Binary relations and preference modelling

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1 Introduction

This volume is dedicated to concepts, results, procedures and software aiming at helping one or several person in making a decision. It is then natural to investigate how the various courses of action that are involved in this decision compare in terms of preference. The aim of this paper is to propose a brief survey of the main tools and results that can be useful to do so.

The literature on preference modelling is vast. This can fist be explained by the fact that the question of modelling preferences occurs in several disciplines, for instance:

- in *Economics* where one tries to model the preferences of a "rational consumer" (see, e.g, Debreu 1959),
- in *Psychology* in which the study of preference judgments collected in experiments is quite common (see Kahneman and Tversky 1979, Kahneman, Slovic, and Tversky 1981),
- in *Political Science* in which the question of defining a collective preference on the basis of the opinion of several voters is central (see Sen 1986),
- in *Operational Research* in which optimizing an objective function implies the definition of a direction of preference (seer Roy 1985),
- in *Artificial Intelligence* in which the creation of autonomous agents able to take decisions implies the modelling of their vision of what is desirable and what is less so (see Doyle and Wellman 1992)

Moreover, the question of preference modelling can be studied from a variety of perspectives (see Bell, Raiffa, and Tversky 1988), among which:

- a *normative* perspective one investigates preference models that are likely to lead to a "rational behavior",
- a *descriptive* perspective trying to find adequate models to capture judgements obtained in experiments,
- a *prescriptive* perspective in which one tries to build a preference model that is able to lead to an adequate recommendation.

Finally, the preferences that are to be modelled can be expressed on a variety of objects depending on the underlying decision problem. For instance, one may compare:

- vectors in $\mathbb{R}^p$ indicating the consumption of p perfectly divisible goods,
- candidates in an election,
- probability distributions modelling the possible financial results of various investment prospects,
- alternatives evaluated on several criteria expressed in incommensurable units when comparing sites for a new factory,
- projects evaluated on a monetary scale conditionally on the occurrence of various events or on the actions of other players.
- etc.

It would be impossible within the scope of this introductory paper to exhaustively summarize the immense literature on the subject. More realistically, we will try here to present in a simple way the main concepts used in building models of preference. This will give the reader the necessary background to tackle the remaining chapters in this book. The reader willing to deepen his understanding of the subject is referred to Aleskerov, Bouyssou, and Monjardet (2006), Fishburn (1970, 1985) Krantz, Luce, Suppes, and Tversky (1971), Pirlot and Vincke (1997), Roberts (1979), or Roubens and Vincke (1985).

This paper is organized as follows. Section 2 is devoted to the concept of *binary relation* since this is the central tool in most models of preference. Section 3 defines a "preference structure". Section 4 introduces two classical preference structures: complete orders and weak orders. Sections 5 and 6 introduce several more general preference structures. Section 7 concludes with the mention of several important questions that we could not tackle here.

2 Binary relations

2.1 Definitions

A binary relation T on a set A is a subset of the Cartesian product $A \times A$, i.e., a set of ordered pairs (a, b) of elements of A. If the ordered pair (a, b) belongs to the set T, we will often write $a\ T\ b$ instead of $(a, b) \in T$. In the opposite case, we write $(a, b) \notin T$ or $a \neg T\ b$. Except when explicitly mentioned otherwise, we will suppose in all what follows that the set A is *finite*.

Remark 1
Since binary relations are sets, we can apply to them the classical operations of set theory. For instance, given any two binary relations T_1 and T_2 on A, we will write:

$$\begin{aligned} T_1 \subset T_2 &\text{ iff } a\ T_1\ b \Rightarrow a\ T_2\ b, \forall a, b \in A, \\ a\ (T_1 \cup T_2)\ b &\text{ iff } a\ T_1\ b \text{ or } a\ T_2\ b, \\ a\ (T_1 \cap T_2)\ b &\text{ iff } a\ T_1\ b \text{ and } a\ T_2\ b. \end{aligned}$$

Moreover the *product* $T_1 \cdot T_2$ will be defined by:

$$a\ T_1 \cdot T_2\ b \text{ iff } \exists c \in A : a\ T_1\ c \text{ and } c\ T_2\ b.$$

We denote by T^2 the relation $T \cdot T$, i.e., the product of the relation T with itself. •

Given a binary relation T on A, we define:

- its inverse relation T^- such that:

$$a\ T^-\ b \text{ iff } b\ T\ a,$$

- its complement, i.e., the binary relation T^c such that:

$$a\ T^c\ b \text{ iff } a\ \neg T\ b,$$

- its dual relation T^d such that:

$$a\ T^d\ b \text{ iff } b\ \neg T\ a,$$

- its symmetric part I_T such that:

$$a\ I_T\ b \text{ iff } [a\ T\ b \text{ and } b\ T\ a],$$

- its asymmetric part P_T such that:

$$a\ P_T\ b \text{ iff } [a\ T\ b \text{ and } b\ \neg T\ a],$$

- its associated equivalence relation E_T such that:

$$a\ E_T\ b \text{ iff } \left\{ \begin{array}{l} a\ T\ c \Leftrightarrow b\ T\ c, \\ c\ T\ a \Leftrightarrow c\ T\ b, \end{array} \right\}, \forall c \in A.$$

Remark 2
It is easy to check that we have:

$$\begin{aligned} T^d &= T^{-c} = T^{c-}, \\ I_T &= T \cap T^-, \\ P_T &= T \cap T^d. \end{aligned}$$

•

2.2 Properties of a binary relation

A binary relation T on A is said to be:

- *reflexive* if $a\ T\ a$,
- *irreflexive* if $a \neg T\ a$,
- *symmetric* if $a\ T\ b \Rightarrow b\ T\ a$,
- *antisymmetric* if $a\ T\ b$ and $b\ T\ a \Rightarrow a = b$,
- *asymmetric* if $a\ T\ b \Rightarrow b \neg T\ a$,
- *weakly complete* if $a \neq b \Rightarrow a\ T\ b$ or $b\ T\ a$,
- *complete* if $a\ T\ b$ or $b\ T\ a$,
- *transitive* if $a\ T\ b$ and $b\ T\ c \Rightarrow a\ T\ c$,
- *negatively transitive* if $a \neg T\ b$ and $b \neg T\ c \Rightarrow a \neg T\ c$,
- de *Ferrers* if $[a\ T\ b$ and $c\ T\ d] \Rightarrow [a\ T\ d$ or $c\ T\ d]$,
- *semi-transitive* if $[a\ T\ b$ and $b\ T\ c] \Rightarrow [a\ T\ d$ or $d\ T\ c]$,

for all $a, b, c, d \in A$.

Remark 3
The above properties are not independent. For instance, it is easy to check that:

- a relation is asymmetric iff it is irreflexive and antisymmetric,

- a relation is complete iff it is weakly complete and reflexive,
- an asymmetric and negatively transitive relation is transitive,
- a complete and transitive relation is negatively transitive.

