Notes on bipolar concordance relations

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7 October 2008

1 Introduction

The purpose of this note is to write down the result on concordance relations that I presented in Luxembourg one year ago. The first two sections are adapted from Bouyssou (1996). They prove a general result on concordance relations. Section 4 shows how this general result applies to bipolar concordance. The final section presents an approximation result on bipolar outranking relations.

2 Definitions and notation

A *valued relation* T on a set X is a function from $X \times X$ into $[0, 1]$.

Remark 1
In this text, we always work between 0 and 1 with 1/2 playing the rôle of the midpoint. A simple rescaling allows to work between -1 and 1 with 0 playing the rôle of a midpoint. •

A valued relation T is said to be *reflexive* if $T(x, x) = 1$, for all $x \in X$. A valued relation T on X such that $T(x, y) \in \{0, 1\}$, for all $x, y \in X$, is said to be *crisp*. As is usual, we write $x \; T \; y$ instead of $T(x, y) = 1$ when T is a crisp relation.

A crisp relation T on a set X is said to be:

- *reflexive* if $x \; T \; x$,
- *complete* if $x \; T \; y$ or $y \; T \; x$,
- *transitive* if $x \; T \; y$ and $y \; T \; z \Rightarrow x \; T \; z$,
- *antisymmetric* if $x \; T \; y$ and $y \; T \; x \Rightarrow x = y$,

- *Ferrers* if $(x\ T\ y$ and $z\ T\ w) \Rightarrow (x\ T\ w$ or $z\ T\ y)$,
- *semi-transitive* if $(x\ T\ y$ and $y\ T\ z) \Rightarrow (x\ T\ w$ or $w\ T\ z)$,

for all $x, y, z, w \in X$.

We say that a crisp relation is:

- a *linear order* if it is complete, antisymmetric and transitive,
- a *weak order* if it is complete and transitive,
- a *semi-order* if it is reflexive, Ferrers and semi-transitive,
- an *interval order* if it is reflexive and Ferrers.

Remark 2
It is easy to check that a reflexive and Ferrers relation is necessarily complete.

If P (resp. I) is the asymmetric (resp. symmetric) part of a semiorder S, it is easy to show that we have $PIP \subseteq P$, $PPI \subseteq P$ and $IPP \subseteq P$ (see, e.g. Roubens and Vincke, 1985, page 35). In particular P is transitive. •

We denote by $\mathcal{LO}_X$ (resp. $\mathcal{WO}_X$, $\mathcal{SO}_X$, $\mathcal{IO}_X$) the set of all linear orders (resp. weak orders, semi-orders, interval orders) on a set X. It is well-known that $\mathcal{LO}_X \subseteq \mathcal{WO}_X \subseteq \mathcal{SO}_X \subseteq \mathcal{IO}_X$ all inclusions being strict as soon as X is large enough.

Throughout this paper $A = \{a, b, c, \dots\}$ will denote a *finite* set with $|A| = m \geq 2$ elements. We interpret the elements of A as "alternatives" to be compared using an outranking method.

3 Generalized concordance

A "Generalized Concordance situation on A" consists in:

- a strictly positive integer n,
- n functions $g_1, g_2, \dots, g_n$ from A into $\mathbb{R}$,
- n functions $t_1, t_2, \dots, t_n$ from $\mathbb{R}^2$ into $[0, 1]$ such that, $\forall i \in \{1, 2, \dots, n\}$, t_i is non-decreasing (resp. non-increasing) in its first (resp. second) argument and $t_i(x, x) = 1$, $\forall x \in \mathbb{R}$,
- n strictly positive real numbers $k_1, k_2, \dots, k_n$.

