

## Chapter 1

# CONJOINT MEASUREMENT TOOLS FOR MCDM

### *A brief introduction*

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**Abstract** This paper offers a brief and nontechnical introduction to the use of conjoint measurement in multiple criteria decision making. The emphasis is on the, central, additive value function model. We outline its axiomatic foundations and present various possible assessment techniques to implement it. Some extensions of this model, e.g. nonadditive models or models tolerating intransitive preferences are then briefly reviewed.

**Keywords:** Conjoint Measurement, Additive Value Function, Preference Modelling.

## 1. Introduction and motivation

Conjoint measurement is a set of tools and results first developed in Economics [44] and Psychology [141] in the beginning of the '60s. Its, ambi-

tious, aim is to provide measurement techniques that would be adapted to the needs of the Social Sciences in which, most often, multiple dimensions have to be taken into account.

Soon after its development, people working in decision analysis realized that the techniques of conjoint measurement could also be used as tools to structure preferences [51, 165]. This is the subject of this paper which offers a brief and nontechnical introduction to conjoint measurement models and their use in multiple criteria decision making. More detailed treatments may be found in [63, 79, 121, 135, 209]. Advanced references include [58, 129, 211].

### 1.1. Conjoint measurement models in decision theory

The starting point of most works in decision theory is a binary relation  $\succsim$  on a set  $A$  of objects. This binary relation is usually interpreted as an “at least as good as” relation between alternative courses of action gathered in  $A$ .

Manipulating a binary relation can be quite cumbersome as soon as the set of objects is large. Therefore, it is not surprising that many works have looked for a *numerical representation* of the binary relation  $\succsim$ . The most obvious numerical representation amounts to associate a real number  $V(a)$  to each object  $a \in A$  in such a way that the comparison between these numbers faithfully reflects the original relation  $\succsim$ . This leads to defining a real-valued function  $V$  on  $A$ , such that:

$$a \succsim b \Leftrightarrow V(a) \geq V(b), \quad (1.1)$$

for all  $a, b \in A$ . When such a numerical representation is possible, one can use  $V$  instead of  $\succsim$  and, e.g. apply classical optimization techniques to find the most preferred elements in  $A$  given  $\succsim$ . We shall call such a function  $V$  a *value function*.

It should be clear that not all binary relations  $\succsim$  may be represented by a value function. Condition (1.1) imposes that  $\succsim$  is complete (i.e.  $a \succsim b$  or  $b \succsim a$ , for all  $a, b \in A$ ) and transitive (i.e.  $a \succsim b$  and  $b \succsim c$  imply  $a \succsim c$ , for all  $a, b, c \in A$ ). When  $A$  is finite or countably infinite, it is well-known [58, 129] that these two conditions are, in fact, not only necessary but also sufficient to build a value function satisfying (1.1).

#### Remark 1

The general case is more complex since (1.1) implies, for instance, that there must be “enough” real numbers to distinguish objects that have to be distinguished. The necessary and sufficient conditions for (1.1) can be found in [58, 129]. An advanced treatment is [13]. Sufficient conditions

that are well-adapted to cases frequently encountered in Economics can be found in [42, 45]; see [34] for a synthesis. •

It is vital to note that, when a value function satisfying (1.1) exists, it is by no means unique. Taking any increasing function  $\phi$  on  $\mathbb{R}$ , it is clear that  $\phi \circ V$  gives another acceptable value function. A moment of reflection will convince the reader that only such transformations are acceptable and that if  $V$  and  $U$  are two real-valued functions on  $A$  satisfying (1.1), they must be related by an increasing transformation. In other words, a value function in the sense of (1.1) defines an *ordinal scale*.

Ordinal scales, although useful, do not allow the use of sophisticated assessment procedures, i.e. of procedures that allow an analyst to assess the relation  $\succsim$  through a structured dialogue with the decision-maker. This is because the knowledge that  $V(a) \geq V(b)$  is strictly equivalent to the knowledge of  $a \succsim b$  and no inference can be drawn from this assertion besides the use of transitivity.

It is therefore not surprising that much attention has been devoted to numerical representations leading to more constrained scales. Many possible avenues have been explored to do so. Among the most well-known, let us mention:

- the possibility to compare *probability distributions* on the set  $A$  [58, 207]. If it is required that, not only (1.1) holds but that the numbers attached to the objects should be such that their expected values reflect the comparison of probability distributions on the set of objects, a much more constrained numerical representation clearly obtains,
- the introduction of “preference difference” comparisons of the type: the difference between  $a$  and  $b$  is larger than the difference between  $c$  and  $d$ , see [44, 81, 123, 129, 159, 180, 199]. If it is required that, not only (1.1) holds, but that the differences between numbers also reflect the comparisons of preference differences, a more constrained numerical representation obtains.

When objects are evaluated according to several dimensions, i.e. when  $\succsim$  is defined on a product set, new possibilities emerge to obtain numerical representations that would specialize (1.1). The purpose of conjoint measurement is to study such kinds of models.

There are many situations in decision theory which call for the study of binary relations defined on product sets. Among them let us mention:

- *Multiple criteria decision making* using a preference relation comparing alternatives evaluated on several attributes [16, 121, 162, 173, 209],

- *Decision under uncertainty* using a preference relation comparing alternatives evaluated on several states of nature [68, 107, 177, 184, 210, 211],
- *Consumer theory* manipulating preference relations for bundles of several goods [43],
- *Intertemporal decision making* using a preference relation between alternatives evaluated at several moments in time [121, 125, 126],
- *Inequality measurement* comparing distributions of wealth across several individuals [5, 17, 18, 217].

The purpose of this paper is to give an introduction to the main models of conjoint measurement useful in multiple criteria decision making. The results and concepts that are presented may however be of interest in all of the afore-mentioned areas of research.

**Remark 2**

Restricting ourselves to applications in multiple criteria decision making will not allow us to cover every aspect of conjoint measurement. Among the most important topics left aside, let us mention: the introduction of statistical elements in conjoint measurement models [54, 108] and the test of conjoint measurement models in experiments [135]. •

Given a binary relation  $\succsim$  on a product set  $X = X_1 \times X_2 \times \dots \times X_n$ , the theory of conjoint measurement consists in finding conditions under which it is possible to build a convenient numerical representation of  $\succsim$  and to study the uniqueness of this representation. The central model is the *additive value function* model in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^n v_i(x_i) \geq \sum_{i=1}^n v_i(y_i) \quad (1.2)$$

where  $v_i$  are real-valued functions, called *partial value functions*, on the sets  $X_i$  and it is understood that  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$ . Clearly if  $\succsim$  has a representation in model (1.2), taking any common increasing transformation of the  $v_i$  will *not* lead to another representation in model (1.2).

Specializations of this model in the above-mentioned areas give several central models in decision theory:

- The Subjective Expected Utility model, in the case of decision-making under uncertainty,
- The discounted utility model for dynamic decision making,

- Inequality measures *à la* Atkinson/Sen in the area of social welfare.

The axiomatic analysis of this model is now quite firmly established [44, 129, 211]; this model forms the basis of many decision analysis techniques [79, 121, 209, 211]. This is studied in sections 3 and 4 after we introduce our main notation and definitions in section 2.

**Remark 3**

One possible objection to the study of model (1.2) is that the choice of an *additive* model seems arbitrary and restrictive. It should be observed here that the functions  $v_i$  will precisely be assessed so that additivity holds. Furthermore, the use of a simple model may be seen as an advantage in view of the limitations of the cognitive abilities of most human beings.

It is also useful to notice that this model can be reformulated so as to make addition disappear. Indeed if there are partial value functions  $v_i$  such that (1.2) holds, it is clear that  $V = \sum_{i=1}^n v_i$  is a value function satisfying (1.1). Since  $V$  defines an ordinal scale, taking the exponential of  $V$  leads to another valid value function  $W$ . Clearly  $W$  has now a multiplicative form:

$$x \succsim y \Leftrightarrow W(x) = \prod_{i=1}^n w_i(x_i) \geq W(y) = \prod_{i=1}^n w_i(y_i).$$

where  $w_i(x_i) = e^{v_i(x_i)}$ .

The reader is referred to chapter XXX (Chapter 6, Dyer) for the study of situations in which  $V$  defines a scale that is more constrained than an ordinal scale, e.g. because it is supposed to reflect preference differences or because it allows to compute expected utilities. In such cases, the additive form (1.2) is no more equivalent to the multiplicative form considered above. •

In section 5 we present a number of extensions of this model going from nonadditive representations of transitive relations to model tolerating intransitive indifference and, finally, nonadditive representations of non-transitive relations.

**Remark 4**

In this paper, we shall restrict our attention to the case in which alternatives may be evaluated on the various attributes without risk or uncertainty. Excellent overviews of these cases may be found in [121, 209]; recent references include [142, 150]. •

Before starting our study of conjoint measurement oriented towards MCDM, it is worth recalling that conjoint measurement aims at establishing measurement models in the Social Sciences. To many, the very

notion of “measurement in the Social Sciences” may appear contradictory. It may therefore be useful to briefly consider how the notion of measurement can be modelled in Physics, an area in which the notion of “measurement” seems to arise quite naturally, and to explain how a “measurement model” may indeed be useful in order to structure preferences.

## 1.2. An aside: measuring length

Physicists usually take measurement for granted and are not particularly concerned with the technical and philosophical issues it raises (at least when they work within the realm of Newtonian Physics). However, for a Social Scientist, these questions are of utmost importance. It may thus help to have an idea of how things appear to work in Physics before tackling more delicate cases.

Suppose that you are on a desert island and that you want to “measure” the length of a collection of rigid straight rods. Note that we do not discuss here the “pre-theoretical” intuition that “length” is a property of these rods that can be measured, as opposed, say, to their softness or their beauty.

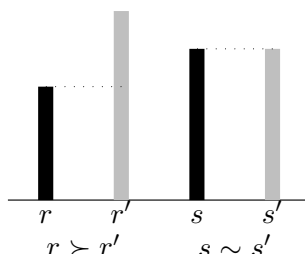


Figure 1.1. Comparing the length of two rods.

A first simple step in the construction of a measure of length is to place the two rods side by side in such a way that one of their extremities is at the same level (see Figure 1.1). Two things may happen: either the upper extremities of the two rods coincide or not. This seems to be the simplest way to devise an experimental procedure leading to the discovery of which rod “has more length” than the other. Technically, this leads to defining two binary relations  $\succ$  and  $\sim$  on the set of rods in the following way:

- $r \succ r'$  when the extremity of  $r$  is higher than the extremity of  $r'$ ,
- $r \sim r'$  when the extremities of  $r$  and  $r'$  are at the same level,

Clearly, if length is a quality of the rods that can be measured, it is expected that these pairwise comparisons are somehow consistent, e.g.,

- if  $r \succ r'$  and  $r' \succ r''$ , it should follow that  $r \succ r''$ ,
- if  $r \sim r'$  and  $r' \sim r''$ , it should follow that  $r \sim r''$ ,
- if  $r \sim r'$  and  $r' \succ r''$ , it should follow that  $r \succ r''$ .

Although quite obvious, these consistency requirements are stringent. For instance, the second and the third conditions are likely to be violated if the experimental procedure involves some imprecision, e.g if two rods that slightly differ in length are nevertheless judged “equally long”. They represent a form of *idealization* of what could be a perfect experimental procedure.

With the binary relations  $\succ$  and  $\sim$  at hand, we are still rather far from a full-blown measure of length. It is nevertheless possible to assign numbers to each of the rods in such a way that the comparison of these numbers reflects what has been obtained experimentally. When the consistency requirements mentioned above are satisfied, it is indeed generally possible to build a real-valued function  $\Phi$  on the set of rods that would satisfy:

$$\begin{aligned} r \succ r' &\Leftrightarrow \Phi(r) > \Phi(r') \text{ and} \\ r \sim r' &\Leftrightarrow \Phi(r) = \Phi(r'). \end{aligned}$$

If the experiment is costly or difficult to perform, such a numerical assignment may indeed be useful because it summarizes, once for all, what has been obtained in experiments. Clearly there are many possible ways to assign numbers to rods in this way. Up to this point, they are equally good for our purposes. The reader will easily check that defining  $\succsim$  as  $\succ$  or  $\sim$ , the function  $\Phi$  is nothing else than a “value function” for length: any increasing transformation may therefore be applied to  $\Phi$ .

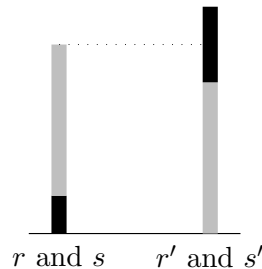


Figure 1.2. Comparing the length of composite rods.

The next major step towards the construction of a measure of length is the realization that it is possible to form new rods by simply placing two or more rods “in a row”, i.e. you may *concatenate* rods. From the point of view of length, it seems obvious to expect this concatenation operation  $\circ$  to be “commutative” ( $r \circ s$  has the same length as  $s \circ r$ ) and associative ( $(r \circ s) \circ t$  has the same length as  $r \circ (s \circ t)$ ).