Whatever the properties of T, it is clear that:

- P_T is always asymmetric,
- I_T is always symmetric,
- E_T is always reflexive, symmetric and transitive. •

Remark 4
It is possible to reformulate the above properties in a variety of ways. For instance, observe that: que:

- T is complete $\Leftrightarrow T \cup T^{-} = A \times A$,
- T is asymmetric $\Leftrightarrow T \cap T^{-} = \varnothing$,
- T is transitive $\Leftrightarrow T^2 \subset T$,
- T is Ferrers $\Leftrightarrow T \cdot T^d \cdot T \subset T$,
- T is semi-transitive $\Leftrightarrow T \cdot T \cdot T^d \subset T$. •

An *equivalence* is a reflexive, symmetric and transitive binary relation (hence, the binary relation E_T defined earlier is an equivalence whatever the properties of T). Let E be an equivalence on A. Given an element $a \in A$, the equivalence class associated to a, denoted by $[a]_E$, is the set $\{b \in A : a\ E\ b\}$. It is always true that $a \in [a]_E$. It is easy to show that $\forall a, b \in A$, either $[a]_E = [b]_E$ or $[a]_E \cap [b]_E = \varnothing$. An equivalence therefore partitions A into *equivalence classes*. The set of all these equivalence classes is called the quotient A for E and is denoted A/E.

2.3 Graphical representation of a binary relation

A binary relation T on A can be represented as a directed graph (A, T) where A is the set of vertices of the graph and T is the set of the arcs of the graph. (i.e., ordered pair of vertices). The particular properties of a binary relation can easily be interpreted using the sagittal representation of the graph (A, T). The reflexivity of T implies the presence of a loop on each vertex. The symmetry of T means that when there is an arc going from a to

b, there is also an arc going from b vers a. The transitivity of T means that as soon as there is path of length 2 going from a to b, there is an arc from a vers b. Taking the inverse relation is tantamount to inverting the orientation of all arc in the graph. Taking the complement consists in adding all missing arcs and deling all existing ones. Observe that a symmetric relation can be more conveniently represented using a non-oriented graph, in which the ordered pairs (a, b) and (b, a) of the relation are represented using a single edge between the vertices a and b.

2.4 Matrix representation of a binary relation

Another way to represent a binary relation T on A is to associate to each element of de A a row and a column of a square matrix M^T of dimension $|A|$. The element M^T_{ab} of this matrix being at the intersection of the row associated to a and to the column associated to b is 1 if $a\ T\ b$ and 0 otherwise.

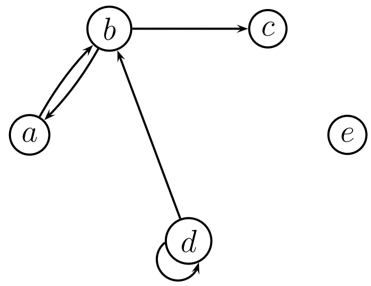
With such a representation, the reflexivity of T implies the presence of 1 on the diagonal of the matrix, provided that the elements of A have been associated in the order to the row and columns of the matrix. Under this hypothesis, the symmetry of T is equivalent to the fact that M^T is equal to its transpose. Taking the inverse relation consists in transposing the matrix M^T. The matrix associated to the product of two binary relations is the boolean matrix product of the two corresponding matrices.

2.5 Example

Let $A = \{a, b, c, d, e\}$. Consider the binary relation $T = \{(a, b), (b, a), (b, c), (d, b), (d, d)\}$. A matrix representation of T is the following:

↻	a	b	c	d	e
a	0	1	0	0	0
b	1	0	1	0	0
c	0	0	0	0	0
d	0	1	0	1	0
e	0	0	0	0	0

A sagittal representation of the graph (A, T) is:



3 Binary relations and preference structures

Consider an ordered pair (a, b) of objects. It is classically supposed that there can only be two types of answer to the question "is object a at least as good as object b?": "YES" or "NO", these two answers being exclusive. Asking such a question for all ordered pais of objects leads to defining a *binary relation* S on the set A of all objects letting $a\ S\ b$ if and only if the answer to the question "is a at least as good as b?" is YES. In view of its definition, it is natural to consider that S is reflexive; we will do so in all what follows.

Definition 1
A preference structure *on* A *is a reflexive binary relation* S *on* A*.*

Remark 5
The preceding definition raises a question of *observability.* If the idea of preference is to be based on observable behavior, the primitive may be taken to be choices made on various subsets of objects. This change of primitive is at the heart of "revealed preference" theory in which the relation S is inferred from choice that are observable. Such an inference requires that choices are essentially "binary", i.e., that choices made on pairs of objects are sufficient to infer choice made on larger sets of objects. The condition allowing such a rationalization of a choice function through a binary relation are classical (see, e.g., Sen 1970, 1977). They have recently been severely questioned (see Malishevski 1993, Sen 1993, Sugden 1985). •

Remark 6
In some cases, one may envisage other answers that YES or NO to the question "is a at least as good as b?", for instance:

- answers such that "I do not know",
- answers including information on the *intensity* of the preference, e.g., "a is strongly — weakly, moderately — preferred to b",

- answers including information on the *credibility* of the proposition "a is at least as good as b", e.g., "the credibility of the 'a is least as good as b' is greater than the credibility of the proposition 'c is at least as good as d'" or even "the credibility of the proposition 'a is at least as good as b' is $\alpha \in [0; 1]$".

Admitting such answers implies using a language that is richer than that of binary relations, for instance:

- the language of *fuzzy relations* (see Doignon, Monjardet, Roubens, and Vincke 1986, Fodor and Roubens 1994, Perny and Roy 1992), each assertion of the type $a\ S\ b$ having a *degree of credibility*,
- languages tolerating hesitation (see, e.g.,, Roy and Vincke 1987)
- languages using the idea of *intensity of preference* (see Bana e Costa and Vansnick 1994, Doignon 1987), an assertion such that $a\ S\ b$ and $b\ \neg S\ a$ being further qualified (weak, strong or extreme preference, for instance) or,
- languages making use of *non-classical logics* (see Tsoukiàs and Vincke 1992, 1995, 1997)) allowing to capture the absence of information or, on the contrary, the existence of contradictory information; with such languages, the truth value of the assertion $a\ S\ b$ can take values different from just "true" or "false" and include "unknown" and "contradictory".

We do not consider such extensions in this paper. •

Let us consider a preference S on a set A. For all pair of objects $\{a, b\}$, we are in one of the following four situations (see Figure 1):

1. $[a\ S\ b$ and $b\ S\ a]$, denoted by $a\ I_S\ b$, interpreted as "a is *indifferent* to b",
2. $[a\ \neg S\ b$ and $b\ \neg S\ a]$, denoted by $a\ J_S\ b$, interpreted as "a is *incomparable* to b",
3. $[a\ S\ b$ and $b\ \neg S\ a]$, denoted by $a\ P_S\ b$, interpreted as "a is *strictly preferred* to b" and
4. $[a\ \neg S\ b$ and $b\ S\ a]$, denoted by $b\ P_S\ a$, interpreted as "b is *strictly preferred* to a".

	$b\ S\ a$	$b\ \neg S\ a$
$a\ S\ b$	$a\ I\ b$	$a\ P\ b$
$a\ \neg S\ b$	$b\ P\ a$	$a\ J\ b$

Figure 1: Four exhaustive and mutually exclusive situations

When there is no risk of ambiguity, we use I, J and P instead of I_S, J_S and P_S. By construction, I and J are symmetric and P is asymmetric. Since S is reflexive, I is reflexive and J is irreflexive. The three relations P, I and I are:

- mutually exclusive, i.e., $P \cap I = P \cap J = I \cap J = \varnothing$ and
- exhaustive, i.e., $P \cup P^{-} \cup I \cup J = A^2$.