Remark 3
It is useful to have the following interpretation in mind:

- n is the number of criteria,
- $g_1, g_2, \ldots, g_n$ are the criteria,
- the function t_i transforms the comparisons of alternatives on criterion g_i into a valued relation. •

On the basis of such a "situation", the "Generalized Concordance" method or more briefly the GC method (inspired by Perny, 1992, Perny and Roy, 1992) leads to a valued relation T on A letting, $\forall a, b \in A$:

$$T(a,b) = \frac{\sum_{i=1}^{n} k_i \, t_i(g_i(a), g_i(b))}{\sum_{i=1}^{n} k_i}$$

We denote by $\mathcal{GC}_A$ the set of all valued relations on A that can be obtained with the GC method on the basis of a "Generalized Concordance situation on A". Since it was supposed that $t_i(x, x) = 1$, all relations in $T \in \mathcal{GC}_A$ are obviously reflexive (i.e., $T(a, a) = 1$, $\forall a \in A$).

In what follows we study the structural properties of relations in $\mathcal{GC}_A$, besides reflexivity. The following definitions will prove useful for this purpose.

Definition 1
Let T be a valued relation on a finite set A. The relation T is said to be a t-g relation if there are:

- *a real-valued function g on A and*
- *a function t from $g[A] \times g[A]$ into $[0, 1]$ being non-decreasing (resp. non-increasing) in its first (resp. second) argument and such that $t(x, x) = 1, \forall x \in g[A]$,*

such that,

$$T(a, b) = t(g(a), g(b)),$$

for all $a, b \in A$.

The notion of t-g relation is very closely related to that of "monotone scalability" used in Monjardet (1984) (after Fishburn, 1973), the only difference being the addition here of a restriction on $t(x, x)$. By construction, relations in $\mathcal{GC}_A$ are "convex mixtures" of t-g relations.

Definition 2
Let T be a valued relation on a finite set A. We say that T is lower diagonal stepped *if $T(a, a) = 1$, for all $a \in A$ and there is a linear order V on A such that, for all $a, b \in A$,*

$$a \; V \; b \Rightarrow T(a, c) \geq T(b, c) \text{ and } T(c, a) \leq T(c, b), \forall c \in A.$$

Apart from the restriction that $T(a,a) = 1$, a lower diagonal stepped relation is identical to a relation having a "monotone board" as defined in Monjardet (1984).

Definition 3
Let T be a valued relation on a finite set A. We say that T is linear *if,*

$$[T(a,c) > T(b,c) \text{ or } T(c,a) < T(c,b)] \Rightarrow [T(a,d) \geq T(b,d) \text{ and } T(d,a) \leq T(d,b)],$$

for all $a,b,c,d \in A$.

The following lemma is a direct consequence of Monjardet (1984, Theorem 13). For the sake of completeness we outline its proof.

Lemma 1
Let T be a valued relation on a finite set A. The following statements are equivalent:

1. *T is a t-g relation,*
2. *T is reflexive and linear,*
3. *T is lower diagonal stepped.*

Proof
The part $[1 \Rightarrow 2]$ is obvious.

$[2 \Rightarrow 3]$. Define the crisp relation $\succsim_T$ (called the trace of T), letting, for all $a,b \in A$,

$$a \succsim_T b \Rightarrow [T(a,c) \geq T(b,c) \text{ and } T(c,a) \leq T(c,b) \text{ for all } c \in A].$$

The relation $\succsim_T$ is reflexive and transitive. It is easy to see that it is also complete when T is linear. Consider now any linear order V extending $\succsim_T$, i.e., such that $V \subseteq \succsim_T$ (such a linear order exists by Szpilrajn's lemma). Using the reflexivity and the linearity of T, it is easy to prove that T is lower diagonal stepped using such a linear order.

$[3 \Rightarrow 1]$. Since A is a finite set and V is a linear order, there is a real-valued function g on A such that, for all $a,b \in A$, $g(a) \geq g(b) \Leftrightarrow a \; V \; b$. Given such a function g define $t(g(a),g(b)) = T(a,b)$. The valued relation T being lower diagonal stepped, it is easy to prove that t is a well-defined real-valued function on $g[A] \times g[A]$, has the required monotonicity property and is such that $t(g(a),g(a)) = 1$, for all $a \in A$ (for details, see Fishburn, 1973, Theorem 1 and A). □

Definition 4
Let $\mathcal{K}$ be a set of crisp relations on a finite set A. The valued relation T on A is said to be representable *in $\mathcal{K}$ if there is a function ϕ from $\mathcal{K}$ into* $[0, 1]$ *such that:*

$$\sum_{K \in \mathcal{K}} \phi(K) = 1$$

for which:

$$T(a, b) = \sum_{K \in \mathcal{K}} \phi(K) K(a, b),$$

for all $a, b \in A$.

