in such a way that

$$x \succsim_*^\ell y \Leftrightarrow u(x) \geq u(y)$$
$$x \succsim_*^r y \Leftrightarrow v(x) \geq v(y)$$

General case

- A subset $\mathcal{X}^* \subseteq X$ is dense for the pair T and F if, for all $x, y \in X$,

$$x\ T\ y \Rightarrow [x \succsim_*^\ell x^* \text{ and } x^*\ T\ y]$$
$$x\ R^c\ y \Rightarrow [x^*\ R^c\ y \text{ and } x^* \succsim_*^\ell x]$$

  for some $x^* \in \mathcal{X}^*$

Semiorder with frontier

Representations with no nesting

- let T and F be two disjoint relations on X
- let $R = T \cup F$ and $I = R^{sc}$ (symmetric complement of R)

$$x\ T\ y \Leftrightarrow u(x) > v(y)$$
$$x\ F\ y \Leftrightarrow u(x) = v(y)$$
$$u(x) < v(x)$$
$$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$$

Necessary conditions

- all conditions used for the case of representations with no proper nesting remain necessary
- a stronger version of thinness is needed

Strong thinness

- F is *strongly upper thin* if

$$\left.\begin{matrix} x \; F \; z \\ y \; F \; z \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x \; T \; w \Leftrightarrow y \; T \; w \\ x \; F \; w \Leftrightarrow y \; F \; w \\ w \; F \; x \Leftrightarrow w \; F \; y \\ w \; T \; x \Leftrightarrow w \; T \; y \end{matrix}\right\} \Leftrightarrow x \sim_* y$$

- F is *strongly lower thin* if

$$\left.\begin{matrix} z \; F \; x \\ z \; F \; y \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x \; T \; w \Leftrightarrow y \; T \; w \\ x \; F \; w \Leftrightarrow y \; F \; w \\ w \; F \; x \Leftrightarrow w \; F \; y \\ w \; T \; x \Leftrightarrow w \; T \; y \end{matrix}\right\} \Leftrightarrow x \sim_* y$$

Representations with no nesting

Proposition (B & M, 2008)

Let T and F be two disjoint relations on a *finite or countably infinite* set X. The following statements are equivalent:

1. the pair of relations has a representation with no nesting
2. T is a semiorder, R is a semiorder, $TFI \subseteq T$, $IFT \subseteq T$, F is strongly upper thin, and F is strongly lower thin

We can always choose u and v in such a way that

$$x \succsim_* y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$$

General case

- open question

Semiorder with frontier

Representation with constant threshold

$$x \; T \; y \Leftrightarrow u(x) > u(y) + 1$$
$$x \; F \; y \Leftrightarrow u(x) = u(y) + 1$$

Proposition (B & M, 2008)

Let T and F be two disjoint relations on a *finite* set X. The following statements are equivalent:

1. this pair of relations has a constant threshold representation
2. T is a semiorder, R is a semiorder, $TFI \subseteq T$, $IFT \subseteq T$, F is strongly upper thin, and F is strongly lower thin

We can always choose u and v in such a way that

$$x \succsim_* y \Leftrightarrow u(x) \geq u(y)$$

Bi-semiorder

Definition (Ducamp & Falmagne, 1969)

Let $\mathcal{T}$ and $\mathcal{P}$ be two relations between the sets A and Z

$$a \; \mathcal{P} \; p \Leftrightarrow f(a) > g(p) + 1$$
$$a \; \mathcal{T} \; p \Leftrightarrow f(a) > g(p)$$

More general models

- many possible variants

$$a \; \mathcal{P} \; p \Leftrightarrow f(a) > h(p)$$
$$a \; \mathcal{T} \; p \Leftrightarrow f(a) > g(p)$$
$$h(p) > g(p)$$

Notation

Traces

- trace of $\mathcal{T}$ on A (resp. Z) is denoted by $\succsim_{\mathcal{T}}^{A}$ (resp. $\succsim_{\mathcal{T}}^{Z}$)
- trace of $\mathcal{P}$ on A (resp. Z) is denoted by $\succsim_{\mathcal{P}}^{A}$ (resp. $\succsim_{\mathcal{P}}^{Z}$)
- $\succsim_{\circ}^{A} = \succsim_{\mathcal{T}}^{A} \cap \succsim_{\mathcal{P}}^{A}$ and $\succsim_{\circ}^{Z} = \succsim_{\mathcal{T}}^{Z} \cap \succsim_{\mathcal{P}}^{Z}$