You clearly want to be able to measure the length of these composite objects and you can always include them in our experimental procedure outlined above (see Figure 1.2). Ideally, you would like your numerical assignment  $\Phi$  to be somehow compatible with the concatenation operation: knowing the numbers assigned to two rods, you want to be able to deduce the number assigned to their concatenation. The most obvious way to achieve that is to require that the numerical assignment of a composite object can be deduced by addition from the numerical assignments of the objects composing it, i.e. that

$$\Phi(r \circ r') = \Phi(r) + \Phi(r').$$

This clearly places many additional constraints on the results of your experiment. An obvious one is that  $\succ$  and  $\sim$  should be compatible with the concatenation operation  $\circ$ , e.g.

$$r \succ r' \text{ and } t \sim t' \text{ should lead to } r \circ t \succ r' \circ t'.$$

These new constraints may or may not be satisfied. When they are, the usefulness of the numerical assignment  $\Phi$  is even more apparent: a simple arithmetic operation will allow to infer the result of an experiment involving composite objects.

Let us take a simple example. Suppose that you have 5 rods  $r_1, r_2, \dots, r_5$  and that, because space is limited, you can only concatenate at most two rods and that not all concatenations are possible. Let us suppose, for the moment, that you do not have much technology available so that you may only experiment using *different* rods. You may well collect the following information, using obvious notation exploiting the transitivity of  $\succ$  which holds in this experiment,

$$r_1 \circ r_5 \succ r_3 \circ r_4 \succ r_1 \circ r_2 \succ r_5 \succ r_4 \succ r_3 \succ r_2 \succ r_1.$$

Your problem is then to find a numerical assignment  $\Phi$  to rods such that using an addition operation, you can infer the numerical assignment of composite objects consistently with your observations. Let us consider the following three assignments:



	$\Phi$	$\Phi'$	$\Phi''$
$r_1$	14	10	14
$r_2$	15	91	16
$r_3$	20	92	17
$r_4$	21	93	18
$r_5$	28	100	29

These three assignments are equally valid to reflect the comparisons of single rods. Only the first and the third allow to capture the comparisons of composite objects that were performed. Note that, going from  $\Phi$  to  $\Phi''$  does not involve just changing the “unit of measurement”: since  $\Phi(r_1) = \Phi''(r_1)$  this would imply that  $\Phi = \Phi''$ , which is clearly false.

Such numerical assignments have limited usefulness. Indeed, it is tempting to use them to predict the result of comparisons that we have not been able to perform. But this turns out to be quite disappointing: using  $\Phi$  you would conclude that  $r_2 \circ r_3 \sim r_1 \circ r_4$  since  $\Phi(r_2) + \Phi(r_3) = 15 + 20 = 35 = \Phi(r_1) + \Phi(r_4)$ , but, using  $\Phi''$ , you would conclude that  $r_2 \circ r_3 \succ r_1 \circ r_4$  since  $\Phi''(r_2) + \Phi''(r_3) = 16 + 17 = 33$  while  $\Phi''(r_1) + \Phi''(r_4) = 14 + 18 = 32$ .

Intuitively, “measuring” calls for some kind of a *standard* (e.g. the “Mètre-étalon” that can be found in the Bureau International des Poids et Mesures in Sèvres, near Paris). This implies choosing an appropriate “standard” rod *and* being able to prepare perfect copies of this standard rod (we say here “appropriate” because the choice of a standard should be made in accordance with the lengths of the objects to be measured: a tiny or a huge standard will not facilitate experiments). Let us call  $s_0$  the standard rod. Let us suppose that you have been able to prepare a large number of perfect copies  $s_1, s_2, \dots$  of  $s_0$ . We therefore have:

$$s_0 \sim s_1, s_0 \sim s_2, s_0 \sim s_3, \dots$$

Let us also agree that the length of  $s_0$  is 1. This is your, arbitrary, unit of length. How can you use  $s_0$  and its perfect copies so as to determine unambiguously the length of any other (simple or composite) object? Quite simply, you may prepare a “standard sequence of length  $n$ ”,  $S(n) = s_1 \circ s_2 \circ \dots \circ s_{n-1} \circ s_n$ , i.e. a composite object that is made by concatenating  $n$  perfect copies of our standard rod  $s_0$ . The length of a standard sequence of length  $n$  is exactly  $n$  since we have concatenated  $n$  objects that are perfect copies of the standard rod of length 1. Take any rod  $r$  and let us compare  $r$  with several standard sequences of increasing length:  $S(1), S(2), \dots$

Two cases may arise. There may be a standard sequence  $S(k)$  such that  $r \sim S(k)$ . In that case, we know that the number  $\Phi(r)$  assigned to  $r$  must be exactly  $k$ . This is unlikely however. The most common situation is that we will find two consecutive standard sequences  $S(k-1)$  and  $S(k)$

such that  $r \succ S(k-1)$  and  $S(k) \succ r$  (see Figure 1.3). This means that  $\Phi(r)$  must be such that  $k-1 < \Phi(r) < k$ . We seem to be in trouble here since, as before,  $\Phi(r)$  is not exactly determined. How can you proceed? This depends on your technology for preparing perfect copies.

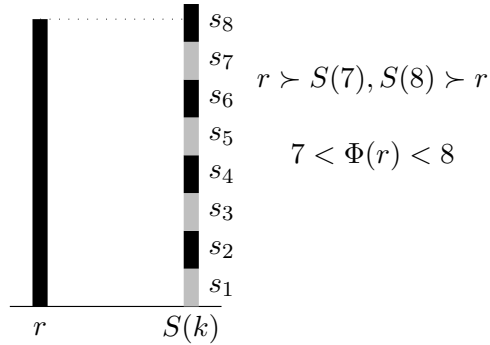


Figure 1.3. Using standard sequences.

Imagine that you are able to prepare perfect copies not only of the standard rod but also of any object. You may then prepare several copies ( $r_1, r_2, \dots$ ) of the rod  $r$ . You can now compare a composite object made out of two perfect copies of  $r$  with your standard sequences  $S(1), S(2), \dots$ . As before, you shall eventually arrive at locating  $\Phi(r_1 \circ r_2) = 2\Phi(r)$  within an interval of width 1. This means that the interval of imprecision surrounding  $\Phi(r)$  has been divided by two. Continuing this process, considering longer and longer sequences of perfect copies of  $r$ , you will keep on reducing the width of the interval containing  $\Phi(r)$ . This means that you can approximate  $\Phi(r)$  with any given level of precision. Mathematically, a unique value for  $\Phi(r)$  will be obtained using a simple argument.

Supposing that you are in position to prepare perfect copies of any object is a strong technological requirement. When this is not possible, there still exists a way out. Instead of preparing a perfect copy of  $r$  you may also try to increase the granularity of your standard sequence. This means building an object  $t$  that you would be able to replicate perfectly and such that concatenating  $t$  with one of its perfect replicas gives an object that has exactly the length of the standard object  $s_0$ , i.e.  $\Phi(t) = 1/2$ . Considering standard sequences based on  $t$ , you will be able to increase by a factor 2 the precision with which we measure the length of  $r$ . Repeating the process, i.e. subdividing  $t$ , will lead, as before, to a unique limiting value for  $\Phi(r)$ .

The mathematical machinery underlying the measurement process informally described above (called “extensive measurement”) rests on the theory of ordered groups. It is beautifully described and illustrated in [129]. Although the underlying principles are simple, we may expect complications to occur e.g. when not all concatenations are feasible, when there is some level (say the velocity of light if we were to measure speed) that cannot be exceeded or when it comes to relate different measures. See [129, 140, 168] for a detailed treatment.

Clearly, this was an overly detailed and unnecessary complicated description of how length could be measured. Since our aim is to eventually deal with “measurement” in the Social Sciences, it may however be useful to keep the above process in mind. Its basic ingredients are the following:

- well-behaved relations  $\succ$  and  $\sim$  allowing to compare objects,
- a concatenation operation  $\circ$  allowing to consider composite objects,
- consistency requirements linking  $\succ$ ,  $\sim$  and  $\circ$ ,
- the ability to prepare perfect copies of some objects in order to build standard sequences.

Basically, conjoint measurement is a quite ingenious way to perform related measurement operations when no concatenation operation is available. This will however require that objects can be evaluated along several dimensions. Before explaining how this might work, it is worth explaining the context in which such measurement might prove useful.

### Remark 5

It is often asserted that “measurement is impossible in the Social Sciences” precisely because the Social Scientist has no way to define a concatenation operation. Indeed, it would seem hazardous to try to concatenate the intelligence of two subjects or the pain of two patients (see [56, 106]). Under certain conditions, the power of conjoint measurement will precisely be to provide a means to bypass this absence of readily available concatenation operation when the objects are evaluated on several dimensions.

Let us remark that, even when there seems to be a concatenation operation readily available, it does not always fit the purposes of extensive measurement. Consider for instance an individual expressing preferences for the quantity of the two goods he consumes. The objects therefore take the well structured form of points in the positive orthant of  $\mathbb{R}^2$ . There seems to be an obvious concatenation operation here:  $(x, y) \circ (z, w)$  might simply be taken to be  $(x + y, z + w)$ . However a fairly rational person,

consuming pants and jackets, may indeed prefer  $(3, 0)$  (3 pants and no jacket) to  $(0, 3)$  (no pants and 3 jackets) but at the same time prefer  $(3, 3)$  to  $(6, 0)$ . This implies that these preferences cannot be explained by a measure that would be additive with respect to the concatenation operation consisting in adding the quantities of the two goods consumed. Indeed  $(3, 0) \succ (0, 3)$  implies  $\Phi(3, 0) > \Phi(0, 3)$ , which implies  $\Phi(3, 0) + \Phi(3, 0) > \Phi(0, 3) + \Phi(3, 0)$ . Additivity with respect to concatenation should then imply that  $(3, 0) \circ (3, 0) \succ (0, 3) \circ (3, 0)$ , that is  $(6, 0) \succ (3, 3)$ .

### 1.3. An example: Even swaps

The even swaps technique described and advocated in [120, 121, 165] is a simple way to deal with decision problems involving several attributes that does not have recourse to a formal representation of preferences, which will be the subject of conjoint measurement. Because this technique is simple and may be quite useful, we describe it below using the same example as in [120]. This will also allow to illustrate the type of problems that are dealt with in decision analysis applications of conjoint measurement.

#### Example 6 (Even swaps technique)

A consultant considers renting a new office. Five different locations have been identified after a careful consideration of many possibilities, rejecting all those that do not meet a number of requirements.

His feeling is that five distinct characteristics, we shall say five attributes, of the possible locations should enter into his decision: his daily commute time (expressed in minutes), the ease of access for his clients (expressed as the percentage of his present clients living close to the office), the level of services offered by the new office (expressed on an ad hoc scale with three levels:  $A$  (all facilities available),  $B$  (telephone and fax),  $C$  (no facilities)), the size of the office expressed in square feet, and the monthly cost expressed in dollars.

The evaluation of the five offices is given in Table 1.1. The consultant

	$a$	$b$	$c$	$d$	$e$
Commute	45	25	20	25	30
Clients	50	80	70	85	75
Services	A	B	C	A	C
Size	800	700	500	950	700
Cost	1850	1700	1500	1900	1750

Table 1.1. Evaluation of the 5 offices on the 5 attributes.

has well-defined preferences on each of these attributes, independently of what is happening on the other attributes. His preference increases with the level of access for his clients, the level of services of the office and its size. It decreases with commute time and cost. This gives a first easy way to compare alternatives through the use of *dominance*.

An alternative  $y$  is dominated by an alternative  $x$  if  $x$  is at least as good as  $y$  on *all* attributes while being strictly better for at least one attribute. Clearly dominated alternatives are not candidate for the final choice and may, thus, be dropped from consideration. The reader will easily check that, on this example, alternative  $b$  dominates alternative  $e$ :  $e$  and  $b$  have similar size but  $b$  is less expensive, involves a shorter commute time, an easier access to clients and a better level of services. We may therefore forget about alternative  $e$ . This is the only case of “pure dominance” in our table. It is however easy to see that  $d$  is “close” to dominating  $a$ , the only difference in favor of  $a$  being on the cost attribute (50 \$ per month). This is felt more than compensated by the differences in favor of  $d$  on all other attributes: commute time (20 minutes), client access (35 %) and size (150 sq. feet).

Dropping all alternatives that are not candidate for choice, this initial investigation allows to reduce the problem to:

	$b$	$c$	$d$
Commute	25	20	25
Clients	80	70	85
Services	$B$	$C$	$A$
Size	700	500	950
Cost	1700	1500	1900

A natural way to proceed is then to assess tradeoffs. Observe that all alternatives but  $c$  have a common evaluation on commute time. We may therefore ask the consultant, starting with office  $c$ , what gain on client access would compensate a loss of 5 minutes on commute time. We are looking for an alternative  $c'$  that would be evaluated as follows:

	$c$	$c'$
Commute	20	<b>25</b>
Clients	70	<b>70 + <math>\delta</math></b>
Services	$C$	$C$
Size	500	500
Cost	1500	1500

and judged indifferent to  $c$ . Although this is not an easy question, it is clearly crucial in order to structure preferences.