Remark 7
Many works use $\succsim$ instead of S, $\succ$ instead of P and $\sim$ instead of I. •

Remark 8
Given a preference structure on S sur A, it may be useful to consider the relation induced by S on the quotient set A/E_S, where E_S denotes the equivalence associated to S. This allows to simplify many results. •

Remark 9
A preference structure being a reflexive binary relation, we can use the graphical and matrix representations introduced earlier to represent it. In order to simplify graphical representations, we will systematically omit reflexivity loops and we will use the conventions introduced in Figure 2

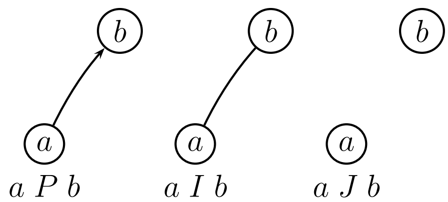


Figure 2: Graphical conventions

Example 1
Let $A = \{a, b, c, d, e\}$ and the preference structure $S = \{$ (a, a), (a, b), (a, c), (a, e), (b, a), (b, b), (b, c), (c, b), (c, c), (d, a), (d, b), (d, c), (d, d), (e, a), (e, c), (e, e), $\}$. We have:

- $P = \{(a,c),(d,a),(d,b),(d,c),(e,c)\}$,
- $I = \{(a,a),(a,b),\ (a,e),\ (b,a),\ (b,b),\ (b,c),\ (c,b),\ (c,c),\ (d,d),\ (e,a),\ (e,e)\}$,
- $J = \{(b,e),(d,e),(e,b),(e,d)\}$.

Using the above conventions, we obtain the matrix representation (Figure 3) and the graphical representation (Figure 4) of T. ◇

↻	a	b	c	d	e
a	1	1	1	0	1
b	1	1	1	0	0
c	0	1	1	0	0
d	1	1	1	1	0
e	1	0	1	0	1

Figure 3: Matrix representation

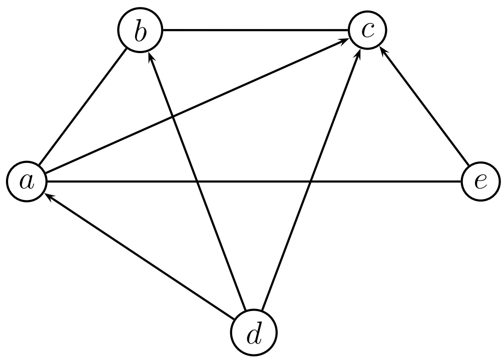


Figure 4: Graphical representation

4 Classical preference structures

4.1 Total order

4.1.1 Definition

A preference structure S is a total order if:

- S is complete,
- S is transitive,
- S is antisymmetric.

In a total order, the incomparability relation is empty ($J = \varnothing$) and the indifference relation I is limited to pairs of identical objects ($I = \{(a, a) : a \in A\}$). The strict preference is P is weakly complete and transitive. A total order therefore consist in a ranking of the objects from A from best to worst (using the relation P) without the possibility of ex aequo.

Remark 10
It is easy to check that an equivalent definition of a total order consists in saying that S is complete and the only circuits in this relation are loops.

It is clear that if S is a total order:

- P is weakly complete and transitive,
- I is transitive,
- $I \cdot P \subset P$,
- $P \cdot I \subset P$. •

Remark 11
Checking if a preference structure is a total order is quite simple using the matric representation of S. Indeed, labelling rows and columns of the matrix according to P, we obtain a matrix that has only 0 below the diagonal and 1 elsewhere. The relation P corresponds to off-diagonal 1's. In the graphical representation if vertices are ranked according to P, all arcs are going from left to right.

Example 2
Let $A = \{a, b, c, d, e\}$. Consider the preference structure $S = \{(a, a)$, (a, b), (a, c), (a, d), (a, e), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, d), (d, e), $(e, e)\}$.

It is easy to check that it is a total order using the matrix representation shown on Figure 5 or its graphical representation shown on Figure 6. ◇

$\circlearrowleft$	a	b	c	d	e
a	1	1	1	1	1
b	0	1	1	1	1
c	0	0	1	1	1
d	0	0	0	1	1
e	0	0	0	0	1

Figure 5: Matrix representation of a total order

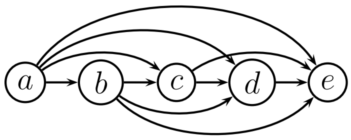


Figure 6: Graphical representation of a total order

4.1.2 Numerical representation

Let S be a total order on A. One may associate to each object a rank in such a way that this rank reflects the position of the object in the relation S. We leave to the reader the easy proof of the following result.

Theorem 1
A preference structure S on a finite set A is a total order if and only if (iff) there is a function $g : A \to \mathbb{R}$ such that, $\forall a, b \in A$:

$$\begin{cases} a \; S \; b \Leftrightarrow g(a) \geq g(b), \\ g(a) = g(b) \Rightarrow a = b. \end{cases}$$

Remark 12
The numerical representation of a total order is not unique. It is easy to show that given a numerical representation g satisfying the conditions of Theorem 1, any increasing transformation applied to g leads to another admissible representation. Conversely, if g and h are two numerical representations of the same total order in the sense of Theorem 1, there is an increasing function ϕ such that $g = \phi \circ h$. The scale g is said to be an *ordinal scale*.

Let g be a function satisfying the condition of the above theorem. It is possible to compare "differences" such as $g(a) - g(b)$ and $g(c) - g(d)$. These comparisons are nevertheless clearly dependent upon the choice of the particular function g: another legitimate choice can lead to other comparisons of differences. Hence, in general, it is impossible to give a particular meaning to these comparisons. •

Remark 13
Theorem 1 remains true if A is countably infinite (g is defined by an easy induction argument). It is clear that the result is no more true in the general case. Let us illustrate this fact by two exemples.

1. It is well know that the cardinality of $\mathcal{P}(\mathbb{R})$ (i.e;, the set of subsets of $\mathbb{R}$) is strictly greater than that of $\mathbb{R}$. Any total order on $\mathcal{P}(\mathbb{R})$ cannot have a numerical representation in the sense of Theorem 1. A natural question arises. Is Theorem 1 true when attention is restricted to sets A having at most the cardinality of $\mathbb{R}$. This is not so, as shown by the following famous exemple.

2. Let $A = \mathbb{R} \times \{0; 1\}$. It is easy to show that there are one-to-one functions between these two sets that have the same cardinality. Consider the lexicographic order defined letting:

$$(x, y)\ P\ (z, w) \Leftrightarrow \begin{cases} x > z \text{ or} \\ x = z \text{ and } y > w, \end{cases}$$

and

$$(x, y)\ I\ (z, w) \Leftrightarrow x = z \text{ and } y = w.$$

It is easy to show that the structure $S = P \cup I$ is a total order. It does not have a numerical representation in the sense of Theorem refth:rep:num:ordre:total. Indeed, suppose that g is such a representation. We would have, $\forall x \in \mathbb{R}$, $(x, 1)\ P\ (x, 0)$ so that $g(x, 1) > g(x, 0)$. There exists a rational number $\mu(x)$ such that $g(x, 1) > \mu(x) > g(x, 0)$. We have $(y, 1)\ P\ (y, 0)\ P\ (x, 1)\ P\ (x, 0)$ iff $y > x$. Hence, $y > x$ implies $\mu(y) > \mu(x)$. The function μ built above is therefore a bijection between $\mathbb{R}$ and $\mathbb{Q}$, a contradiction.