From Lemma 1, we know that relations in $\mathcal{GC}_A$ are "convex mixtures" of lower diagonal stepped relations. We proceed by showing that these relations are particular convex mixtures of elements of $\mathcal{SO}_A$, i.e. crisp semi-orders. We shall use two lemmas.

Lemma 2
Let T be a crisp relation on a finite set A. The relation T is lower diagonal stepped if and only if $T \in \mathcal{SO}_A$.

Proof
Results immediately from the classical properties of semi-orders, see, e.g., Fishburn (1970) or Roubens and Vincke (1985). □

Lemma 3
Let T be a valued relation on a finite set A. If T is lower diagonal stepped then it is representable in $\mathcal{SO}_A$.

Proof
Since A is finite and T is lower diagonal stepped, it takes up to $m(m-1)/2 = r$ values in $(0, 1)$. These r, non-necessarily distinct, values are such that:

$$0 < q_1 \leq q_2 \leq \cdots \leq q_r < 1.$$

For any $i = 1, 2, \ldots, r$, define the crisp relation T_i on A letting, for all $a, b \in A$, $a\ T_i\ b \Leftrightarrow T(a, b) \geq q_i$. Since T is lower diagonal stepped, it is easy to see that T_i is lower diagonal stepped, for $i = 1, 2, \ldots, r$. Thus, we know from Lemma 2, that $T_i \in \mathcal{SO}_A$. Let T_* be the crisp relation on A such that, for all $a, b \in A$, $a\ T_*\ b \Leftrightarrow T(a, b) \geq 1$. It is clear that $T_* \in \mathcal{SO}_A$.

Define a function ϕ from $\mathcal{SO}_A$ into $[0,1]$ letting, for all $W \in \mathcal{SO}_A$:

$$\phi(W) = \begin{cases} q_1 & \text{if } W = T_1, \\ q_2 - q_1 & \text{if } W = T_2, \\ \vdots & \\ q_r - q_{r-1} & \text{if } W = T_r, \\ 1 - q_r & \text{if } W = T_*, \end{cases}$$

It is easy to see that:

$$\sum_{W \in \mathcal{SO}_A} \phi(W) = 1,$$

and that, with this function ϕ, T is representable in $\mathcal{SO}_A$. □

Combining Lemmas 1, 2 and 3 allows us to state a characterization of the elements of $\mathcal{GC}_A$.

Proposition 1
Let T be a valued relation on a finite set A. Then $T \in \mathcal{GC}_A$ if and only if *it representable in the set $\mathcal{SO}_A$ of all semi-orders on A.*

PROOF
[T is representable in $\mathcal{SO}_A \Rightarrow T \in \mathcal{GC}_A$].

Let $\mathcal{K} = \{T_1, T_2, \ldots, T_\ell\}$ be the set of all semi-orders T in $\mathcal{SO}_A$ such that $\phi(T) > 0$ (since A is finite, so is $\mathcal{SO}_A$ and, hence, $\mathcal{K}$). The set A being finite, for any T_i, $i = 1, 2, \ldots, \ell$, there is a function u_i from A into $\mathbb{R}$ such that, $\forall a, b \in A$, $a\ T_i\ b \Leftrightarrow u_i(a) \geq u_i(b) - 1$.

Consider a "situation" involving ℓ criteria and let: $k_i = \phi(T_i)$, $g_i = u_i$, $t_i(x, y) = 1$ if $x \geq y - 1$ and 0 otherwise. With such a function t_i, we clearly have that $t_i(x, x) = 1$. Furthermore, t_i is nondecreasing in its first argument and nondecreasing in its second argument. It is obvious that applying the GC method to this "situation" leads to T.

[$T \in \mathcal{GC}_A \Rightarrow T$ is representable in $\mathcal{SO}_A$].