#### Remark 7

In this paper, we do not consider the possibility of lexicographic preferences, in which such tradeoffs do not occur, see [59, 60, 160]. Lexico-

graphic preferences may also be combined with the possibility of “local” tradeoffs, see [22, 64, 136]. •

**Remark 8**

Since tradeoffs questions may be difficult, it is wise to start with an attribute requiring few assessments (in the example, all alternatives but one have a common evaluation on commute time). Clearly this attribute should be traded against one with an underlying “continuous” structure (cost, in the example). •

Suppose that the answer is that for  $\delta = 8$ , it is reasonable to assume that  $c$  and  $c'$  would be indifferent. This means that the decision table can be reformulated as follows:

	$b$	$c'$	$d$
Commute	25	25	25
Clients	80	78	85
Services	$B$	$C$	$A$
Size	700	500	950
Cost	1700	1500	1900

It is then apparent that all alternatives have a similar evaluation on the first attribute which, therefore, is not useful to discriminate between alternatives and may be forgotten. The reduced decision table is as follows:

	$b$	$c'$	$d$
Clients	80	78	85
Services	$B$	$C$	$A$
Size	700	500	950
Cost	1700	1500	1900

There is no case of dominance in this reduced table. Therefore further simplification calls for the assessment of new tradeoffs. Using cost as the reference attribute, we then proceed to “neutralize” the service attribute. Starting with office  $c'$ , this means asking for the increase in monthly cost that the consultant would just be prepared to pay to go from level “ $C$ ” of service to level “ $B$ ”. Suppose that this increase is roughly 250 \$. This defines alternative  $c''$ . Similarly, starting with office  $d$  we ask for the reduction of cost that would exactly compensate a reduction of services from “ $A$ ” to “ $B$ ”. Suppose that the answer is 100 \$ a month, which defines alternative  $d'$ . The decision table is reshaped as:

	$b$	$c''$	$d'$
Clients	80	78	85
Services	$B$	<b>B</b>	<b>B</b>
Size	700	500	950
Cost	1700	<b>1750</b>	<b>1800</b>

We may forget about the second attribute which does not discriminate any more between alternatives. When this is done, it is apparent that  $c''$  is dominated by  $b$  and can be suppressed. Therefore, the decision table at this stage looks like the following:

	$b$	$d'$
Clients	80	85
Size	700	950
Cost	1700	1800

Unfortunately, this table reveals no case of dominance. New tradeoffs have to be assessed. We may now ask, starting with office  $b$ , what additional cost the consultant would be ready to incur to increase its size by 250 square feet. Suppose that the rough answer is 250 \$ a month, which defines  $b'$ . We are now facing the following table:

	$b'$	$d'$
Clients	80	85
Size	<b>950</b>	950
Cost	<b>1950</b>	1800

Attribute size may now be dropped from consideration. But, when this is done, it is clear that  $d'$  dominates  $b'$ . Hence it seems obvious to recommend office  $d$  as the final choice.  $\diamond$

The above process is simple and looks quite obvious. If this works, why be interested at all in “measurement” if the idea is to help someone to come up with a decision?

First observe that in the above example, the set of alternatives was relatively small. In many practical situations, the set of objects to compare is much larger than the set of alternatives in our example. Using the even swaps technique could then require a considerable number of difficult tradeoff questions. Furthermore, as the output of the technique is not a preference model but just the recommendation of an alternative in a given set, the appearance of new alternatives (e.g. because a new office is for rent) would require starting a new round of questions. This is likely to be highly frustrating. Finally, the informal even swaps technique may not be well adapted to the, many, situations, in which the decision under study takes place in a complex organizational environment. In such situations, having a formal model to be able to communicate and to convince is an invaluable asset. Such a model will furthermore allow to conduct extensive sensitivity analysis and, hence, to deal with imprecision both in the evaluations of the objects to compare and in the answers to difficult questions concerning tradeoffs.

This clearly leaves room for a more formal approach to structure preferences. But where can “measurement” be involved in the process? It

should be observed that, beyond surface, there are many analogies between the even swaps process and the measurement of length considered above.

First, note that, in both cases, objects are compared using binary relations. In the measurement of length, the binary relation  $\succ$  reads “is longer than”. Here it reads “is preferred to”. Similarly, the relation  $\sim$  reading before “has equal length” now reads “is indifferent to”. We supposed in the measurement of length process that  $\succ$  and  $\sim$  would nicely combine in experiments: if  $r \succ r'$  and  $r' \sim r''$  then we should observe that  $r \succ r''$ . Implicitly, a similar hypothesis was made in the even swaps technique. To realize that this is the case, it is worth summarizing the main steps of the argument.

We started with Table 1.1. Our overall recommendation was to rent office  $d$ . This means that we have reasons to believe that  $d$  is preferred to all other potential locations, i.e.  $d \succ a$ ,  $d \succ b$ ,  $d \succ c$ , and  $d \succ e$ . How did we arrive logically at such a conclusion?

Based on the initial table, using dominance and quasi-dominance, we concluded that  $b$  was preferable to  $e$  and that  $d$  was preferable to  $a$ . Using symbols, we have  $b \succ e$  and  $d \succ a$ . After assessing some tradeoffs, we concluded, using dominance, that  $b \succ c''$ . But remember,  $c''$  was built so as to be indifferent to  $c'$  and, in turn,  $c'$  was built so as to be indifferent to  $c$ . That is, we have  $c'' \sim c'$  and  $c' \sim c$ . Later, we built an alternative  $d'$  that is indifferent to  $d$  ( $d \sim d'$ ) and an alternative  $b'$  that is indifferent to  $b$  ( $b \sim b'$ ). We then concluded, using dominance, that  $d'$  was preferable to  $b'$  ( $d' \succ b'$ ). Hence, we know that:

$$\begin{aligned} d &\succ a, b \succ e, \\ c'' &\sim c', c' \sim c, b \succ c'', \\ d &\sim d', b \sim b', d' \succ b'. \end{aligned}$$

Using the consistency rules linking  $\succ$  and  $\sim$  that we considered for the measurement of length, it is easy to see that the last line implies  $d \succ b$ . Since  $b \succ e$ , this implies  $d \succ e$ . It remains to show that  $d \succ c$ . But the second line leads to, combining  $\succ$  and  $\sim$ ,  $b \succ c$ . Therefore  $d \succ b$  leads to  $d \succ c$  and we are home. Hence, we have used the same properties for preference and indifference as the properties of “is longer than” and “has equal length” that we hypothesized in the measurement of length.

Second it should be observed that expressing tradeoffs leads, indirectly, to equating the “length” of “preference intervals” on different attributes. Indeed, remember how  $c'$  was constructed above: saying that  $c$  and  $c'$  are indifferent more or less amounts to saying that the interval [25, 20] on commute time has exactly the same “length” as the interval [70, 78] on client access. Consider an alternative  $f$  that would be iden-



tical to  $c$  except that it has a client access at 78%. We may again ask which increase in client access would compensate a loss of 5 minutes on commute time. In a tabular form we are now comparing the following two alternatives:

	$f$	$f'$
Commute	20	25
Clients	78	$78 + \delta$
Services	C	C
Size	500	500
Cost	1500	1500

Suppose that the answer is that for  $\delta = 10$ ,  $f$  and  $f'$  would be indifferent. This means that the interval  $[25, 20]$  on commute time has exactly the same length as the interval  $[78, 88]$  on client access. Now, we know that the preference intervals  $[70, 78]$  and  $[78, 88]$  have the same “length”. Hence, tradeoffs provide a means to equate two preference intervals on the same attribute. This brings us quite close to the construction of standard sequences. This, we shall shortly do.

How does this information about the “length” of preference intervals relate to judgements of preference or indifference? Exactly as in the even swaps technique. You can use this measure of “length” modifying alternatives in such a way that they only differ on a single attribute and then use a simple dominance argument.

Conjoint measurement techniques may roughly be seen as a formalization of the even swaps technique that leads to building a numerical model of preferences much in the same way that we built a numerical model for length. This will require assessment procedures that will rest on the same principles as the standard sequence technique used for length. This process of “measuring preferences” is not an easy one. It will however lead to a numerical model of preference that will not only allow us to make a choice within a limited number of alternatives but that can serve as an input of computerized optimization algorithms that will be able to deal with much more complex cases.

## 2. Definitions and notation

Before entering into the details of how conjoint measurement may work, a few definitions and notation will be needed.

### 2.1. Binary relations

A *binary relation*  $\succsim$  on a set  $A$  is a subset of  $A \times A$ . We write  $a \succsim b$  instead of  $(a, b) \in \succsim$ . A binary relation  $\succsim$  on  $A$  is said to be:

- *reflexive* if  $[a \succsim a]$ ,

- *complete* if  $[a \succsim b \text{ or } b \succsim a]$ ,
- *symmetric* if  $[a \succsim b] \Rightarrow [b \succsim a]$ ,
- *asymmetric* if  $[a \succsim b] \Rightarrow [Not[b \succsim a]]$ ,
- *transitive* if  $[a \succsim b \text{ and } b \succsim c] \Rightarrow [a \succsim c]$ ,
- *negatively transitive* if  $[Not[a \succsim b] \text{ and } Not[b \succsim c]] \Rightarrow Not[a \succsim c]$ ,

for all  $a, b, c \in A$ .

The *asymmetric* (resp. *symmetric*) part of  $\succsim$  is the binary relation  $\succ$  (resp.  $\sim$ ) on  $A$  defined letting, for all  $a, b \in A$ ,  $a \succ b \Leftrightarrow [a \succsim b \text{ and } Not(b \succsim a)]$  (resp.  $a \sim b \Leftrightarrow [a \succsim b \text{ and } b \succsim a]$ ). A similar convention will hold when  $\succsim$  is subscripted and/or superscripted.

A *weak order* (resp. an *equivalence relation*) is a complete and transitive (resp. reflexive, symmetric and transitive) binary relation. For a detailed analysis of the use of binary relation as tools for preference modelling we refer to [4, 58, 66, 161, 167, 169]. The weak order model underlies the examples that were presented in the introduction. Indeed, the reader will easily prove the following.

**Proposition 9**

Let  $\succsim$  be a weak order on  $A$ . Then:

- $\succ$  is transitive,
- $\succ$  is negatively transitive,
- $\sim$  is transitive,
- $[a \succ b \text{ and } b \sim c] \Rightarrow a \succ c$ ,
- $[a \sim b \text{ and } b \succ c] \Rightarrow a \succ c$ ,

for all  $a, b, c \in A$ .

## 2.2. Binary relations on product sets

In the sequel, we consider a set  $X = \prod_{i=1}^n X_i$  with  $n \geq 2$ . Elements  $x, y, z, \dots$  of  $X$  will be interpreted as alternatives evaluated on a set  $N = \{1, 2, \dots, n\}$  of attributes. A typical binary relation on  $X$  is still denoted as  $\succsim$ , interpreted as an “at least as good as” preference relation between multi-attributed alternatives with  $\sim$  interpreted as indifference and  $\succ$  as strict preference.

For any nonempty subset  $J$  of the set of attributes  $N$ , we denote by  $X_J$  (resp.  $X_{-J}$ ) the set  $\prod_{i \in J} X_i$  (resp.  $\prod_{i \notin J} X_i$ ). With customary abuse of notation,  $(x_J, y_{-J})$  will denote the element  $w \in X$  such that  $w_i = x_i$

if  $i \in J$  and  $w_i = y_i$  otherwise. When  $J = \{i\}$  we shall simply write  $X_{-i}$  and  $(x_i, y_{-i})$ .

**Remark 10**

Throughout this paper, we shall work with a binary relation defined on a product set. This setup conceals the important work that has to be done in practice to make it useful:

- the structuring of objectives [3, 15, 16, 117, 118, 119, 157, 163],
- the definition of adequate attributes to measure the attainment of objectives [80, 96, 116, 122, 173, 208, 216],
- the definition of an adequate family of attributes [24, 121, 173, 174, 209],
- the modelling of uncertainty, imprecision and inaccurate determination [23, 27, 121, 171].

The importance of this “preliminary” work should not be forgotten in what follows. •

### 2.3. Independence and marginal preferences

In conjoint measurement, one starts with a preference relation  $\succsim$  on  $X$ . It is then of vital importance to investigate how this information makes it possible to define preference relations on attributes or subsets of attributes.

Let  $J \subseteq N$  be a nonempty set of attributes. We define the *marginal relation*  $\succsim_J$  induced on  $X_J$  by  $\succsim$  letting, for all  $x_J, y_J \in X_J$ :

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with asymmetric (resp. symmetric) part  $\succ_J$  (resp.  $\sim_J$ ). When  $J = \{i\}$ , we often abuse notation and write  $\succsim_i$  instead of  $\succsim_{\{i\}}$ . Note that if  $\succsim$  is reflexive (resp. transitive), the same will be true for  $\succsim_J$ . This is clearly not true for completeness however.

**Definition 11 (Independence)**

Consider a binary relation  $\succsim$  on a set  $X = \prod_{i=1}^n X_i$  and let  $J \subseteq N$  be a nonempty subset of attributes. We say that  $\succsim$  is independent for  $J$  if, for all  $x_J, y_J \in X_J$ ,

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If  $\succsim$  is independent for all nonempty subsets of  $N$ , we say that  $\succsim$  is independent. If  $\succsim$  is independent for all subsets containing a single attribute, we say that  $\succsim$  is weakly independent.