Beardon, Candeal, Herden, Induráin, and Mehta (2002) propose a detailed analysis of the various situations in which a total order does not have a numerical representation. The necessary and sufficient conditions ensuring that a total order has a numerical representation are known. (Bridges and Mehta 1995, Debreu 1954, Fishburn 1970, Krantz et al. 1971). They amount to supposing that S on A has a behavior that is "close" from that of $\geq$ in $\mathbb{R}$. •

4.2 Weak orders

4.2.1 Definition

A preference structure S is a weak order if:

- S is complete,
- S is transitive.

Weak orders generalize total orders since they do not have to antisymmetric. Hence, indifference between distinct elements is allowed in weak orders.

Remark 14
An equivalent definition of a weak order is that S is complete and any circuit of S has no P arc.

It is clear that if S is a weak order:

- P is transitive,
- P is negatively transitive,
- I is transitive (I is therefore an equivalence),
- $I \cdot P \subset P$,
- $P \cdot I \subset P$,
- the relation S indices a total order on the quotient set A/I. •

Remark 15
Let T be an asymmetric and negatively transitive binary relation on A. Let $S = T \cup (T^- \cap T^d)$. It is easy to show that S is a weak order. •

Remark 16
If the rows and columns of the matrix representation of a weak order are ordered according to a relation that is compatible with P (the ordering of the rows and columns for indifferent elements being unimportant), we obtain a matric in which the 1's are separated from the 0's by a stepped frontier that is below the diagonal and touches the diagonal. In a similar way, the graphical representation of a weak order generalizes that of a total order.

Example 3
Let $A = \{a, b, c, d, e\}$. Consider the preference structure $S = (a, a), (a, b)$, (a, c), (a, d), (a, e), (b, a), (b, b), (b, c), (b, d), (b, e), (c, c), (c, d), (c, e), (d, c), (d, d), (d, e), $(e, e)\}$. It is easy to check that this is a weak order considering the matrix representation shown on Figure 7 or the graphical representation depicted on Figure 8. ◇

↺	a	b	c	d	e
a	1	1	1	1	1
b	1	1	1	1	1
c	0	0	1	1	1
d	0	0	1	1	1
e	0	0	0	0	1

Figure 7: Matrix representation of a weak order

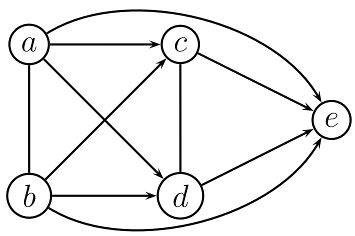


Figure 8: Graphical representation of a weak order

4.2.2 Numerical representation

Remembering that weak order induces a total order on the quotient set A/I, it is easy to prove the following result.

Theorem 2
A preference structure S on a finite set A is a weak order iff there is a function $g : A \to \mathbb{R}$ such that, $\forall a, b \in A$:

$$a \; S \; b \Leftrightarrow g(a) \geq g(b).$$

Remark 17
As above, the numerical representation of a weak order is defined up to an increasing transformation. The function g is an ordinal scale and most of the assertions that can be obtained using arithmetic operations on the values of g have a truth value that depends on the function g that was chosen: they are not meaningful in the sense of Roberts (1979). •

Remark 18
It is clear that the above result remain true when A is countably infinite (since in this case a total order structure always has a numerical representation) As was the case with total orders, extending this result to arbitrary sets implies the introduction of additional conditions. •

4.3 Classical problems

In most studies involving preferences, the weak order model is used: the function g representing the weak order is the function that should be maximized and that is called depending on the context: the value function, the objective function, the criterion, the value function, etc. It is striking that decision problem have been dealt with so often in this way without much investigation on the adequateness of g as a model of preference.

We discuss here a few classical question that has been dealt with using the weak order model.

4.3.1 Choosing on the basis of binary relation

Suppose that we have a weak order S on a set A and consider the situation (common in Economics) in which a choice must be made in a subset $B \subseteq A$. How to use the information contained in S to guide such a choice? A natural way to define the set $C(B, S)$ of chosen objects (remark that since we do not require $C(B, S)$ to be a singleton, it would be more adequate to speak of objects that are susceptible to be chosen) in B on the basis of S is to let:

$$C(B, S) = \{b \in B : Not[\, a \; P \; b\,] \text{ for all } a \in B\},$$

An object a belongs to the choice set as soon as there is no other object that is strictly preferred to a. It is not difficult to show that $C(B, S)$ is always nonempty as soon as B is finite (the general case raises difficult question, see Bergstrom 1975) and S is a weak order. Let us observe that, when B is finite, imposing that S is a weak order is only a sufficient condition for the nonemptyness of $C(B, S)$.

A classic result (see Sen 1970)) says that, when B is finite, $C(B, S)$ is nonempty as soon as P is acyclic. in B (it is never true that, for all $a_1, a_2, \ldots, a_k$ in B, $a_1 \; P \; a_2, a_2 \; P \; a_3, \ldots, a_{k-1} \; P \; a_k$ and $a_k \; P \; a_1$). The use of structures that are more general than the weak order also allows to give a simple answer to the problem deal with here.

Let us mention that there are situations (e.g., a competitive exam) in which it is desirable to rank order all elements in a subset $B \subseteq A$ and not only to define the choice set $C(B, S)$. The weak order model allows to give a trivial answer to this problem since the restriction of a weak order on A to a subset $B \subseteq A$ is a weak order on B.

4.3.2 Aggregating preferences

Suppose that you have collected $n \geq 2$ preference structures on A, for instance because the objects are evaluated according to various points of views (voters, criteria or experts). In such a situation, it is natural to try to build a "collective" preference structure S that aggregates the information contained in $(S_1, S_2, \ldots, S_n)$. In general, one looks for a mechanism (e.g., an electoral system or an aggregation method) that is able to aggregate *any* n-tuple of preference structures on A into a collective preference structure. When the weak order model is used, defining such a mechanism amounts to defining an aggregation function F from $\mathcal{WO}(A)^n$ dans $\mathcal{WO}(A)$, where $\mathcal{WO}(A)$ is the set of all weak orders on A. The work of Arrow (1963) has clearly shown the difficulty of such a problem. Imposing a small number of apparently reasonable conditions on F (unanimity, independence with respect to irrelevant alternatives, absence of dictator) leads to a logical impossibility: it is impossible to simultaneously satisfy all these principles (for a rich synthesis of such results, see Campbell and Kelly 2002 and Sen 1986). The simple majority method can be used to illustrate the problem uncovered by Arrow(s result. This method consist in declaring that "a is collectively at least as good as b" if there are more weak orders in which "a is at least as good as b" than weak orders for which "b is at least as good as a". Such a method seems highly reasonable and in line with our intuitive conception of democracy. It does not always lead to to a collective weak order; it may even lead to a collective relation having cycle in its asymmetric part. This is the famous "Condorcet paradox"; $A = \{a, b, c\}$, $n = 3$, $a\ P_1\ b\ P_1\ c, c\ P_2\ a\ P_2\ b$ and $b\ P_3\ c\ P_3\ a$ gives the simplest example of such a situation. Using a collective preference structure in which strict preference may be cyclic in order to choose and/or to rank order is far from being an easy task. Many works have investigated the question (see Laslier 1997, Moulin 1986, Schwartz 1986).