Necessary conditions

- the six relations $\succsim_{\mathcal{T}}^{A}$, $\succsim_{\mathcal{T}}^{Z}$, $\succsim_{\mathcal{P}}^{A}$, $\succsim_{\mathcal{P}}^{Z}$, $\succsim_{\circ}^{A}$ and $\succsim_{\circ}^{Z}$ are complete
- $\succsim_{\circ}^{A}$ is complete iff $\succsim_{\mathcal{T}}^{A}$ and $\succsim_{\mathcal{P}}^{A}$ are complete and compatible
- $\succsim_{\circ}^{Z}$ is complete iff $\succsim_{\mathcal{T}}^{Z}$ and $\succsim_{\mathcal{P}}^{Z}$ are complete and compatible

Necessary conditions

Conditions

- $\mathcal{P} \subseteq \mathcal{T}$
- $\mathcal{T}$ is Ferrers ($\succsim_{\mathcal{T}}^{A}$ and $\succsim_{\mathcal{T}}^{Z}$ are complete)
- $\mathcal{P}$ is Ferrers ($\succsim_{\mathcal{P}}^{A}$ and $\succsim_{\mathcal{P}}^{Z}$ are complete)
- compatibility of traces

$$\left.\begin{array}{c} a\,\mathcal{P}\,p \\ \text{and} \\ b\,\mathcal{T}\,q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} b\,\mathcal{P}\,p \\ \text{or} \\ a\,\mathcal{T}\,q \end{array}\right. \qquad \left.\begin{array}{c} a\,\mathcal{P}\,p \\ \text{and} \\ b\,\mathcal{T}\,q \end{array}\right\} \Rightarrow \left\{\begin{array}{c} a\,\mathcal{P}\,q \\ \text{or} \\ b\,\mathcal{T}\,p \end{array}\right.$$

Result

Theorem, Ducamp & Falmagne (1969)

Let A and Z be *finite* sets. Let $\mathcal{T}$ and $\mathcal{P}$ be two relations between A and Z. The following statements are equivalent:

1. $\mathcal{P}$ and $\mathcal{T}$ are biorders satisfying conditions the two compatibility conditions
2. the pair of relations $\mathcal{P}$ and $\mathcal{T}$ has a constant threshold representation

We may always choose the functions f and g such that

$$a \succsim_\circ^A b \Leftrightarrow f(a) \geq f(b)$$
$$p \succsim_\circ^Z q \Leftrightarrow g(p) \geq g(q)$$

skip proof

Outline of proof

$$\alpha \; Q_\circ \; \beta \Leftrightarrow \begin{cases} \alpha, \beta \in A & \text{and } \alpha \succsim_\circ^A \beta \\ \alpha, \beta \in Z & \text{and } \alpha \succsim_\circ^Z \beta \\ \alpha \in A, \beta \in Z & \text{and } \alpha \; \mathcal{T} \; \beta \\ \alpha \in Z, \beta \in A & \text{and } Not[\beta \; \mathcal{T} \; \alpha] \end{cases}$$

$Q_\circ$ is a weak order

$$\alpha \; H_\circ \; \beta \Leftrightarrow \begin{cases} \alpha, \beta \in A & \text{and } \alpha \; \mathcal{P} \; \gamma \text{ and } Not[\beta \; \mathcal{T} \; \gamma], \text{ for some } \gamma \in Z \\ \alpha, \beta \in Z & \text{and } Not[\gamma \; \mathcal{T} \; \alpha] \text{ and } \gamma \; \mathcal{P} \; \beta, \text{ for some } \gamma \in A \\ \alpha \in A, \beta \in Z & \text{and } \alpha \; \mathcal{P} \; \beta \\ \alpha \in Z, \beta \in A & \text{and } Not[\gamma \; \mathcal{T} \; \alpha], \gamma \; \mathcal{P} \; \delta, \text{ and } Not[\beta \; \mathcal{T} \; \delta] \\ & \quad \text{for some } \gamma \in A, \delta \in Z \end{cases}$$

$H_\circ$ is a semiorder and $Q_\circ$ refines the weak order underlying $H_\circ$

$$\alpha \; H_\circ \; \beta \Leftrightarrow F(\alpha) > F(\beta) + 1$$
$$\alpha \; Q_\circ \; \beta \Leftrightarrow F(\alpha) \geq F(\beta)$$

Bi-semiorder with frontiers

Notation

- four relations $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{F}$ between the sets A and Z
- define $\mathcal{S} = \mathcal{P} \cup \mathcal{J}$, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$
- we suppose that $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{S} \subseteq \mathcal{T}$
- consequence: $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{R}$

Constant threshold representation

$$\begin{aligned} a \,\mathcal{P}\, p &\Leftrightarrow f(a) > g(p) + 1 \\ a \,\mathcal{J}\, p &\Leftrightarrow f(a) = g(p) + 1 \\ a \,\mathcal{T}\, p &\Leftrightarrow f(a) > g(p) \\ a \,\mathcal{F}\, p &\Leftrightarrow f(a) = g(p) \end{aligned}$$