In view of (1.2), it is clear that the additive value model will require that  $\succsim$  is independent. This crucial condition says that common evaluations on some attributes do not influence preference. Whereas independence implies weak independence, it is well-known that the converse is not true [211].

**Remark 12**

Under certain conditions, e.g. when  $X$  is adequately “rich”, it is not necessary to test that a weak order  $\succsim$  is independent for  $J$ , for *all*  $J \subseteq N$  in order to know that  $\succsim$  is independent, see [21, 89, 121]. This is often useful in practice. •

**Remark 13**

Weak independence is referred to as “weak separability” in [211]; in section 5, we use “weak separability” (and “separability”) with a different meaning. •

**Remark 14**

Independence, or at least weak independence, is an almost universally accepted hypothesis in multiple criteria decision making. It cannot be overemphasized that it is easy to find examples in which it is inadequate.

If a meal is described by the two attributes, main course and wine, it is highly likely that most gourmets will violate independence, preferring red wine with beef and white wine with fish. Similarly, in a dynamic decision problem, a preference for variety will often lead to violating independence: you may prefer Pizza to Steak, but your preference for meals today (first attribute) and tomorrow (second attribute) may well be such that (Pizza, Steak) preferred to (Pizza, Pizza), while (Steak, Pizza) is preferred to (Steak, Steak).

Many authors [119, 173, 209] have argued that such failures of independence were almost always due to a poor structuring of attributes (e.g. in our choice of meal example above, preference for variety should be explicitly modelled). •

When  $\succsim$  is a weakly independent weak order, marginal preferences are well-behaved and combine so as to give meaning to the idea of dominance that we already encountered. The proof of the following is left to the reader as an easy exercise.

**Proposition 15**

Let  $\succsim$  be a weakly independent weak order on  $X = \prod_{i=1}^n X_i$ . Then:

- $\succsim_i$  is a weak order on  $X_i$ ,
- $[x_i \succsim_i y_i, \text{ for all } i \in N] \Rightarrow x \succsim y$ ,

▪  $[x_i \succsim_i y_i, \text{ for all } i \in N \text{ and } x_j \succ_j y_j \text{ for some } j \in N] \Rightarrow x \succ y,$   
for all  $x, y \in X$ .

### 3. The additive value model in the “rich” case

The purpose of this section and the following is to present the conditions under which a preference relation on a product set may be represented by the additive value function model (1.2) and how such a model can be assessed. We begin here with the case that most closely resembles the measurement of length described in section 1.2.

#### 3.1. Outline of theory

When the structure of  $X$  is supposed to be “adequately rich”, conjoint measurement is a quite clever adaptation of the process that we described in section 1.2 for the measurement of length. What will be measured here are the “length” of preference intervals on an attribute using a preference interval on another attribute as a standard.

**3.1.1 The case of two attributes.** Consider first the two attribute case. Hence the relation  $\succsim$  is defined on a set  $X = X_1 \times X_2$ . Clearly, in view of (1.2), we need to suppose that  $\succsim$  is an *independent weak order*. Consider two levels  $x_1^0, x_1^1 \in X_1$  on the first attribute such that  $x_1^1 \succ_1 x_1^0$ , i.e.  $x_1^1$  is preferable to  $x_1^0$ . This makes sense because, we supposed that  $\succsim$  is *independent*. Note also that we shall have to exclude the case in which all levels on the first attribute would be indifferent in order to be able to find such levels.

Choose any  $x_2^0 \in X_2$ . The, arbitrarily chosen, element  $(x_1^0, x_2^0) \in X$  will be our “reference point”. The basic idea is to use this reference point and the “unit” on the first attribute given by the reference preference interval  $[x_1^0, x_1^1]$  to build a standard sequence on the preference intervals on the second attribute. Hence, we are looking for an element  $x_2^1 \in X_2$  that would be such that:

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0). \quad (1.3)$$

Clearly this will require the structure of  $X_2$  to be adequately “rich” so as to find the level  $x_2^1 \in X_2$  such that the reference preference interval on the first attribute  $[x_1^0, x_1^1]$  is exactly matched by a preference interval of the same “length” on the second attribute  $[x_2^0, x_2^1]$ . Technically, this calls for a solvability assumption or, more restrictively, for the supposition that  $X_2$  has a (topological) structure that is close to that of an interval of  $\mathbb{R}$  and that  $\succsim$  is “somehow” continuous.

If such a level  $x_2^1$  can be found, model (1.2) implies:

$$\begin{aligned} v_1(x_1^0) + v_2(x_2^1) &= v_1(x_1^1) + v_2(x_2^0) \text{ so that} \\ v_2(x_2^1) - v_2(x_2^0) &= v_1(x_1^1) - v_1(x_1^0). \end{aligned} \tag{1.4}$$

Let us fix the origin of measurement letting:

$$v_1(x_1^0) = v_2(x_2^0) = 0,$$

and our unit of measurement letting:

$$v_1(x_1^1) = 1 \text{ so that } v_1(x_1^1) - v_1(x_1^0) = 1.$$

Using (1.4), we therefore obtain  $v_2(x_2^1) = 1$ . We have therefore found an interval between levels on the second attribute ( $[x_2^0, x_2^1]$ ) that exactly matches our reference interval on the first attribute ( $[x_1^0, x_1^1]$ ). We may proceed to build our standard sequence on the second attribute (see Figure 1.4) asking for levels  $x_2^2, x_2^3, \dots$  such that:

$$\begin{aligned} (x_1^0, x_2^2) &\sim (x_1^1, x_2^1), \\ (x_1^0, x_2^3) &\sim (x_1^1, x_2^2), \\ &\dots \\ (x_1^0, x_2^k) &\sim (x_1^1, x_2^{k-1}). \end{aligned}$$

As above, using (1.2) leads to:

$$\begin{aligned} v_2(x_2^2) - v_2(x_2^1) &= v_1(x_1^1) - v_1(x_1^0), \\ v_2(x_2^3) - v_2(x_2^2) &= v_1(x_1^1) - v_1(x_1^0), \\ &\dots \\ v_2(x_2^k) - v_2(x_2^{k-1}) &= v_1(x_1^1) - v_1(x_1^0), \end{aligned}$$

so that:

$$v_2(x_2^2) = 2, v_2(x_2^3) = 3, \dots, v_2(x_2^k) = k.$$

This process of building a standard sequence of the second attribute therefore leads to defining  $v_2$  on a number of, carefully, selected elements of  $X_2$ .

Remember the standard sequence that we built for length in section 1.2. An implicit hypothesis was that the length of any rod could be exceeded by the length of a composite object obtained by concatenating a sufficient number of perfect copies of a standard rod. Such an hypothesis is called ‘‘Archimedean’’ since it mimics the property of the real numbers

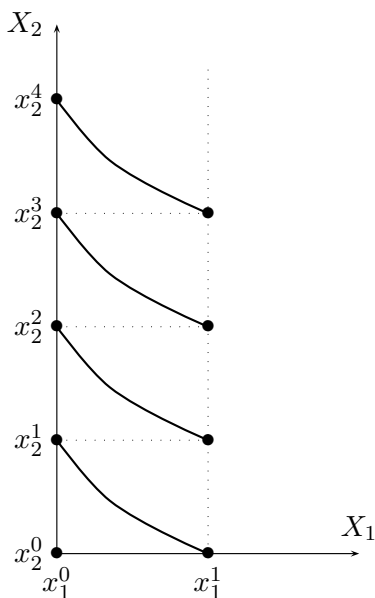


Figure 1.4. Building a standard sequence on  $X_2$ .

saying that for any positive real numbers  $x, y$  it is true that  $nx > y$  for some integer  $n$ , i.e.  $y$ , no matter how large, may always be exceeded by taking any  $x$ , no matter how small, and adding it with itself and repeating the operation a sufficient number of times. Clearly, we will need a similar hypothesis here. Failing it, there might exist a level  $y_2 \in X_2$  that will never be “reached” by our standard sequence, i.e. such that  $y_2 \succ_2 x_2^k$ , for  $k = 1, 2, \dots$ . For measurement models in which this Archimedean condition is omitted, see [155, 193].

**Remark 16**

At this point a good exercise for the reader is to figure out how we may extend the standard sequence to cover levels of  $X_2$  that are “below” the reference level  $x_2^0$ . This should not be difficult. •

Now that a standard sequence is built on the second attribute, we may use any part of it to build a standard sequence on the first attribute.

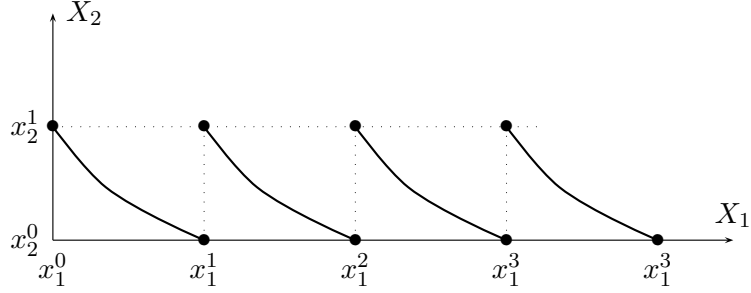


Figure 1.5. Building a standard sequence on  $X_1$ .

This will require finding levels  $x_1^2, x_1^3, \dots \in X_1$  such that (see Figure 1.5):

$$\begin{aligned} (x_1^2, x_2^0) &\sim (x_1^1, x_2^1), \\ (x_1^3, x_2^0) &\sim (x_1^2, x_2^1), \\ &\dots \\ (x_1^k, x_2^0) &\sim (x_1^{k-1}, x_2^1). \end{aligned}$$

Using (1.2) leads to:

$$\begin{aligned} v_1(x_1^2) - v_1(x_1^1) &= v_2(x_2^1) - v_2(x_2^0), \\ v_1(x_1^3) - v_1(x_1^2) &= v_2(x_2^1) - v_2(x_2^0), \\ &\dots \\ v_1(x_1^k) - v_1(x_1^{k-1}) &= v_2(x_2^1) - v_2(x_2^0), \end{aligned}$$

so that:

$$v_1(x_1^2) = 2, v_1(x_1^3) = 3, \dots, v_1(x_1^k) = k.$$

As was the case for the second attribute, the construction of such a sequence will require the structure of  $X_1$  to be adequately rich, which calls for a solvability assumption. An Archimedean condition will also be needed in order to be sure that all levels of  $X_1$  can be reached by the sequence.

We have defined a “grid” in  $X$  (see Figure 1.6) and we have  $v_1(x_1^k) = k$  and  $v_2(x_2^k) = k$  for all elements of this grid. Intuitively such numerical assignments seem to define an adequate additive value function on the grid. We have to prove that this intuition is correct. Let us first verify that, for all integers  $\alpha, \beta, \gamma, \delta$ :

$$\alpha + \beta = \gamma + \delta = \epsilon \Rightarrow (x_1^\alpha, x_2^\beta) \sim (x_1^\gamma, x_2^\delta). \quad (1.5)$$



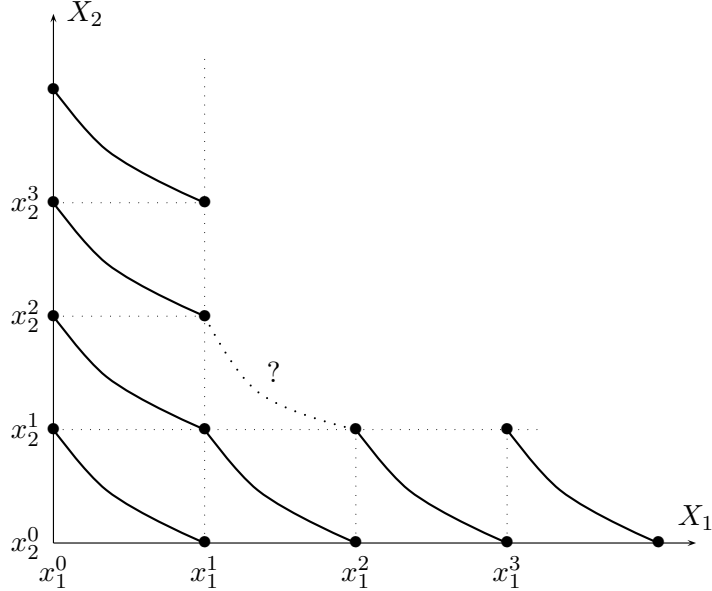


Figure 1.6. The grid.

When  $\epsilon = 1$ , (1.5) holds by construction because we have:  $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$ . When  $\epsilon = 2$ , we know that  $(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$  and  $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$  and the claim is proved using the transitivity of  $\sim$ .

Consider the  $\epsilon = 3$  case. We have  $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$  and  $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$ . It remains to be shown that  $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$  (see the dotted arc in Figure 1.6). This does not seem to follow from the previous conditions that we more or less explicitly used: transitivity, independence, “richness”, Archimedean. Indeed, it does not. Hence, we have to suppose that:  $(x_1^2, x_2^0) \sim (x_1^0, x_2^2)$  and  $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$  imply  $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$ . This condition, called the Thomsen condition, is clearly necessary for (1.2). The above reasoning easily extends to all points on the grid, using weak ordering, independence and the Thomsen condition. Hence, (1.5) holds on the grid.