4.3.3 Particular structure of the set of objects

In many situations, it is natural to suppose that the set of objects A has a particular structure. This will be the case in:

- decision with multiple criteria in which the elements of A are vectors of evaluations on several dimensions, attributes or criteria. In this case, we have $A \subseteq A_1 \times A_2 \times \cdots \times A_n$ where A_i is the set of possible evaluations of the objects on the ith dimension,

- décision under risk in which the elements on A are viewed as probability distribution on a set of consequences. In this case we have $A \subseteq \mathcal{P}(C)$

where $\mathcal{P}(C)$ is a set of probability distributions on a set of consequences C,

- décision under uncertainty in which the elements of A are characterized by consequences occurring contingently upon the occurrence of "several states of nature". In this case, we have $A \subseteq C^n$ where C is a set of consequences, supposing that n distinct states of nature are distinguished.

In all these cases it is tempting to add to the weak order model additional conditions that will allow to take advantage of the particular structure of the set A. Among these condition, let us mention:

- *preference independence* (Keeney and Raiffa 1976, Krantz et al. 1971, Wakker 1989) in the case of decision-making with multiple criteria, implying that the comparison of two objects only differing on a subset of criteria is independent from the their common evaluations:

$$(a_I, c_{-I})\ S\ (b_I, c_{-I}) \Leftrightarrow (a_I, d_{-I})\ S\ (b_I, d_{-I})$$

where I is a subset of criteria $\{1, 2, \ldots, n\}$ and where (a_I, c_{-I}) denotes the object $e \in A$ such that $e_i = a_i$ if $i \in I$ and $e_i = c_i$ otherwise.

- *independence with respect to probabilistic mixing* (Fishburn 1970, 1988) in the case of decision-making under risk, implying that the preference relation between two probability distribution is not altered when they are both mixed with a common probability distribution:

$$a\ S\ b \Leftrightarrow (a\alpha c)\ S\ (b\alpha c)$$

where $(a\alpha b)$ denotes the convex combination of the probability distributions a and b with the coefficient $\alpha \in (0; 1)$,

- *the sure thing principle* (Fishburn 1970, Savage 1954, Wakker 1989) in the case of decision-making under uncertainty implying that the preference between two acts does not depend on common consequences obtained in some states of nature:

$$(a_I, c_{-I})\ S\ (b_I, c_{-I}) \Leftrightarrow (a_I, d_{-I})\ S\ (b_I, d_{-I})$$

where I is a subset of states of nature and (a_I, c_{-I}) denotes the act $e \in A$ such that $e_i = a_i$ if $i \in I$ and $e_i = c_i$ otherwise.

When these conditions are applied to sets of objects that are sufficiently "rich" (and when it is required that S behaves coherently with this richness, see Fishburn (1970), Wakker (1989)) we obtain famous model particularizing the one of the classical theory:

- the model of *additive value functions* in the case of decision with multiple criteria:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} u_i(a_i) \geq \sum_{i=1}^{n} u_i(b_i)$$

  where u_i is a real-valued fonction on A_i, denoting by a_i the evaluation of object a on the ith criterion,

- the *expected utility* model in the case of decision-making under risk:

$$a\ S\ b \Leftrightarrow \sum_{c \in C} p_a(c)u(c) \geq \sum_{c \in C} p_b(c)u(c)$$

  where u is a real-valued function on C and $p_a(c)$ is the probability to obtain consequence $c \in C$ with object a,

- the *subjective expected utility* model in the case of decision-making under uncertainty:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i)$$

  where u is a real-valued function on C and the p_i's are non-negative numbers adding up to 1 that cas be interpreted as the subjective probabilities of the various states of nature.

On the major interest of these models is to allow a numerical representation g of S that is much more specific that given by Theorem 2. The additional conditions mentioned above imply that, when A is adequately rich (e.g.,that $A = A_1 \times A_2 \times \cdots \times A_n$ in the case of decision-making with multiple criteria and that each A_i has a rich structure, see Wakker (1989)), imply that g can be additively decomposed. The numerical representation obtained is an interval scale (unique up to the choice of origin and unit) It is then possible to use sophisticated elicitation techniques to assess g and, therefore, structure a preference model (see Keeney and Raiffa 1976, Krantz et al. 1971, Wakker 1989).

These additional conditions were subjected to many empirical tests. In the fields of decision-making under risk and uncertainty, it was show that the conditions at the heart of the expected utility model (independence axiome and sure thing principle) were falsified in a predictable and reproducible way (see Allais 1953, Ellsberg 1961, Kahneman and Tversky 1979, McCrimmon and Larsson 1979). This has generated numerous studies investigating models only using weakening of these additional conditions (see Fishburn 1988, Machina 1982, Quiggin 1982, 1993, Yaari 1987 for decision under risk and Dubois, Prade, and Sabbadin 2001, Gilboa 1987, Gilboa and Schmeidler 1989, Schmeidler 1989, Wakker 1989 for decision under uncertainty).

Dutch book-like arguments (adhering to these generalized models may transform an individual into a "money pump") have often been used to criticize these models (see Raiffa 1970). The validity of such arguments nevertheless raises difficult questions (see Machina 1989, McClennen 1990, for a criticisl of such arguments for decision-making under risk).

Let us finally mention that other structures for A can be usefully studied. For instance, when A is endowed with a topological structure, it is natural to investigate numerical representation having nice continuity properties (see, e.g., Bosi and Mehta 2002, Bridges and Mehta 1995, Jaffray 1975). Similarly, if A is endowed with a binary operation allowing to combine its elements (this is the case in decision under risk using "probabilistic mixing" of two objects) a numerical representation is sought that is somehow compatible (most often through addition) with this operation (see Krantz et al. 1971).

5 Semi-orders and Interval orders

In weak orders, the indifference relation I is transitive. This hypothesis is sometimes inadequate since it amounts to supposing a perfect discrimination between close but distinct objects. Luce (1956) was the first to suggest a preference structure in which indifference may be intransitive (see Pirlot and Vincke 1997, for earlier references). He suggested the following example.

Example 4
Consider a set A consisting in 101 cups of coffee numbered from 0 à 100 and identical except that there are i grains of sugar in the ith cup. It is likely that an individual comparing these cups will not be able to detect a difference between two consecutive cups. Hence, it is likely that we obtain:

$$a_0 \ I \ a_1, a_1 \ I \ a_2, \ldots, a_{99} \ I \ a_{100}.$$

If the relation I is supposed to be transitive, we should have $a_0 \ I \ a_{100}$, which seems unlikely as soon as the individual is supposed to prefer sugared coffee. ◇

The two preference structures introduced in this section aim at modelling situations in which indifference is not transitive while maintaining our other hypotheses (transitivity of P, no incomparability) made so far.

5.1 Semi-order

5.1.1 Definition

A preference structure S is a semi-order if:

- S is complete,
- S is Ferrers,
- S is semi-transitive.

Remark 19
It is easy to check that an equivalent definition of a semi-order is to suppose that S is complete and all circuits of S have more I arcs than P arcs.

Moreover, it is easy to prove that if S is a semi-order:

- P is transitive,
- P is Ferrers,
- P is semi-transitive,
- $P \cdot I \cdot P \subset P$,
- $P \cdot P \cdot I \subset P$,
- $I \cdot P \cdot P \subset P$,
- $P^2 \cap I^2 = \varnothing$. •

As will become apparent later, semi-orders arise when an indifference threshold is introduced when comparing objects evaluated on a numerical scale. As an easy exercise, the reader may wish to check that any weak order is a semi-order.