More general models

- many possible variants

Bi-semiorder with frontiers

Traces

$$a \succsim_{\diamond}^{A} b \Leftrightarrow \left\{ \begin{array}{c} b \,\mathcal{P}\, r \Rightarrow a \,\mathcal{P}\, r \\ b \,\mathcal{S}\, r \Rightarrow a \,\mathcal{S}\, r \\ b \,\mathcal{T}\, r \Rightarrow a \,\mathcal{T}\, r \\ b \,\mathcal{R}\, r \Rightarrow a \,\mathcal{R}\, r \end{array} \right\} \text{ for all } r \in Z$$

$$p \succsim_{\diamond}^{Z} q \Leftrightarrow \left\{ \begin{array}{c} c \,\mathcal{P}\, p \Rightarrow c \,\mathcal{P}\, q \\ c \,\mathcal{S}\, p \Rightarrow c \,\mathcal{S}\, q \\ c \,\mathcal{T}\, p \Rightarrow c \,\mathcal{T}\, q \\ c \,\mathcal{R}\, p \Rightarrow c \,\mathcal{R}\, q \end{array} \right\} \text{ for all } c \in A$$

Necessary conditions

- $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{R}$ must be biorders
- all conditions (12 in total) necessary to imply that $\succsim^A_\diamond$ on A and $\succsim^Z_\diamond$ on Z are complete
- new thinness conditions

$$\left.\begin{array}{c} a\ \mathcal{F}\ p \\ \text{and} \\ b\ \mathcal{F}\ p \end{array}\right\} \Rightarrow a \sim^A_\diamond b \qquad \left.\begin{array}{c} a\ \mathcal{F}\ p \\ \text{and} \\ a\ \mathcal{F}\ q \end{array}\right\} \Rightarrow p \sim^Z_\diamond q$$

$$\left.\begin{array}{c} a\ \mathcal{J}\ p \\ \text{and} \\ b\ \mathcal{J}\ p \end{array}\right\} \Rightarrow a \sim^A_\diamond b \qquad \left.\begin{array}{c} a\ \mathcal{J}\ p \\ \text{and} \\ a\ \mathcal{J}\ q \end{array}\right\} \Rightarrow p \sim^Z_\diamond q$$

- when thinness conditions are imposed, some of the 12 compatibility conditions become redundant. In total 8 of them must be imposed

Result: finite case

Proposition (B & M, 2008)

Let A and Z be *finite sets*. Let $\mathcal{P}$, $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{F}$ be four relations between the sets A and Z such that $\mathcal{P} \cap \mathcal{J} = \varnothing$, $\mathcal{T} \cap \mathcal{F} = \varnothing$, $\mathcal{J} \cap \mathcal{F} = \varnothing$, and $\mathcal{P} \cup \mathcal{J} = \mathcal{S} \subseteq \mathcal{T}$. The following statements are equivalent:

1. there is a constant threshold representation of $\langle \mathcal{P}, \mathcal{J}, \mathcal{T}, \mathcal{F} \rangle$
2. $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{R} = \mathcal{T} \cup \mathcal{F}$ are biorders satisfying the 8 compatibility conditions and such that thinness$^\diamond$ holds for both $\mathcal{J}$ and $\mathcal{F}$ on both A and Z

The functions f and g above can always be chosen so that, for all $a, b \in A$ and $p, q \in Z$,

$$a \succsim^A_\diamond b \Leftrightarrow f(a) \geq f(b)$$
$$p \succsim^Z_\diamond q \Leftrightarrow g(p) \geq g(q)$$

Idea of proof

- tedious ...
- ... but closely follows the strategy of Ducamp & Falmagne (1969)

Summary

Considering preference structures with frontier

- leads to interesting questions ...
- ...that are simple but not trivial

Results

- N & S conditions for the representation of biorders with frontier
  - intervals order with frontier
  - semiorders with frontier (representation with no proper nesting)
  - semiorders with frontier with no nesting (countable case)
  - semiorders with frontier with constant threshold (finite case)
- N & S conditions for the representation of bi-semiorders with frontiers with constant threshold in the finite case

Applications

Conjoint measurement with ordered categories

- N & S conditions for an additive representation with two attributes and two categories with or without frontier
- N & S conditions for an additive representation with two attributes and three categories with or without frontiers in the finite case

Temporal logic

- interval orders are used to deal with the problem of locating "events" on a "time scale" given information on the fact that some entirely precede or follow others
- interval orders with frontier is the adequate model if one wishes to include "immediate succession" relations as in Golumbic & Shamir (1993)

Future research

Open problems & future research

- semiorder with frontier
  - representation with no nesting in the general case
- study the many variants of bi-semiorder with or without frontier

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