It remains to show that:

$$\epsilon = \alpha + \beta > \epsilon' = \gamma + \delta \Rightarrow (x_1^\alpha, x_2^\beta) \succ (x_1^\gamma, x_2^\delta). \quad (1.6)$$

Using transitivity, it is sufficient to show that (1.6) holds when  $\epsilon = \epsilon' + 1$ . By construction, we know that  $(x_1^1, x_2^0) \succ (x_1^0, x_2^0)$ . Using independence this implies that  $(x_1^1, x_2^k) \succ (x_1^0, x_2^k)$ . Using (1.5) we have  $(x_1^1, x_2^k) \sim (x_1^{k+1}, x_2^0)$  and  $(x_1^0, x_2^k) \sim (x_1^k, x_2^0)$ . Therefore we have  $(x_1^{k+1}, x_2^0) \succ (x_1^k, x_2^0)$ , the desired conclusion.

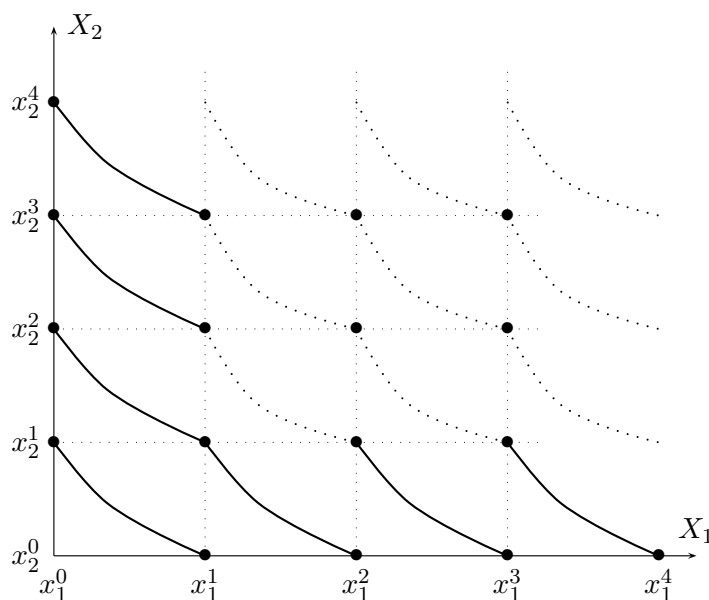


Figure 1.7. The entire grid.

Hence, we have built an additive value function of a suitably chosen grid (see Figure 1.7). The logic of the assessment procedure is then to assess more and more points somehow considering more finely grained standard sequences. The two techniques evoked for length may be used here depending on the underlying structure of  $X$ . Going to the limit then unambiguously defines the functions  $v_1$  and  $v_2$ . Clearly such  $v_1$  and  $v_2$  are intimately related. Once we have chosen an arbitrary reference point  $(x_1^0, x_2^0)$  and a level  $x_1^1$  defining the unit of measurement, the process just described entirely defines  $v_1$  and  $v_2$ . It follows that the only possible transformations that can be applied to  $v_1$  and  $v_2$  is to multiply both by the same positive number  $\alpha$  and to add to both a, possibly different, constant. This is usually summarized saying that  $v_1$  and  $v_2$  define interval scales with a common unit.

The above reasoning is a rough sketch of the proof of the existence of an additive value function when  $n = 2$ , as well as a sketch of how it could be assessed. Careful readers will want to refer to [58, 129, 211].

**Remark 17**

The measurement of length through standard sequences described above leads to a scale that is unique once the unit of measurement is chosen. At this point, a good exercise for the reader is to find an intuitive ex-

planation to the fact that, when measuring the “length” of preference intervals, the origin of measurement becomes arbitrary. The analogy with the the measurement of duration on the one hand and dates, as given in a calendar, on the other hand should help. •

**Remark 18**

As was already the case with the even swaps technique, it is worth emphasizing that this assessment technique makes no use of the vague notion of the “importance” of the various attributes. The “importance” is captured here in the lengths of the preference intervals on the various attributes.

A common but critical mistake is to confuse the additive value function model (1.2) with a weighted average and to try to assess weights asking whether an attribute is “more important” than another. This makes no sense. •

**3.1.2 The case of more than two attributes.** The good news is that the process is exactly the same when there are more than two attributes. With one surprise: the Thomsen condition is no more needed to prove that the standard sequences defined on each attribute lead to an adequate value function on the grid. A heuristic explanation of this strange result is that, when  $n = 2$ , there is no difference between independence and weak independence. This is no more true when  $n \geq 3$  and assuming independence is much stronger than just assuming weak independence.

**3.2. Statement of results**

We use below the “algebraic approach” [127, 129, 141]. A more restrictive approach using a topological structure on  $X$  is given in [44, 58, 211]. We formalize below the conditions informally introduced in the preceding section. The reader not interested in the precise statement of the results or, better, having already written down his own statement, may skip this section.

**Definition 19 (Thomsen condition)**

Let  $\succsim$  be a binary relation on a set  $X = X_1 \times X_2$ . It is said to satisfy the Thomsen condition if

$$(x_1, x_2) \sim (y_1, y_2) \text{ and } (y_1, z_2) \sim (z_1, x_2) \Rightarrow (x_1, z_2) \sim (z_1, y_2),$$

for all  $x_1, y_1, z_1 \in X_1$  and all  $x_2, y_2, z_2 \in X_2$ .

Figure 1.8 shows how the Thomsen condition uses two “indifference curves” (i.e. curves linking points that are indifferent) to place a constraint on a third one. This was needed above to prove that an additive

value function existed on our grid. Remember that the Thomsen condition is only needed when  $n = 2$ ; hence, we only stated it in this case.

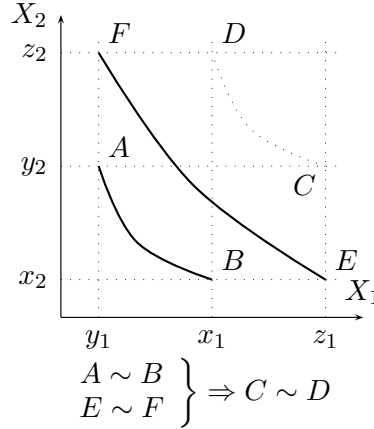


Figure 1.8. The Thomsen condition.

### Definition 20 (Standard sequences)

A standard sequence on attribute  $i \in N$  is a set  $\{a_i^k : a_i^k \in X_i, k \in K\}$  where  $K$  is a set of consecutive integers (positive or negative, finite or infinite) such that there are  $x_{-i}, y_{-i} \in X_{-i}$  satisfying  $\text{Not}[x_{-i} \sim_{-i} y_{-i}]$  and  $(a_i^k, x_{-i}) \sim (a_i^{k+1}, y_{-i})$ , for all  $k \in K$ .

A standard sequence on attribute  $i \in N$  is said to be *strictly bounded* if there are  $b_i, c_i \in X_i$  such that  $b_i \succ_i a_i^k \succ_i c_i$ , for all  $k \in K$ . It is then clear that, when model (1.2) holds, any strictly bounded standard sequence must be finite.

### Definition 21 (Archimedean)

For all  $i \in N$ , any strictly bounded standard sequence on  $i \in N$  is finite.

The following condition rules out the case in which a standard sequence cannot be built because all levels are indifferent.

### Definition 22 (Essentiality)

Let  $\succsim$  be a binary relation on a set  $X = X_1 \times X_2 \times \cdots \times X_n$ . Attribute  $i \in N$  is said to be *essential* if  $(x_i, a_{-i}) \succ (y_i, a_{-i})$ , for some  $x_i, y_i \in X_i$  and some  $a_{-i} \in X_{-i}$ .

### Definition 23 (Restricted Solvability)

Let  $\succsim$  be a binary relation on a set  $X = X_1 \times X_2 \times \cdots \times X_n$ . Restricted solvability is said to hold with respect to attribute  $i \in N$  if, for all  $x \in$

$X$ , all  $z_{-i} \in X_{-i}$  and all  $a_i, b_i \in X_i$ ,  $[(a_i, z_{-i}) \succsim x \succsim (b_i, z_{-i})] \Rightarrow [x \sim (c_i, z_{-i}), \text{ for some } c_i \in X_i]$ .

**Remark 24**

Restricted solvability is illustrated in Figure 1.9 in the case where  $n = 2$ . It says that, given any  $x \in X$ , if it is possible find two levels  $a_i, b_i \in X_i$  such that when combined with a certain level  $z_{-i} \in X_{-i}$  on the other attributes,  $(a_i, z_{-i})$  is preferred to  $x$  and  $x$  is preferred to  $(b_i, z_{-i})$ , it should be possible to find a level  $c_i$ , “in between”  $a_i$  and  $b_i$ , such that  $(c_i, z_{-i})$  is exactly indifferent to  $x$ .

A much stronger hypothesis is unrestricted solvability asserting that for all  $x \in X$  and all  $z_{-i} \in X_{-i}$ ,  $x \sim (c_i, z_{-i})$ , for some  $c_i \in X_i$ . Its use leads however to much simpler proofs [58, 86].

It is easy to imagine situations in which restricted solvability might hold while unrestricted solvability would fail. Suppose, e.g. that a firm has to choose between several investment projects, two attributes being the Net Present Value (NPV) of the projects and their impact on the image of the firm in the public. Consider a project consisting in investing in the software market. It has a reasonable NPV and no adverse consequences on the image of the firm. Consider another project that could have dramatic consequences on the image of the firm, because it leads to investing the market of cocaine. Unrestricted solvability would require that by sufficiently increasing the NPV of the second project it would become indifferent to the more standard project of investing in the software market. This is not required by restricted solvability. •

We are now in position to state the central results concerning model (1.2). Proofs may be found in [129, 213].

**Theorem 25 (Additive value function when  $n = 2$ )**

Let  $\succsim$  be a binary relation on a set  $X = X_1 \times X_2$ . If restricted solvability holds on all attributes and each attribute is essential then  $\succsim$  has a representation in model (1.2) if and only if  $\succsim$  is an independent weak order satisfying the Thomsen and the Archimedean conditions

Furthermore in this representation,  $v_1$  and  $v_2$  are interval scales with a common unit, i.e. if  $v_1, v_2$  and  $w_1, w_2$  are two pairs of functions satisfying (1.2), there are real numbers  $\alpha, \beta_1, \beta_2$  with  $\alpha > 0$  such that, for all  $x_1 \in X_1$  and all  $x_2 \in X_2$

$$v_1(x_1) = \alpha w_1(x_1) + \beta_1 \text{ and } v_2(x_2) = \alpha w_2(x_2) + \beta_2.$$

When  $n \geq 3$  and at least three attributes are essential, the above result simplifies in that the Thomsen condition can now be omitted.

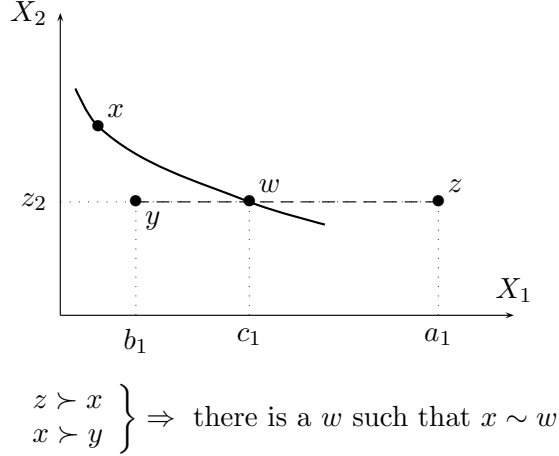


Figure 1.9. Restricted Solvability on  $X_1$ .

**Theorem 26 (Additive value function when  $n \geq 3$ )**

Let  $\succsim$  be a binary relation on a set  $X = X_1 \times X_2 \times \dots \times X_n$  with  $n \geq 3$ . If restricted solvability holds on all attributes and at least 3 attributes are essential then  $\succsim$  has a representation in model (1.2) if and only if  $\succsim$  is an independent weak order satisfying the Archimedean condition.

Furthermore in this this representation  $v_1, v_2, \dots, v_n$  are interval scales with a common unit.

**Remark 27**

As mentioned in introduction, the additive value model is central to several fields in decision theory. It is therefore not surprising that much energy has been devoted to analyze variants and refinements of the above results. Among the most significant ones, let us mention:

- the study of cases in which solvability holds only on some or none of the attributes [75, 85, 86, 87, 88, 112, 113, 154],
- the study of the relation between the “algebraic approach” introduced above and the topological one used in [44], see e.g. [115, 124, 211, 213].

The above results are only valid when  $X$  is the entire Cartesian product of the sets  $X_i$ . Results in which  $X$  is a subset of the whole Cartesian product  $X_1 \times X_2 \times \dots \times X_n$  are not easy to obtain, see [37, 181] (the situation is “easier” in the special case of homogeneous product sets, see [214, 215]).