Remark 20
The graphical representation of a semiorder is characterized by the fact that the four configurations depicted in Figure 9 are forbidden (whatever appears on the diagonal and with the possibility that two indifferent elements may be identical). •

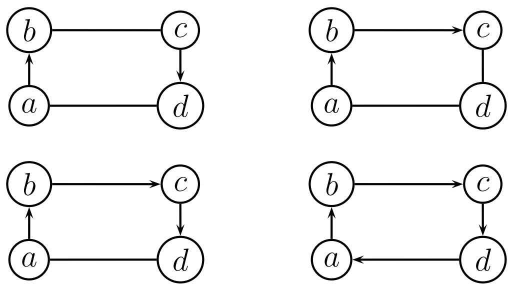


Figure 9: Forbidden configurations in a semi-order

5.1.2 Weak order associated with a semi-order

Let S is be a binary relation on A. The binary relation $S^{\pm}$ on A defined by

$$a\ S^{\pm}\ b \Leftrightarrow \left\{ \begin{array}{l} b\ S\ c \Rightarrow a\ S\ c, \\ c\ S\ a \Rightarrow c\ S\ b, \end{array} \right\} \forall c \in A$$

is called the "trace" of S. It is clear that the trace of a relation is always reflexive and transitive. We leave to the reader the easy proof of the following result.

Theorem 3
Let S be a reflexive binary relation on A. S is a semi-order if and only if its trace $S^{\pm}$ is complete.

Remark 21
When S is a semi-order, the weak order $S^{\pm}$ is obtained ranking the elements of A according to their degree in S (i.e., number of arcs leaving a vertex minus the number of arcs entering it). One can check that a weak order is always identical to its trace. •

5.1.3 Matrix representation(Jacquet-Lagrèze 1978)

Ordering the row and columns of the matrix representation of a semi-order using an order that is compatible with the trace of the relation, we obtain a matrix in which the 1 are separated from the 0 by a frontiers that is stepped and located below the diagonal. This follows immediately from the definition of the trace. In contrast with what happens with weak orders, the frontier separating the 1 and the 0 does not necessarily touch the diagonal.

Example 5
Let $A = \{a, b, c, d, e, f\}$. Consider the preference structure $S = \{(a, a), (a, b), (a, c), (a, d), (a, e), (a, f)\ (b, a), (b, b), (b, c), (b, d), (b, e), (b, f), (c, b), (c, c),$

(c, d), (c, e), (c, f), (d, c), (d, d), (d, e), (d, f), (e, c), (e, d), (e, e), (e, f), (f, e), (f, f) }. We obtain the matric representation shown on Figure 10. This relation is not a weak order: we have, e.g., $e \, S \, c$ and $c \, S \, b$ but $e \, \neg S \, b$. $\diamond$

$\circlearrowleft$	a	b	c	d	e	f
a	1	1	1	1	1	1
b	1	1	1	1	1	1
c	0	1	1	1	1	1
d	0	0	1	1	1	1
e	0	0	1	1	1	1
f	0	0	0	0	1	1

Figure 10: Matrix representation of a semi-order

5.1.4 Numerical representation

Theorem 4

Lett A be a finite set. The following proposition are equivalent:

1. *S is a semi-order on A,*
2. *there is a function $g : A \to \mathbb{R}$ and a constant $q \geq 0$ such that, $\forall a, b \in A$:*

$$a \, S \, b \Leftrightarrow g(a) \geq g(b) - q,$$

3. *there is function $g : A \to \mathbb{R}$ and a function $q : \mathbb{R} \to \mathbb{R}^+$ such that, $\forall a, b \in A$:*

$$g(a) > g(b) \Rightarrow g(a) + q(g(a)) \geq g(b) + q(g(b)),$$

and

$$a \, S \, b \Leftrightarrow g(a) \geq g(b) - q(g(b)).$$

Proof

See Fishburn (1985), Pirlot and Vincke (1997, Theorème 3.1), Scott and Suppes (1958) or Suppes, Krantz, Luce, and Tversky (1989, Chapitre 16) $\square$

This result shows that semi-orders naturally arise when objects evaluated on a numerical scale are compared on the basis of the scale but differences that are less than a constant threshold are not perceived or are not considered to be significant. The threshold is not necessarily constant provided that we never have $g(a) > g(b)$ and $g(b) + q(g(b)) > g(a) + q(g(a))$. Let us observe that the generalization of this result to arbitrary sets raises delicate problems (see Beja and Gilboa 1992, Candeal, Induráin, and Zudaire 2002, Fishburn 1973, 1985).

Remark 22
Let us build the numerical representation of the semi-order for which we gave the matrix representation earlier. Having chosen an arbitrary positive value for q, e.g., $q = 1$, the function g is built associating increasing values to the elements f, e, d, c, b, a (i.e., considering the lower elements in the weak order $S^{\pm}$ first) while satisfying the desired numerical representation. In such a way, we obtain: $g(f) = 0$, $g(e) = 0,5$, $g(d) = 1,1$, $g(c) = 1,2$, $g(b) = 2,15$ and $g(a) = 3$. •

Remark 23
The numerical representation of a semi-order is not unique. All increasing transformation applied to g gives another acceptable representation provided that the same transformation is applied to q. However all representations of a semi-order cannot be obtained in this way as shown by the following example. The scale that is built is more complex than an ordinal scale.

Example 6
Let $A = \{a, b, c, d\}$. Consider the preference structure $S = \{(a, d), (a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c)\}$. It is easy to check, e.g., using a matrix representation, that this structure is a semi-order. The following table give two numerical representation of S that cannot be obtained from one another by an increasing transformation.

	a	b	c	d	threshold
g	2	1,1	1	0	1.5
g'	2	1	1	0	1.5

◇

•

5.2 Interval order

5.2.1 Definition

A preference structure S is an interval order if:

- S is complete,

- S is Ferrers.

This structure generalized all structures introduced so far. As we will later see, it arises naturally when one wishes to compare interval on an ordinal scale.

Remark 24
It is easy to check that an equivalent definition of an interval order consists in saying that S is complete and that all circuits in S have at least two consecutive I arcs.

It is easily checked that if S is an interval order:

- P is transitive,
- P is Ferrers,
- $P \cdot I \cdot P \subset P$. •

Remark 25
The graphical representation of an interval order is characterized by the fact that the three configurations depicted on Figure 11 are forbidden (whatever appears on the diagonal and with the possibility that two indifferent elements may be identical). •

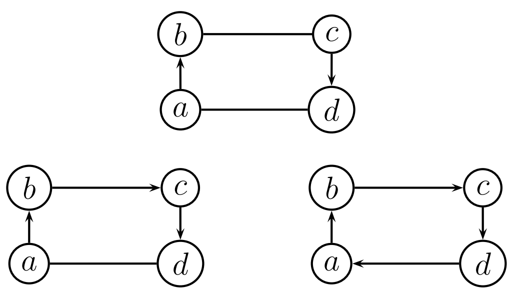


Figure 11: Forbidden configurations in an interval order

5.2.2 Weak orders associated to an interval order

Let S be a binary relation on A. Let us define a relation S^+ on A letting:

$$a\ S^+\ b \Leftrightarrow [b\ S\ c \Rightarrow a\ S\ c, \forall c \in A].$$

Similarly, we define the relation S^- letting:

$$a\ S^-\ b \Leftrightarrow [c\ S\ a \Rightarrow c\ S\ b, \forall c \in A].$$

The relation S^+ (resp. S^-) is called the right trace (resp. left trace) of S. It is clear that S^+ and S^- are always reflexives and transitives.