To prove that T is representable in $\mathcal{SO}_A$, it is sufficient to prove that the relations defined by $T_i(a, b) = t_i(g_i(a), g_i(b))$, $\forall a, b \in A$, are representable in $\mathcal{SO}_A$, since T is a convex mixture of the relations T_i. From lemma 1, we know that the relations T_i are lower diagonal stepped and the use of lemma 3 completes the proof. □

Example 1
Let $A = \{a, b, c, d\}$. Let T be the valued relation such that:

	a	b	c	d
a	1.0	1.0	1.0	0.6
b	1.0	1.0	0.6	1.0
c	0.6	0.6	1.0	1.0
d	0.6	1.0	0.6	1.0

Let us show that this valued relation does not belong to $\mathcal{GC}_A$.

Suppose that $T \in \mathcal{GC}_A$ and let us show that this leads to a contradiction.

Let $\mathcal{K} \subseteq \mathcal{SO}_A$ be the set of semiorders allowing to represent T. If $S_i \in \mathcal{K}$, we denote by P_i (resp. I_i) the asymmetric (resp. symmetric) part of S_i.

Because $T(a, b) = T(b, a) = 1$, we must have $a\ S_i\ b$ and $b\ S_i\ a$, for all $S_i \in \mathcal{K}$. Similarly, $T(b, d) = T(d, b) = 1$ implies that $b\ S_i\ d$ and $d\ S_i\ b$, for all $S_i \in \mathcal{K}$.

Because $T(c, d) = 1$ and $T(d, c) = 0.6$, all semiorders in $\mathcal{K}$ are such that $c\ S_i\ d$. The sum of the weights of the semiorders such that $c\ P_i\ d$ must be 0.4. The sum of the weights of the remaining semiorders (for which $c\ I_i\ d$) must be 0.6. Similarly, because $T(a, c) = 1$ and $T(c, a) = 0.6$, all semiorders in $\mathcal{K}$ are such that $a\ S_i\ c$. The sum of the weights of the semiorders such that $a\ P_i\ c$ must be 0.4. The sum of the weights of the remaining semiorders (for which $a\ I_i\ c$) must be 0.6.

We have $T(b, c) = 0.6$ and $T(c, b) = 0.6$. Hence, the sum of the weights of the semiorders such that $b\ P_i\ c$ must be 0.4. The sum of the weights of the semiorders such that $c\ P_i\ b$ must be 0.4. The remaining semiorders are such that $b\ I_i\ c$ and total weight of 0.2. Similarly, because $T(a, d) = 0.6$ and $T(d, a) = 0.6$, the sum of the weights of the semiorders such that $a\ P_i\ d$ must be 0.4, the sum of the weights of the semiorders such that $d\ P_i\ a$ must be 0.4 and the remaining semiorders are such that $b\ I_i\ c$ and a total weight of 0.2.

Any semiorder such that $b\ P_i\ c$ cannot have $c\ P_i\ d$ since this would imply $b\ P_i\ d$, using the transitivity of P_i. Hence all the semiorders such that $b\ P_i\ c$ must have $c\ I_i\ d$. For these semiorders, it is impossible to have $d\ P_i\ a$ since this would imply, using $P_i I_i P_i \subseteq P_i$, $b\ P_i\ a$. These semiorders have a total weight of 0.4

Any semiorder such that $c\ P_i\ b$ cannot have $a\ P_i\ c$ since this would imply $a\ P_i\ b$, using the transitivity of P_i. Hence the semiorders such that $c\ P_i\ b$ must have $a\ I_i\ c$. For these semiorders, it is impossible to have $d\ P_i\ a$ since this would imply, using $P_i I_i P_i \subseteq P_i$, $d\ P_i\ b$. These semiorders have a total weight of 0.4.

Hence, it is impossible that the total weight of the semiorders such that $d\ P_i\ a$ is 0.4, as required. $\diamond$

4 Application to bipolar concordance relations

4.1 Definitions

We consider a finite set A of alternatives. Each alternative $a \in A$ is supposed to be evaluated on a set of n criteria. Each criterion uses two thresholds.