•

### 3.3. Implementation: Standard sequences and beyond

We have already shown above how additive value functions can be assessed using the standard sequence technique. It is worth recalling here some of the characteristics of this assessment procedure:

- It requires the set  $X_i$  to be *rich* so that it is possible to find a preference interval on  $X_i$  that will exactly match a preference interval on another attribute. This excludes using such an assessment procedure when some of the sets  $X_i$  are discrete.
- It relies on *indifference* judgements which, a priori, are less firmly established than preference judgements.
- It relies on judgements concerning fictitious alternatives which, a priori, are harder to conceive than judgements concerning real alternatives.
- The various assessments are thoroughly intertwined and, e.g., an imprecision on the assessment of  $x_2^1$ , i.e. the endpoint of the first interval in the standard sequence on  $X_2$  (see Figure 1.4) will propagate to many assessed values,
- The assessment of tradeoffs may be plagued with cognitive biases, see [46, 197].

The assessment procedure based on standard sequences is therefore rather demanding; this should be no surprise given the proximity between this form of measurement and extensive measurement illustrated above on the case of length. Hence, the assessment procedure based on standard sequences seems to be seldom used in the practice of decision analysis [121, 209]. The literature on the experimental assessment of additive value functions, see e.g. [197, 208, 216], suggests that this assessment is a difficult task that may be affected by several cognitive biases.

Many other simplified assessment procedures have been proposed that are less firmly grounded in theory. In many of them, the assessment of the partial value functions  $v_i$  relies on *direct* comparison of preference differences without recourse to an interval on another attribute used as a “meter stick”. We refer to [50] for a theoretical analysis of these techniques. They are also studied in detail in Chapter XXX of this volume (Chapter 6, Dyer).

These procedures include:

- *direct rating* techniques in which values of  $v_i$  are directly assessed with reference to two arbitrarily chosen points [52, 53],

- procedures based on *bisection*, the decision-maker being asked to assess a point that is “half way” in terms of preference two reference points [209],
- procedures trying to build *standard sequences* on each attribute in terms of “preference differences” [129, ch. 4].

An excellent overview of these techniques may be found in [209].

## 4. The additive value model in the “finite” case

### 4.1. Outline of theory

In this section, we suppose that  $\succsim$  is a binary relation on a finite set  $X \subseteq X_1 \times X_2 \times \cdots \times X_n$  (contrary to the preceding section, dealing with subsets of product sets will raise no difficulty here). The finiteness hypothesis clearly invalidates the standard sequence mechanism used till now. On each attribute there will only be finitely many “preference intervals” and exact matches between preference intervals will only happen exceptionally, see [212].

Clearly, independence remains a necessary condition for model (1.2) as before. Given the absence of structure of the set  $X$ , it is unlikely that this condition is sufficient to ensure (1.2). The following example shows that this intuition is indeed correct.

#### Example 28

Let  $X = X_1 \times X_2$  with  $X_1 = \{a, b, c\}$  and  $X_2 = \{d, e, f\}$ . Consider the weak order on  $X$  such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ af \succ be \succ cd \succ ce \succ bf \succ cf.$$

It is easy to check that  $\succsim$  is independent. Indeed, we may for instance check that:

$$\begin{aligned} ad \succ bd \text{ and } ae \succ be \text{ and } af \succ bf, \\ ad \succ ae \text{ and } bd \succ be \text{ and } cd \succ ce. \end{aligned}$$

This relation cannot however be represented in model (1.2) since:

$$\begin{aligned} af \succ be &\Rightarrow v_1(a) + v_2(f) > v_1(b) + v_2(e), \\ be \succ cd &\Rightarrow v_1(b) + v_2(e) > v_1(c) + v_2(d), \\ ce \succ bf &\Rightarrow v_1(c) + v_2(e) > v_1(b) + v_2(f), \\ bd \succ ae &\Rightarrow v_1(b) + v_2(d) > v_1(a) + v_2(e). \end{aligned}$$

Summing the first two inequalities leads to:

$$v_1(a) + v_2(f) > v_1(c) + v_2(d).$$



Summing the last two inequalities leads to:

$$v_1(c) + v_2(d) > v_1(a) + v_2(f),$$

a contradiction.

Note that, since no indifference is involved, the Thomsen condition is trivially satisfied. Although it is clearly necessary for model (1.2), adding it to independence will therefore not solve the problem.  $\diamond$

The conditions allowing to build an additive value model in the finite case were investigated in [1, 2, 179]. Although the resulting conditions turn out to be complex, the underlying idea is quite simple. It amounts to finding conditions under which a system of linear inequalities has a solution.

Suppose that  $x \succ y$ . If model (1.2) holds, this implies that:

$$\sum_{i=1}^n v_i(x_i) > \sum_{i=1}^n v_i(y_i). \quad (1.7)$$

Similarly if  $x \sim y$ , we obtain:

$$\sum_{i=1}^n v_i(x_i) = \sum_{i=1}^n v_i(y_i). \quad (1.8)$$

The problem is then to find conditions on  $\succsim$  such that the system of finitely many equalities and inequalities (1.7-1.8) has a solution. This is a classical problem in Linear Algebra [83].

**Definition 29 (Relation  $E^m$ )**

Let  $m$  be an integer  $\geq 2$ . Let  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m \in X$ . We say that

$$(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m)$$

if, for all  $i \in N$ ,  $(x_i^1, x_i^2, \dots, x_i^m)$  is a permutation of  $(y_i^1, y_i^2, \dots, y_i^m)$ .

Suppose that  $(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m)$  then model (1.2) implies that

$$\sum_{j=1}^m \sum_{i=1}^n v_i(x_i^j) = \sum_{j=1}^m \sum_{i=1}^n v_i(y_i^j).$$

Therefore if  $x^j \succsim y^j$  for  $j = 1, 2, \dots, m-1$ , it cannot be true that  $x^m \succ y^m$ . This condition must hold for all  $m = 2, 3, \dots$

**Definition 30 (Condition  $C^m$ )**

Let  $m$  be an integer  $\geq 2$ . We say that condition  $C^m$  holds if

$$[x^j \succsim y^j \text{ for } j = 1, 2, \dots, m-1] \Rightarrow \text{Not}[x^m \succ y^m]$$

for all  $x^1, x^2, \dots, x^m, y^1, y^2, \dots, y^m \in X$  such that

$$(x^1, x^2, \dots, x^m) E^m (y^1, y^2, \dots, y^m).$$

**Remark 31**

It is not difficult to check that:

- $C^{m+1} \Rightarrow C^m$ ,
- $C^2 \Rightarrow \succsim$  is independent,
- $C^3 \Rightarrow \succsim$  is transitive. •

We already observed that  $C^m$  was implied by the existence of an additive representation. The main result for the finite case states that requiring that  $\succsim$  is complete and that  $C^m$  holds for  $m = 2, 3, \dots$  is also sufficient. Proofs can be found in [58, 129].

**Theorem 32**

Let  $\succsim$  be a binary relation on a finite set  $X \subseteq X_1 \times X_2 \times \dots \times X_n$ . There are real-valued functions  $v_i$  on  $X_i$  such that (1.2) holds if and only if  $\succsim$  is complete and satisfies  $C^m$  for  $m = 2, 3, \dots$

**Remark 33**

Contrary to the “rich” case considered in the preceding section, we have here necessary and sufficient conditions for the additive value model (1.2). However, it is important to notice that the above result uses a denumerable scheme of conditions. It is shown in [180] that this denumerable scheme cannot be truncated: for all  $m \geq 2$ , there is a relation  $\succsim$  on a finite set  $X$  such that  $C^m$  holds but violating  $C^{m+1}$ . This is studied in more detail in [139, 201, 218]. Therefore, no finite scheme of axioms is sufficient to characterize model (1.2) for all finite sets  $X$ .

Given a finite set  $X$  of given cardinality, it is well-known that the denumerable scheme of condition can be truncated. The precise relation between the cardinality of  $X$  and the number of conditions needed raises difficult combinatorial questions that are studied in [77, 78]. •

**Remark 34**

It is clear that, if a relation  $\succsim$  has a representation in model (1.2) with functions  $v_i$ , it also has a representation using functions  $v'_i = \alpha v_i + \beta_i$  with  $\alpha > 0$ . Contrary to the rich case, the uniqueness of the functions  $v_i$  is more complex as shown by the following example.

**Example 35**

Let  $X = X_1 \times X_2$  with  $X_1 = \{a, b, c\}$  and  $X_2 = \{d, e\}$ . Consider the weak order on  $X$  such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ cd \succ be \succ ce.$$

This relation has a representation in model (1.2) with

$$v_1(a) = 3, v_1(b) = 1, v_1(c) = 0, v_2(d) = 3, v_2(e) = 0.5.$$

An equally valid representation would be given taking  $v_1(b) = 2$ . Clearly this new representation cannot be deduced from the original one applying a positive affine transformation.  $\diamond$

**Remark 36**

Theorem 32 has been extended to the case of an arbitrary set  $X$  in [113, 112], see also [75, 81]. The resulting conditions are however quite complex. This explains why we spent time on this “rich” case in the preceding section.  $\bullet$

**Remark 37**

The use of a denumerable scheme of conditions in theorem 32 does not facilitate the interpretation and the test of conditions. However it should be noticed that, on a given set  $X$ , the test of the  $C^m$  conditions amounts to finding if a system of finitely many linear inequalities has a solution. It is well-known that Linear Programming techniques are quite efficient for such a task.  $\bullet$

**4.2. Implementation: LP-based assessment**

We show how to use LP techniques in order to assess an additive value model (1.2), without supposing that the sets  $X_i$  are rich. For practical purposes, it is not restrictive to assume that we are only interested in assessing a model for a limited range on each  $X_i$ . We therefore assume that the sets  $X_i$  are bounded so that, using independence, there is a worst value  $x_{i*}$  and a most preferable value  $x_i^*$ . Using the uniqueness properties of model (1.2), we may always suppose, after an appropriate normalization, that:

$$v_1(x_{1*}) = v_2(x_{2*}) = \dots = v_n(x_{n*}) = 0 \text{ and} \quad (1.9)$$

$$\sum_{i=1}^n v_i(x_i^*) = 1. \quad (1.10)$$

Two main cases arise (see Figures 1.10 and 1.11):

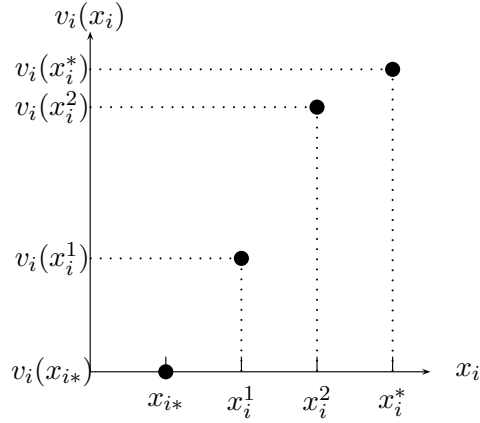


Figure 1.10. Value function when  $X_i$  is discrete.

- attribute  $i \in N$  is discrete so that the evaluation of any conceivable alternative on this attribute belongs to a finite set. We suppose that  $X_i = \{x_{i*}, x_i^1, x_i^2, \dots, x_i^{r_i}, x_i^*\}$ . We therefore have to assess  $r_i + 1$  values of  $v_i$ ,
- the attribute  $i \in N$  has an underlying continuous structure. It is hardly restrictive in practice to suppose that  $X_i \subset \mathbb{R}$ , so that the evaluation of an alternative on this attribute may take any value between  $x_{i*}$  and  $x_i^*$ . In this case, we may opt for the assessment of a piecewise linear approximation of  $v_i$  partitioning the set  $X_i$  in  $r_i + 1$  intervals and supposing that  $v_i$  is linear on each of these intervals. Note that the approximation of  $v_i$  can be made more precise simply by increasing the number of these intervals.

With these conventions, the assessment of the model (1.2) amounts to giving a value to  $\sum_{i=1}^n (r_i + 1)$  unknowns. Clearly any judgment of preference linking  $x$  and  $y$  translate into a *linear inequality* between these unknowns. Similarly any judgment of indifference linking  $x$  and  $y$  translate into a *linear equality*. Linear Programming (LP) offers a powerful tool for testing whether such a system has solutions. Therefore, an assessment procedure can be conceived on the following basis:

- obtain judgments in terms of preference or indifference linking several alternatives in  $X$ ,
- convert these judgments into linear (in)equalities,
- test, using LP, whether this system has a solution.

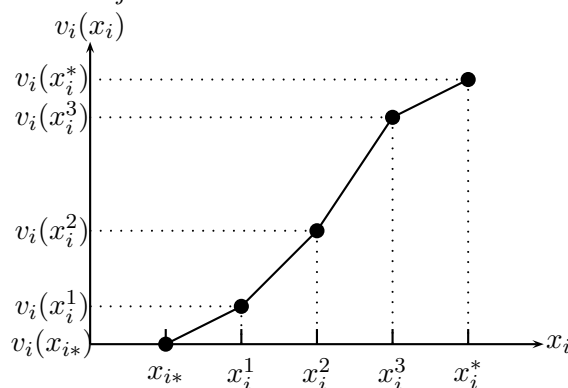


Figure 1.11. Value function when  $X_i$  is continuous.