The proof of the following result is easy and left to the reader.

Theorem 5
Let S be a reflexive binary relation on A. The following three propositions are equivalent:

1. *S is an interval order,*
2. *S^+ is complete,*
3. *S^- is complete.*

Remark 26
When S is an interval order, the weak order S^+ (resp. S^-) can be obtained ranking the elements of A according to their out-degree (resp. in-degree) in S. •

5.2.3 Matrix representation

Let us rank the rows of the matric representation in a way that is compatible with S^+ taking care of ranking indifferent element according to S^+ using an order that is compatible with S^-. Let us perform a similar operation on the columns of the matrix, permuting the roles of S^+ and S^-. We obtain a matrix in which the 1 are separated from the 0 by a stepped frontier that is below the diagonal.

Example 7
Let $A = \{a, b, c, d, e, f\}$. Consider the following structure: $S = \{\ (a, a)$, (a, b), (a, c), (a, d), (a, e), (a, f), (b, a), (b, b), (b, c), (b, d), (b, e), (b, f), (c, b), (c, c), (c, d), (c, e), (c, f), (d, c), (d, d), (d, e), (d, f), (e, c), (e, d), (e, e), (e, f), (f, e), $(f, f)\ \}$.

We obtain the following matrix representation:

⟲	a	b	d	c	e	f
a	1	1	1	1	1	1
b	1	1	1	1	1	1
c	0	1	1	1	1	1
d	0	0	1	1	1	1
e	0	0	1	1	1	1
f	0	0	0	1	1	1

This structure is an interval order. It is not a semi-order since $f\ S\ c$ and $c\ S\ b$ but $f\ \neg S\ d$ and $d\ \neg S\ b$. It is therefore impossible to represent this structure using a stepped matrix with a similar order on rows and columns. ◇

5.2.4 Numerical representation

The proof of the following result can be found in Pirlot and Vincke (1997, Theorème 3.11) or Fishburn (1985).

Theorem 6
Let A be a finite set. The following propositions are equivalent:

1. *S is an interval order on A,*
2. *there are two functions $g : A \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}^+$ such that, $\forall a, b \in A$:*

$$a\ S\ b \Leftrightarrow g(a) + q(g(a)) \geq g(b).$$

We refer to Bridges and Mehta (1995), Chateauneuf (1987), Fishburn (1973, 1985), Nakamura (2002), Oloriz, Candeal, and Induráin (1998) for the problems involved in generalizing this result to arbitrary sets.

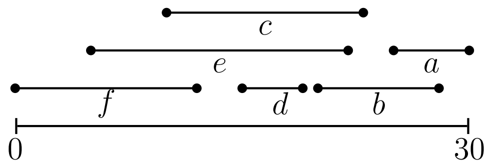
Remark 27
For instance, it is possible to build the numerical representation of the interval order presented earlier as follows. The values of g are arbitrarily chosen provided they increase from the first to the last row of the matrix. The values of $g+q$ are then defined in such a way that they increase from the first to the last column of the matrix and they satisfy the desired representation. For instance, we successively obtain:

$$\begin{gathered} g(f) = 0, g(e) = 5, g(c) = 10, g(d) = 15, g(b) = 20, g(a) = 25, \\ (g+q)(f) = 12, (g+q)(e) = 17, (g+q)(d) = 19, \\ (g+q)(c) = 23, (g+q)(b) = 28, (g+q)(a) = 30. \end{gathered}$$

Letting $\underline{g} = g$ and $\overline{g} = (g+q)$, it is clear that the numerical representation of an interval order amounts to associating with each $a \in A$ an interval $[\underline{g}, \overline{g}]$ such that:

$$\begin{cases} a\ P\ b \Leftrightarrow \underline{g}(a) > \overline{g}(b), \\ a\ I\ b \Leftrightarrow \begin{cases} \underline{g}(a) \leq \overline{g}(b), \\ \underline{g}(b) \leq \overline{g}(a), \end{cases} \end{cases}$$

which leads to the following representation:



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5.3 Remarks

Remark 28
Interval orders may be generalized using a threshold depending on both objects compared. One then obtain a threshold representation of all relations for which the asymmetric part is acyclic (Abbas 1995, Abbas and Vincke 1993, Agaev and Aleskerov 1993, Aleskerov et al. 2006, Diaye 1999, Subiza 1994). We do not tackle such models here. •

Remark 29
In an interval order, the relation P is transitive and, hence, is acyclic. For all nonempty finite subset $B \subset A$, $C(B, S)$ is therefore always nonempty. Using one of the structure introduced in the section does not raise major problems when it comes to linking preferences and choices. •

Remark 30
We saw that when A has a particular structure and that S is a weak order, it is interesting to use such a structure to try to arrive at a numerical representation that is more constrained than an ordinal scale. These extensions make central use of the transitivity of indifference in order to build these more structures representations. It is therefore not simple to do similar things on the basis of a semi-order or an interval order. (see Domotor and Stelzer 1971, Krantz 1967, Luce 1973, Suppes et al. 1989). •

Remark 31
Imposing to build a collective preference that is a semi-order or an interval order does not significantly contribute to solve the aggregation problem of weak orders uncovered by Arrow's theorem (see Sen 1986): as soon as $|A| \geq 4$, the theorem still holds if the collective preference is required to be complete and Ferrers (or complete and semi-transitive). •

6 Preference structures with incomparability

In all the structure envisaged so far, we supposed that S was complete. This hypothesis may seem innocuous, in particular when preferences are inferred from observed choices. It is not without problem however. Indeed, it may well happen that:

- information is poor concerning one or several of the elements of A,
- comparing elements of A implies synthesizing on several conflicting pints of view,
- the objects are not familiar to the individual.

In such cases, it may prove useful for preference modelling to use structures that explicitly include incomparability (Flament 1983, Roy 1985).

6.1 Partial order

A preference structure S is a partial if:

- S is reflexive,
- S is antisymmetric,
- S is transitive.

Intuitively, a partial order is a structure in which given two distinct objects, either one is strictly preferred to the other or the two objects are incomparable, with strict preference being transitive.

Remark 32
It is easy to check that if S is a partial order:

- P is transitive,
- I is limited to loops. •

A fundamental result (see Dushnik and Miller 1941, Fishburn 1985) show that all partial orders on a finite set can be obtained intersecting a finite number of total orders on this set. The minimal number of total orders that are needed for this is called the *dimension* of the partial order. This easily implies the following result.