A criterion g_i is a real-valued function on A. To the criterion g_i, we associate two numbers $it_i \geq 0$ and $pt_i \geq it_i \geq 0$ in such a way that:

$$\begin{aligned} x\ P_i\ y &\Leftrightarrow g_i(x) > g_i(y) + pt_i, \\ x\ Q_i\ y &\Leftrightarrow g_i(y) + pt_i \geq g_i(x) > g_i(y) + it_i, \\ x\ I_i\ y &\Leftrightarrow |g_i(x) - g_i(y)| \leq it_i. \end{aligned}$$

Remark 4
In the above equations, the position of the strict and non-strict inequalities is purely conventional. Since we are dealing here with finite sets, their position could be changed without affecting the analysis below. •

Remark 5
Here we consider constant thresholds. The analysis below is unaffected by the consideration of variable thresholds, provided, they satisfy the usual consistency condition, i.e., for all $a, b \in A$,

$$g_i(a) > g_i(b) \Rightarrow \begin{cases} g_i(a) + it_i(g_i(a)) \geq g_i(b) + it_i(g_i(b)), \\ g_i(a) + pt_i(g_i(a)) \geq g_i(b) + pt_i(g_i(b)). \end{cases}$$

•

To each criterion, we associate a valued preference relation, i.e., a real valued function T_i on A^2 such that, for all $a, b \in A$,

$$\begin{aligned} T_i(a,b) = 0 &\Leftrightarrow g_i(b) > g_i(a) + pt_i, \\ T_i(a,b) = 1/2 &\Leftrightarrow g_i(a) + pt_i \geq g_i(b) > g_i(a) + it_i, \\ T_i(a,b) = 1 &\Leftrightarrow g_i(b) \leq g_i(a) + it_i. \end{aligned} \tag{1}$$

On this basis, a valued relation (the bipolar concordance relation) is built letting, for all $a, b \in A$

$$T(a,b) = \sum_{i=1}^{n} \pi_i\, T_i(a,b), \tag{2}$$

where $\pi_i \in [0,1]$ is the weight associated to criterion g_i, these weights being normalized so that $\sum_{i=1}^{n} \pi_i = 1$.

A valued relation T on A that is build using (1) and (2) is called a bipolar concordance relation.

4.2 Results

Bipolar concordance relations are particular cases of relations that can be obtained with the GC method that uses t-g relations. Indeed, using (1), is equivalent to using a function t_i operating on the criterion g_i that is such that, for all $x, y \in g_i(A)$,

$$t_i(x,y) = \begin{cases} 1 \text{ if } x - y \geq -it_i, \\ 1/2 \text{ if } x - y \in [-pt_i, -it_i), \\ 0 \text{ if } x - y < -pt_i. \end{cases}$$

Such a function t_i satisfies $t_i(x,x) = 1$ and is nondecreasing (resp. nonincreasing) in its first (resp. second) argument.

Therefore, bipolar concordance relations are a particular case of relations build using the GC method and Proposition 1 implies that such relations are representable in $\mathcal{SO}_A$. Conversely, it is easy to see that if a relation is representable in $\mathcal{SO}_A$ it can be obtained using (1). Indeed, it suffices to consider as many criterion as there are relations in $T \in \mathcal{SO}_A$ such that $\phi(T) > 0$, taking for each of these criteria $it_i = pt_i$.

The above argument has proved the following corollary of proposition 1.

Corollary 1
Let A be a finite set.