If the system has no solution then one may either propose a solution that will be “as close as possible” from the information obtained, e.g. violating the minimum number of (in)equalities or suggest the reconsideration of certain judgements. If the system has a solution, one may explore the set of all solutions to this system since they are all candidates for the establishment of model (1.2). These various techniques depend on:

- the choice of the alternatives in  $X$  that are compared: they may be real or fictitious, they may differ on a different number of attributes,
- the way to deal with the inconsistency of the system and to eventually propose some judgments to be reconsidered,
- the way to explore the set of solutions of the system and to use this set as the basis for deriving a prescription.

Linear programming offers of simple and versatile technique to assess additive value functions. All restrictions generating linear constraints of the coefficient of the value function can easily be accommodated. This idea has been often exploited, see [16]. We present below two techniques using it. It should be noticed that rather different techniques have been proposed in the literature on Marketing [35, 103, 104, 114, 132].

**4.2.1 UTA [111].** UTA (“UTilité Additive”, i.e. additive utility in French) is one of the oldest techniques belonging to this family. It is supposed in UTA that there is a subset  $Ref \subset X$  of reference alternatives that the decision-maker knows well either because he/she has experienced them or because they have received particular attention. The technique amounts to asking the DM to provide a weak order on

*Ref.* Each preference or indifference relation contained in this weak order is then translated into a linear constraint:

- $x \sim y$  gives an equality  $v(x) - v(y) = 0$  and
- $x \succ y$  gives an inequality  $v(x) - v(y) > 0$ ,

where  $v(x)$  and  $v(y)$  can be expressed as a linear combination of the unknowns as remarked earlier. Strict inequalities are then translated into large inequalities as is usual in Linear Programming, i.e.  $v(x) - v(y) > 0$  becomes  $v(x) - v(y) \geq \epsilon$  where  $\epsilon > 0$  is a very small positive number that should be chosen according to the precision of the arithmetics used by the LP package.

The test of the existence of a solution to the system of linear constraints is done via standard Goal Programming techniques [36] adding appropriate deviation variables. In UTA, each equation  $v(x) - v(y) = 0$  is translated into an equation  $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- = 0$ , where  $\sigma_x^+, \sigma_x^-, \sigma_y^+$  and  $\sigma_y^-$  are nonnegative deviation variables. Similarly each inequality  $v(x) - v(y) \geq \epsilon$  is written as  $v(x) - v(y) + \sigma_x^+ - \sigma_x^- + \sigma_y^+ - \sigma_y^- \geq \epsilon$ . It is clear that there will exist a solution to the original system of linear constraints if there is a solution of the LP in which all deviation variables are zero. This can easily be tested using the objective function

$$\text{Minimize } Z = \sum_{x \in Ref} \sigma_x^+ + \sigma_x^- \quad (1.11)$$

Two cases arise. If the optimal value of  $Z$  is 0, there is an additive value function that represents the preference information. It should be observed that, except in exceptional cases (e.g. if the preference information collected is identical to the preference information collected with the standard sequence technique), there are infinitely many such additive value functions (that are not related via a simple change of origin and of unit, since we already fixed them through normalization (1.9-1.10)). The one given as the “optimal” one by the LP does not have a special status since it is highly dependent upon the arbitrary choice of the objective function; instead of minimizing the sum of the deviation variables, we could have as well, and still preserving linearity, minimized the largest of these variables. The whole polyhedron of feasible solutions of the original (in)equalities corresponds to adequate additive value functions: we have a whole set  $\mathcal{V}$  of additive value functions representing the information collected on the set of reference alternatives *Ref*.

The size of  $\mathcal{V}$  is clearly dependent upon the choice of the alternatives in *Ref*. Using standard techniques in LP, several functions in  $\mathcal{V}$  may be obtained, e.g. the ones maximizing or minimizing, within  $\mathcal{V}$ ,  $v_i(x_i^*)$

for each attribute [111]. It is often interesting to present them to the decision-maker in the pictorial form of Figures 1.10 and 1.11.

If the optimal value of  $Z$  is strictly greater than 0, there is no additive value function representing the preference information available. The solution given as optimal (note that it is not guaranteed that this solution leads to the minimum possible number of violations w.r.t. the information provided—this would require solving an integer linear programme) is, in general, highly dependent upon the choice of the objective function.

This absence of solution to the system might be due to several factors:

- the piecewise linear approximation of the  $v_i$  for the “continuous” attributes may be too rough. It is easy to test whether an increase in the number of linear pieces on some of these attributes may lead to a nonempty set of additive value functions.
- the information provided by the decision-maker may be of poor quality. It might then be interesting to present to the decision-maker one additive value function (e.g. one may present an average function after some post-optimality analysis) in the pictorial form of Figures 1.10 and 1.11 and to let him react to this information either by modifying his/her initial judgments or even by letting him/her react directly on the shape of the value functions. This is the solution implemented in the well-known PREFCALC system [109].
- the preference provided by the decision-maker might be inconsistent with the conditions implied by an additive value function. The system should then help locate these inconsistencies and allow the DM to think about them. Alternatively, since many alternative attribute descriptions are possible, it may be worth investigating whether a different definition of the various attributes may lead to a preference model consistent with model (1.2). Several examples of such analysis may be found in [119, 121, 209]

When the above techniques fail, the optimal solution of the LP, even if not compatible with the information provided, may still be considered as an adequate model. Again, since the objective function introduced above is somewhat arbitrary and it is recommended in [111] to perform a post-optimality analysis, e.g. considering additive value functions that are “close” to the optimal solution through the introduction of a linear constraint:

$$Z \leq Z^* + \delta,$$

where  $Z^*$  is the optimal value of the objective function of the original LP and  $\delta$  is a “small” positive number. As above, the result of the analysis is

a set  $\mathcal{V}$  of additive value functions defined by a set of linear constraints. A representative sample of additive value functions within  $\mathcal{V}$  may be obtained as above.

It should be noted that many possible variants of UTA can be conceived building on the following comments. They include:

- the addition of monotonicity properties of the  $v_i$  with respect to the underlying continuous attributes,
- the addition of constraints on the shape of the marginal value functions  $v_i$ , e.g. requiring them to be concave, convex or S-shaped,
- the addition of constraints linked to a possible indication of preference intensity for the elements of  $Ref$  given by the DM, e.g. the difference between  $x$  and  $y$  is larger than the difference between  $z$  and  $w$ .

For applications of UTA-like techniques, we refer to [38, 47, 48, 105, 110, 148, 185, 186, 187, 188, 189, 190, 192, 195, 196, 219, 221, 220, 223, 222]. Variants of the method are considered in [19, 20, 191]. This method is also studied in detail in chapter XXX of this volume (Chapter 9, Siskos, UTA).

**4.2.2 MACBETH [12].** It is easy to see that (1.9) and (1.10) may equivalently be written as:

$$x \succsim y \Leftrightarrow \sum_{i=1}^n k_i u_i(x_i) \geq \sum_{i=1}^n k_i u_i(y_i), \quad (1.12)$$

where

$$u_1(x_{1*}) = u_2(x_{2*}) = \dots u_n(x_{n*}) = 0, \quad (1.13)$$

$$u_1(x_1^*) = u_2(x_2^*) = \dots u_n(x_n^*) = 1 \text{ and} \quad (1.14)$$

$$\sum_{i=1}^n k_i = 1. \quad (1.15)$$

With such an expression of an additive value function, it is tempting to break down the assessment into two distinct parts: a value function  $u_i$  is assessed on each attribute and, then, scaling constants  $k_i$  are assessed taking the shape of the value functions  $u_i$  as given. This is the path followed in MACBETH.

**Remark 38**

Again, note that we are speaking here of  $k_i$  as *scaling constants* and not as *weights*. As already mentioned weights that would reflect the ‘‘importance’’ of attributes are irrelevant to assess the additive value function



model. Notice that, under (1.12-1.15) the ordering of the scaling constant  $k_i$  is dependent upon the choice of  $x_{i*}$  and  $x_i^*$ . Increasing the width of the interval  $[x_{i*}, x_i^*]$  will lead to increasing the value of the scaling constant  $k_i$ . The value  $k_i$  has, therefore, nothing to do with the “importance” of attribute  $i$ . This point is unfortunately too often forgotten when using a weighted average of some numerical attributes. In the latter model, changing the units in which the attributes are measured should imply changing the “weights” accordingly. •

The assessment procedure of the  $u_i$  is conceived in such a way as to avoid comparing alternatives differing on more than one attribute. In view of what was said before concerning the standard sequence technique, this is clearly an advantage of the technique. But can it be done? The trick here is that MACBETH asks for judgments related to the difference between the desirability of alternatives and not only judgments in terms of preference or indifference. Partial value functions  $u_i$  are approximated in a similar way than in UTA: for discrete attributes, each point on the function is assessed, for continuous ones, a piecewise linear approximation is used.

MACBETH asks the DM to compare pairs of levels on each attribute. If no difference is felt between these levels, they receive an identical partial value level. If a difference is felt between  $x_i^k$  and  $x_i^r$ , MACBETH asks for a judgment qualifying the strength of this difference. The method and the associated software propose three different semantical categories:

Categories	Description
C1	weak
C2	strong
C3	extreme

with the possibility of using intermediate categories, i.e. between null and weak, weak and strong, strong and extreme (giving a total of six distinct categories). This information is then converted into linear inequations using the natural interpretation that if the “difference” between the levels  $x_i^k$  and  $x_i^r$  has been judged larger than the “difference” between  $x_i^{k'}$  and  $x_i^{r'}$  then it should follow that  $u_i(x_i^k) - u_i(x_i^r) > u_i(x_i^{k'}) - u_i(x_i^{r'})$ . Technically the six distinct categories are delimited by thresholds that are used in the establishment of the constraints of the LP. The software associated to MACBETH offers the possibility to compare all pairs of levels on each attribute for a total of  $(r_i + 1)r_i/2$  comparisons. Using standard Goal Programming techniques, as in UTA, the test of the compatibility of a partial value function with this information is performed via the solution of a LP. If there is a partial value function compatible with the information, a “central” function is proposed to the DM who

has the possibility to modify it. If not, the results of the LP are exploited in such a way to propose modifications of the information that would make it consistent.

The assessment of the scaling constant  $k_i$  is done using similar principles. The DM is asked to compare the following  $(n + 2)$  alternatives by pairs:

$$\begin{aligned} &(x_{1*}, x_{2*}, \dots, x_{n*}), \\ &(x_1^*, x_{2*}, \dots, x_{n*}), \\ &(x_{1*}, x_2^*, \dots, x_{n*}), \\ &\quad \dots \\ &(x_{1*}, x_{2*}, \dots, x_n^*) \text{ and} \\ &(x_1^*, x_2^*, \dots, x_n^*), \end{aligned}$$

placing each pair in a category of difference. This information immediately translates into a set of linear constraints on the  $k_i$ . These constraints are processed as before. It should be noticed that, once the partial value functions  $u_i$  are assessed, it is not necessary to use the levels  $x_{i*}$  and  $x_i^*$  to assess the  $k_i$  since they may well lead to alternatives that are too unrealistic. The authors of MACBETH suggest to replace  $x_{i*}$  by a “neutral” level which appears neither desirable nor undesirable and  $x_i^*$  by a “desirable” level that is judged satisfactory. Although this clearly impacts the quality of the dialogue with the DM, this has no consequence on the underlying technique used to process information.

We refer to [6, 7, 8, 9, 10, 11] for applications of the MACBETH technique. This method is also studied in detail in chapter XXX of this volume (Chapter 10, Bana, MACBETH).

## 5. Extensions

The additive value model (1.2) is the central model for the application of conjoint measurement techniques to decision analysis. In this section, we consider various extensions to this model.

### 5.1. Transitive Decomposable models

The transitive decomposable model has been introduced in [129] as a natural generalization of model (1.2). It amounts to replacing the addition operation by a general function that is increasing in each of its arguments.

#### **Definition 39 (Transitive decomposable model)**

*Let  $\succsim$  be a binary relation on a set  $X = \prod_{i=1}^n X_i$ . The transitive decomposable model holds if, for all  $i \in N$ , there is a real-valued function  $v_i$*

on  $X_i$  and a real-valued function  $g$  on  $\prod_{i=1}^n v_i(X_i)$  that is increasing in all its arguments such that:

$$x \succsim y \Leftrightarrow g(v_1(x_1), \dots, v_n(x_n)) \geq g(v_1(y_1), \dots, v_n(y_n)), \quad (1.16)$$

for all  $x, y \in X$ .

An interesting point with this model is that it admits an intuitively appealing simple characterization. The basic axiom for characterizing the above transitive decomposable model is weak independence, which is clearly implied by (1.16). The following theorem is proved in [129, ch. 7].

**Theorem 40**

*A preference relation  $\succsim$  on a finite or countably infinite set  $X$  has a representation in the transitive decomposable model iff  $\succsim$  is a weakly independent weak order.*

**Remark 41**

This result can be extended to sets of arbitrary cardinality adding a, necessary, condition implying that the weak order  $\succsim$  has a numerical representation, see [42, 45]. •

The weak point of such a model is that the function  $g$  is left unspecified so that the model will be difficult to assess. Furthermore, the uniqueness results for  $v_i$  and  $g$  are clearly much less powerful than what we obtained with model (1.2), see [129, ch. 7]. Therefore, practical applications of this model generally imply specifying the type of function  $g$ , possibly by verifying further conditions on the preference relation that impose that  $g$  belongs to some parameterized family of functions, e.g. some polynomial function of the  $v_i$ . This is studied in detail in [129, ch. 7] and [14, 82, 139, 138, 156, 166, 202]. Since such models have, to the best of our knowledge, never been used in decision analysis, we do not analyze them further.