Theorem 7
Let A be a finite set. The following propositions are equivalent:

1. *S is a partial order on A,*

2. *there is a function* $g : A \to \mathbb{R}$ *such that,* $\forall a, b \in A$*:*

$$\begin{cases} a\ S\ b \Rightarrow g(a) \geq g(b), \\ g(a) = g(b) \Rightarrow a = b. \end{cases}$$

Example 8
Lett $A = \{a, b, c, d, e\}$. Consider the preference structure: $S = \{\ (a,a), (a,b), (a,c), (a,d), (b,b), (b,d), (b,e), (c,c), (c,e), (d,d), (d,e), (e,e)\ \}$. A graphical representation of this structure is shown on Figure 12. It is easy to check

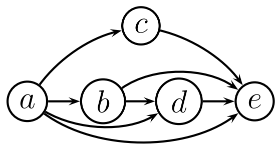

Figure 12: Graphical representation of a partial order

that the structure is partial order having dimension 2 that can be obtained intersecting the two total order (using obvious notation):

$$a > b > d > c > e \text{ and}$$
$$a > c > b > d > e.$$

Let us note that the detection of a partial order of dimension 2 can be done in polynomial time. On the contrary, the determination of the dimension of a partial order is NP-difficult (Doignon, Ducamp, and Falmagne 1984, Fishburn 1985). ◇

6.2 Quasi-order

A preference structure S is a quasi-order if:

- S is reflexive,
- S is transitive.

Quasi-orders generalize partial orders by allowing to have indifference between distinct elements, this indifference relation being transitive.

Remark 33
It is easy to check that if S is a quasi-order:

- P is transitive,
- I is transitive,

- $P \cdot I \subset P$,
- $I \cdot P \subset P$. •

As with partial orders, it is easy to show that any quasi-order on a finite set can be obtained intersecting a finite number of weak orders (see Bossert, Sprumont, and Suzumura 2002, Donaldson and Weymark 1998). This implies the following result.

Theorem 8
Let A be a finite set. The following propositions are equivalent:

1. *S is a quasi-order on A,*
2. *there is a function $g : A \to \mathbb{R}$ such that, $\forall a, b \in A$:*

$$a \; S \; b \Rightarrow g(a) \geq g(b),$$

Remark 34
Alternatively, one can build a numerical representation of a quasi order considering a set of numerical representations of weak orders (see Ok 2002). •

Example 9
Let $A = \{a, b, c, d, e, f\}$. Consider the preference structure $S = \{(a, a)$, (a, b), (a, c), (a, d), (a, e), (a, f) (b, b), (b, d), (b, e), (b, f), (c, c), (c, e), (c, f), (d, b), (d, d), (d, e), (d, f), (e, e), (e, f), (f, e), $(f, f)\}$. It is easy to check that this is a quasi-order. Its graphical representation is given on Figure 13. ◇

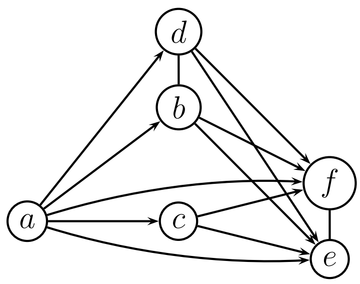

Figure 13: Graphical representation of a quasi-order

Remark 35
It is possible to extend classical models of decision under risk to deal with quasi-orders (Aumann 1962, Fishburn 1970). The multi-attribute case was only studied in the finite case (Fishburn 1970, Scott 1964). Let us also mention that allowing for incomparability in the collective preference does not significantly contribute to solve the problem uncovered by Arrow'theorem (see Weymark 1984). •

Remark 36
Roubens and Vincke (1985) propose definitions of partial semi-orders and interval orders. They allow to have at the same time an intransitive indifference relation together with incomparability situations. We do not detail this point here. •

6.3 Synthesis

We summarize on Figure 14 the properties of preference structures that have been introduced so far.

Structures	**Definition**
Total order total	S complete S antisymmetric S transitive
Weak order	S complete S transitive
Semi-order	S complete S Ferrers S semi-transitive
Interval order	S complete S Ferrers
Partial order	S reflexive S antisymmetric S transitive
Quasi-order	S reflexive S transitive

Figure 14: Common preference structures

7 Conclusion

7.1 Other preference structures

In all the structures introduced so far, the relation P was transitive and, hence, was acyclic. This seems a natural hypothesis. Abandoning it implies reconsidering the links existing between "preference" and "choice" as we already saw. Nevertheless, it is possible to obtain such preferences in experiments (see May 1954, Tversky 1969) when subjects are asked to compare objects evaluated on several dimensions. They are also common in social choice dur to the Condorcet's paradox. Indeed, a famous result (McGarvey 1953) show that with simple majority, any complete preference structure can be be obtained as the result of the aggregation of individual weak orders. With other aggregation methods, all preference structure may occur (Bouyssou 1996).

The literature on Social Choice abound with studies of adequate choice procedure on the basis of such preferences. The particular case of *tournaments* (complete and antisymmetric relations) have been explored in depth (Laslier 1997, Moulin 1986).

More recently, it was shown that it is possible to build numerical representations of such relations (see Bouyssou 1986, Bouyssou and Pirlot 1999, 2002, Fishburn 1982, 1988, 1991a,b, 1992, Tversky 1969, Vind 1991). In the models proposed in Bouyssou and Pirlot (2002), we have sets A being Cartesian products (as in decision under uncertainty or in decision with multiple attributes):

$$a\ S\ b \Leftrightarrow F(p_1(a_1, b_1), p_2(a_2, b_2), \ldots, p_n(a_n, b_n)) \geq 0$$

where the p_i's are functions from A_i^2 to $\mathbb{R}$, F is a function from $\prod_{i=1}^{n} p_i(A_i^2)$ to $\mathbb{R}$ and where, e.g., F can be increasing in all its arguments. This model generalizes the classical additive difference model proposed in Tversky (1969) in which:

$$a\ S\ b \Leftrightarrow \sum_{i=1}^{n} \varphi_i(u_i(a_i) - u_i(b_i)) \geq 0$$

where the u_i's are functions from A_i to $\mathbb{R}$ and the φ_i's are odd increasing functions on $\mathbb{R}$.

Similarly, in the models studied in Fishburn (1982, 1988) for the case of decision-making under risk, the numerical representation is such that:

$$a\ S\ b \Leftrightarrow \sum_{c \in C} \sum_{c' \in C} p_a(c) p_b(c') \phi(c, c') \geq 0$$

where ϕ is a function from C^2 to $\mathbb{R}$ and $p_a(c)$ is the probability to obtain the consequence $c \in C$ with object a.

A common criticism of such models is that cycles leave the door open to apparently "irrational" behavior and makes an individual vulnerable to "Dutch Books" (Raiffa 1970). As in the case of decision under risk mentioned earlier, it is not clear that the arguments are really convincing (see Fishburn 1991b).

7.2 Other problems

This brief survey of classical preference structures used in preference modelling will hopefully give the reader enough clues to tackle a vast and complex literature. It has neglected many important questions, among which:

- the question of the approximation of preference structure by another one, e.g., the search for a total order at minimal distance of a tournament (see Barthélémy, Guénoche, and Hudry 1989, Barthélémy and Monjardet 1981, Bermond 1972, Charon-Fournier, Germa, and Hudry 1992, Hudry and Woirgard 1996, Monjardet 1979, Slater 1961)

- the way to collect and validate preference information in a given context (see von Winterfeldt and Edwards 1986),

- the links between preference modelling the question of meaningfulness in measurement theory (see Roberts 1979),

- the statistical analysis of preference data (see Coombs 1964, Green, Tull, and Albaum 1988),

- deeper questions on the links between value systems and preferences (see Broome 1991, Cowan and Fishburn 1988, Tsoukiàs and Vincke 1992, von Wright 1963).

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