1. *A valued relation T is a bipolar concordance relation iff it belongs to $\mathcal{GC}_A$ iff it belongs to $\mathcal{SO}_A$,*

2. *The set of bipolar concordance relations is identical to the set of bipolar concordance relations obtained with the additional constraint that, for all criteria, $it_i = pt_i$.*

Remark 6
The above gives a complete characterization of the bipolar concordance relations. The analysis in Bouyssou (1996) shows that the hope of improving this characterization is quite limited. The relations lying in $\mathcal{SO}_A$ are a convex polytope in a space of very high dimension. An improved characterization should describe the facets of this polytope. As shown in Bouyssou (1996) this direction of research is quite likely to be hopeless. •

Remark 7
For practical disaggregation purposes, we can always suppose that we are working with criteria such that $it_i = pt_i$. Such criteria are simply semiorders and semiorders are easily characterized using boolean variables. We already observed that in Luxembourg, following Roubens and Vincke (1985). Indeed,

a semiorder on a finite set A with $|A| = m$ can always be described by a set of m^2 binary variables x_{ij}, $i, j = 1, 2, \ldots, m$ such that, for all $i, j, k, \ell \in \{1, 2, \ldots, m\}$

$$\begin{aligned} x_{ij} + x_{ji} &\geq 1, \\ x_{ik} + x_{j\ell} &\geq x_{i\ell} + x_{jk} - 1, \\ x_{ik} + x_{kj} &\geq x_{i\ell} + x_{\ell k} - 1, \end{aligned}$$

as already observed in Roubens and Vincke (1985, page 37) •

Remark 8
Imposing that $it_i = pt_i$ does not restrict the set of bipolar concordance relations. However, it is likely that this constraint will increase the number of criteria needed to recover a given valued relation. Unfortunately, there does not seem to be an easy characterization of homogeneous families of two semiorders, i.e., the structure that is obtained with two thresholds (admitting variable thresholds). Indeed, what would be needed here is a set of variables x_{ij} that can only take three distinct values, e.g., $0, 1/2, 1$, and satisfying the following constraint:

$$\left.\begin{array}{c} x_{ik} > x_{jk} \\ \text{or} \\ x_{ki} < x_{kj} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x_{i\ell} \geq x_{j\ell} \\ \text{and} \\ x_{\ell i} \leq x_{\ell j} \end{array}\right.$$

that captures the linearity of the relation (besides an obvious completeness requirement stating that $x_{ij} = 0$ or $x_{ij} = 1/2$ then $x_{ji} = 1$).

I have tried to look for an efficient formulation of this constraint as a linear constraint in an integer program but I have failed to find anything satisfactory. This is a problem. •

5 What about discordance?

As explained in Luxembourg, the above analysis does not easily generalize to include discordance. Indeed, veto effects does not only affect a pair of alternatives but, in general, have a more global effect.

The general problem is likely to be difficult. Nevertheless, it is possible to show the following.

Let T be a valued relation on A and T' be a bipolar concordance relation on A. We say that T is *below* T' in the sense of bipolar outranking if, for all $a, b \in A$

- $T(a, b) \leq T'(a, b)$,

- if $T(a,b) \neq T'(a,b)$ then either $T(a,b) = 0$ or $T(a,b) = 1/2$ (discordance effects either bring to 0 or the midpoint 1/2).

Clearly bipolar outranking relations are relations that are below a bipolar concordance relation in the above sense. I do not know how to characterize the set of bipolar outranking relations (we know that it includes $\mathcal{SO}_A$). Nevertheless an approximate result is easily achieved. Let T be a valued relation that is below a bipolar concordance relation T'. Then there is a bipolar outranking relation that is "close" to T.

The proof goes as follows. It is always possible to build a bipolar concordance relation T^* that is "close" to T' and that give a positive weight to *all* relations in $\mathcal{SO}_A$. This is obvious. Now every linear order in $\mathcal{SO}_A$ receives a positive weight. With the bipolar concordance relation T^*, discordance effects can be used "at will" since for all $a, b \in A$ there is a criterion such that a is on top and b is at bottom. Hence, it is always possible the two veto thresholds (one bringing at 1/2 and the other bringing at 0) so that it only affects the ordered pair (a, b). This proves the following.

Proposition 2
Let T be a valued relation on A that is below a bipolar concordance relation T'. For all $\epsilon > 0$, there is a bipolar outranking relation T'' that is such that

$$|T(a,b) - T''(a,b)| \leq \epsilon,$$

for all $a, b \in A$.

The above result shows that, in an approximate manner, everything that is below a bipolar concordance relation can be seen as a bipolar outranking relation.

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