The structure of the decomposable model however suggests that assessment techniques for this model could well come from Artificial Intelligence with its “rule induction” machinery. Indeed the function  $g$  in model (1.16) may also be seen as a set of “rules”. We refer to [97, 98, 100, 101] for a thorough study of the potentiality of such an approach.

**Remark 42**

A simple extension of the decomposable model consists in simply asking for a function  $g$  that would be nondecreasing in each of its arguments. The following result is proved in [30] (see also [100]) (it can easily be extended to cover the case of an arbitrary set  $X$ , adding a, necessary, condition implying that  $\succsim$  has a numerical representation).

We say that  $\succsim$  is weakly separable if, for all  $i \in N$  and all  $x_i, y_i \in X_i$ , it is never true that  $(x_i, z_{-i}) \succ (y_i, z_{-i})$  and  $(y_i, w_{-i}) \succ (x_i, w_{-i})$ , for some  $z_{-i}, w_{-i} \in X_{-i}$ . Clearly this is a weakening of weak independence since it tolerates to have at the same time  $(x_i, z_{-i}) \succ (y_i, z_{-i})$  and  $(x_i, w_{-i}) \sim (y_i, w_{-i})$ .

**Theorem 43**

*A preference relation  $\succsim$  on a finite or countably infinite set  $X$  has a representation in the weak decomposable model:*

$$x \succsim y \Leftrightarrow g(u_1(x_1), \dots, u_n(x_n)) \geq g(u_1(y_1), \dots, u_n(y_n))$$

*with  $g$  nondecreasing in all its arguments iff  $\succsim$  is a weakly separable weak order.*

A recent trend of research has tried to characterize special functional forms for  $g$  in the weakly decomposable model, such as max, min or some more complex forms. The main references include [26, 100, 102, 182, 194]. •

**Remark 44**

The use of “fuzzy integrals” as tools for aggregating criteria has recently attracted much attention [49, 90, 91, 93, 94, 95, 143, 145, 144, 146], the Choquet Integral and the Sugeno integral being among the most popular. It should be strongly emphasized that the very definition of these integrals requires to have at hand a weak order on  $\cup_{i=1}^n X_i$ , supposing w.l.o.g. that the sets  $X_i$  are disjoint. This is usually called a “commensurability hypothesis”. Whereas this hypothesis is quite natural when dealing with an homogeneous Cartesian product, as in decision under uncertainty (see e.g. [211]), it is far less so in the area of multiple criteria decision making. A neat conjoint measurement analysis of such models and their associated assessment procedures is an open research question, see [92]. •

**5.2. Intransitive indifference**

Decomposable models form a large family of preferences though not large enough to encompass all cases that may be encountered when asking subjects to express preferences. A major restriction is that not all preferences may be assumed to be weak orders. The example of the sequence of cups of coffee, each differing from the previous one by an imperceptible quantity of sugar added [133], is famous; it leads to the

notions of semiorder and interval order [4, 57, 66, 133, 161], in which indifference is not transitive, while strict preference is.

Ideally, taking intransitive indifference into account, we would want to arrive at a generalization of (1.2) in which:

$$\begin{aligned}x \sim y &\Leftrightarrow |V(x) - V(y)| \leq \epsilon, \\x \succ y &\Leftrightarrow V(x) > V(y) + \epsilon,\end{aligned}$$

where  $\epsilon \geq 0$  and  $V(x) = \sum_{i=1}^n v_i(x_i)$ .

In the finite case, it is not difficult to extend the conditions presented in section 4 to cover such a case. Indeed, we are still looking here for the solution to a system of linear constraints. Although this seems to have never been done, it would not be difficult to adapt the LP-based assessment techniques to this case.

On the contrary, extending the standard sequence technique of section 3 is a formidable challenge. Indeed, remember that these techniques crucially rest on indifference judgments which lead to the determination of “perfect copies” of a given preference interval. As soon as indifference is not supposed to be transitive, “perfect copies” are not so perfect and much trouble is expected. We refer to [84, 128, 134, 161, 198] for a study of these models.

**Remark 45**

Even if the analysis of such models proves difficult, it should be noted that the semi-ordered version of the additive value model may be interpreted as having a “built-in” sensitivity analysis via the introduction of the threshold  $\epsilon$ . Therefore, in practice, we may usefully view  $\epsilon$  not as a parameter to be assessed but as a simple trick to avoid undue discrimination, because of the imprecision inevitably involved in our assessment procedures, between close alternatives •

**Remark 46**

Clearly the above model can be generalized to cope with a possibly non-constant threshold. The literature on the subject remains minimal however, see [161]. •

### 5.3. Nontransitive preferences

Many authors [147, 203] have argued that the reasonableness of supposing that strict preference is transitive is not so strong when it comes to comparing objects evaluated on several attributes. As soon as it is supposed that subjects may use an “ordinal” strategy for comparing objects, examples inspired from the well-known Condorcet paradox [176, 183]

show that intransitivities will be difficult to avoid. Indeed it is possible to observe predictable intransitivities of strict preference in carefully controlled experiments [203]. There may therefore be a descriptive interest to studying such models. When it comes to decision analysis, intransitive preferences are often dismissed on two grounds:

- on a practical level, it is not easy to build a recommendation on the basis of a binary relation in which  $\succ$  would not be transitive. Indeed, social choice theorists, facing a similar problem, have devoted much effort to devising what could be called reasonable procedures to deal with such preferences [41, 62, 130, 131, 149, 158, 178]. This literature does not lead, as was expected, to the emergence of a single suitable procedure in all situations.
- on a more conceptual level, many others have questioned the very rationality of such preferences using some version of the famous “money pump” argument [137, 164].

P. C. Fishburn has forcefully argued [73] that these arguments might not be as decisive as they appear at first sight. Furthermore some MCDM techniques make use of such intransitive models, most notably the so-called outranking methods [25, 172, 204, 205] and chapters XXX in this volume. Besides the intellectual challenge, there might therefore be a real interest in studying such models.

A. Tversky [203] was one of the first to propose such a model generalizing (1.2), known as the *additive difference model*, in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^n \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \quad (1.17)$$

where  $\Phi_i$  are increasing and odd functions.

It is clear that (1.17) allows for intransitive  $\succsim$  but implies its completeness. Clearly, (1.17) implies that  $\succsim$  is independent. This allows to unambiguously define marginal preferences  $\succsim_i$ . Although model (1.17) can accommodate intransitive  $\succsim$ , a consequence of the increasingness of the  $\Phi_i$  is that the marginal preference relations  $\succsim_i$  are weak orders. This, in particular, excludes the possibility of any perception threshold on each attribute which would lead to an intransitive indifference relation on each attribute. Imposing that  $\Phi_i$  are nondecreasing instead of being increasing allows for such a possibility. This gives rise to what is called the “weak additive difference model” in [22].

As suggested in [22, 70, 69, 72, 206], the subtractivity requirement in (1.17) can be relaxed. This leads to *nontransitive additive* conjoint

measurement models in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^n p_i(x_i, y_i) \geq 0 \quad (1.18)$$

where the  $p_i$  are real-valued functions on  $X_i^2$  and may have several additional properties (e.g.  $p_i(x_i, x_i) = 0$ , for all  $i \in \{1, 2, \dots, n\}$  and all  $x_i \in X_i$ ).

This model is an obvious generalization of the (weak) additive difference model. It allows for intransitive and incomplete preference relations  $\succsim$  as well as for intransitive and incomplete marginal preferences  $\succsim_i$ . An interesting specialization of (1.18) obtains when  $p_i$  are required to be *skew symmetric* i.e. such that  $p_i(x_i, y_i) = -p_i(y_i, x_i)$ . This skew symmetric nontransitive additive conjoint measurement model implies that  $\succsim$  is complete and independent.

An excellent overview of these nontransitive models is [73]. Several axiom systems have been proposed to characterize them. P. C. Fishburn gave [70, 69, 72] axioms for the skew symmetric version of (1.18) both in the finite and the infinite case. Necessary and sufficient conditions for a nonstandard version of (1.18) are presented in [76]. [206] gives axioms for (1.18) with  $p_i(x_i, x_i) = 0$  when  $n \geq 4$ . [22] gives necessary and sufficient conditions for (1.18) with and without skew symmetry in the denumerable case when  $n = 2$ .

The additive difference model (1.17) was axiomatized in [74] in the infinite case when  $n \geq 3$  and [22] gives necessary and sufficient conditions for the weak additive difference model in the finite case when  $n = 2$ . Related studies of nontransitive models include [39, 64, 136, 153]. The implications of these models for decision-making under uncertainty were explored in [71] (for a different path to nontransitive models for decision making under risk and/or uncertainty, see [65, 67]).

It should be noticed that even the weakest form of these models, i.e. (1.18) without skew symmetry, involves an addition operation. Therefore it is unsurprising that the axiomatic analysis of these models share some common features with the additive value function model (1.2). Indeed, except in the special case in which  $n = 2$ , this case relating more to ordinal than to conjoint measurement (see [72]), the various axiom systems that have been proposed involve either:

- a denumerable set of cancellation conditions in the finite case or,
- a finite number of cancellation conditions together with unnecessary structural assumptions in the general case (these structural assumptions generally allow us to obtain nice uniqueness results

for (1.18): the functions  $p_i$  are unique up to the multiplication by a common positive constant).

A different path to the analysis of nontransitive conjoint measurement models has recently been proposed in [30, 29, 31]. In order to get a feeling for these various models, it is useful to consider the various strategies that are likely to be implemented when comparing objects differing on several dimensions [40, 151, 152, 175, 200, 203].

Consider two alternatives  $x$  and  $y$  evaluated on a family of  $n$  attributes so that  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$ .

A first strategy that can be used in order to decide whether or not it can be said that “ $x$  is at least as good as  $y$ ” consists in trying to measure the “worth” of each alternative on each attribute and then to combine these evaluations adequately. Giving up all idea of transitivity and completeness, this suggests a model in which:

$$x \succsim y \Leftrightarrow F(u_1(x_1), \dots, u_n(x_n), u_1(y_1), \dots, u_n(y_n)) \geq 0 \quad (1.19)$$

where  $u_i$  are real-valued functions on the  $X_i$  and  $F$  is a real-valued function on  $\prod_{i=1}^n u_i(X_i)^2$ . Additional properties on  $F$ , e.g. its nondecreasingness (resp. nonincreasingness) in its first (resp. last)  $n$  arguments, will give rise to a variety of models implementing this first strategy.

A second strategy relies on the idea of measuring “preference differences” separately on each attribute and then combining these (positive or negative) differences in order to know whether the aggregation of these differences leads to an advantage for  $x$  over  $y$ . More formally, this suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \dots, p_n(x_n, y_n)) \geq 0 \quad (1.20)$$

where  $p_i$  are real-valued functions on  $X_i^2$  and  $G$  is a real-valued function on  $\prod_{i=1}^n p_i(X_i^2)$ . Additional properties on  $G$  (e.g. its oddness or its nondecreasingness in each of its arguments) or on  $p_i$  (e.g.  $p_i(x_i, x_i) = 0$  or  $p_i(x_i, y_i) = -p_i(y_i, x_i)$ ) will give rise to a variety of models in line with the above strategy.

Of course these two strategies are not incompatible and one may well consider using the “worth” of each alternative on each attribute to measure “preference differences”. This suggests a model in which:

$$x \succsim y \Leftrightarrow H(\phi_1(u_1(x_1), u_1(y_1)), \dots, \phi_n(u_n(x_n), u_n(y_n))) \geq 0 \quad (1.21)$$

where  $u_i$  are real-valued functions on  $X_i$ ,  $\phi_i$  are real-valued functions on  $u_i(X_i)^2$  and  $H$  is a real-valued function on  $\prod_{i=1}^n \phi_i(u_i(X_i)^2)$ .

The use of general functional forms, instead of additive ones, greatly facilitate the axiomatic analysis of these models. It mainly relies on the



study of various kinds of *traces* induced by the preference relation on coordinates and does not require a detailed analysis of tradeoffs between attributes.

The price to pay for such an extension of the scope of conjoint measurement is that the number of parameters that would be needed to assess such models is quite high. Furthermore, none of them is likely to possess any remarkable uniqueness properties. Therefore, although proofs are constructive, these results will not give direct hints on how to devise assessment procedures. The general idea here is to use numerical representations as guidelines to understand the consequences of a limited number of cancellation conditions, without imposing any transitivity or completeness requirement on the preference relation and any structural assumptions on the set of objects. Such models have proved useful to:

- understand the ordinal character of some aggregation models proposed in the literature [170, 172], known as the “outranking methods” (see Chapters XXX of this Volume) as shown in [28],
- understand the links between aggregation models aiming at enriching a dominance relation and more traditional conjoint measurement approaches [30],
- to include in a classical conjoint measurement framework, noncompensatory preferences in the sense of [22, 33, 55, 60, 61] as shown in [28, 32, 99].

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