Numerical representations of binary relations with thresholds: A brief survey [1]

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[1] This text is based on a chapter of a book in preparation co-authored with Fuad Aleskerov and Bernard Monjardet (that will be the second edition of Aleskerov and Monjardet, 2002).

Abstract

This purpose of this text is to present in a self-contained way a number of classical results on the numerical representation of binary relations on arbitrary sets involving a threshold. We tackle the case of linear orders, weak orders, biorders, interval orders, semiorders and acyclic relations.

Keywords : utility, threshold, biorder, interval order, semiorder, acyclic relation.

Contents

1 Introduction

1.1 Overview

This aim of this text is to present a number of results concerning the numerical representations of binary relations defined on arbitrary sets that involve a threshold. This text contains no new results. It simply tries to organize in a simple and self-contained way results that are scattered in the literature.

A thorough treatment of the subject would require an entire book (see Bridges and Mehta, 1995). In order to keep this text of reasonable length, we have decided to restrict our study to the numerical representation of the following six classes of binary relations:

1. *linear orders* (section 2),
2. *weak orders* (section 3),
3. *biorders* (section 4),
4. *interval orders* (section 5),
5. *semiorders* (section 6),
6. *acyclic relations* (section 7).

A final section contains remarks and a guide to the literature. Clearly, Sections 2 and 3 are only included here in order to be self-contained. The numerical representations studied in these two sections do not involve thresholds.

We will not cover here:

- the various classes of binary relations uncovered in the study of utility maximization with a context-dependent threshold studied in Aleskerov and Monjardet (2002, ch. 4–5), or the generalizations of semiorders surveyed (in the finite case) in Fishburn (1997),
- the detailed study of numerical representations that do not involve thresholds (e.g., "one-way" representations to be defined later). We will nevertheless try to mention alternative numerical representations whenever they exist in the literature.
- the question of the (semi-)continuity of these numerical representations,
- the problem of the existence of maximal elements of binary relations on infinite sets and the related problem of choice functions on such sets (see Bergstrom, 1975, Horvath and Llinares Ciscar, 1996, Llinares,

1998, Llinares and Sánchez, 1999, Nehring, 1996, Sánchez, Llinares, and Subiza, 2004, Subiza and Peris, 1997, Tian, 1993, Tian and Zhou, 1995, Zhou, 1994, on these topics). Useful results on this topic are collected in Suzumuza (1983, ch. 2)

Moreover, even within such limits, our presentation is not exhaustive. We have sometimes deliberately sacrificed elegance and power in order to remain brief and self-contained. Whenever possible, we mention where more advanced results can be found (the survey of Mehta, 1998, is an excellent starting point).

The rest of this section is devoted to our vocabulary concerning binary relations.

1.2 Definitions and notation

1.2.1 Sets

Let X and Y be two sets. The set X is said to have a larger cardinality than the set Y if there is a function from X *onto* Y. A well-known result in set theory says that the binary relation "has a larger cardinality than" between sets is complete. The sets X and Y are said to have the same *cardinality* if the set X has a larger cardinality than Y and vice versa, i.e., if there is a *one-to-one* function from X *onto* Y. The set X is said to have a strictly larger cardinality than the set Y if X has a larger cardinality than Y and Y does not have a larger cardinality than X.

A set X is said to be *denumerable* is there is a *one-to-one* function from X *onto* the set of integers $\mathbb{N} = \{0, 1, 2, \ldots\}$. It is well-known that the union of two denumerable sets or their Cartesian product is also denumerable. This shows that the sets $\mathbb{Z}$ and $\mathbb{Q}$ are denumerable.

We will say that a set X is *countable* if it is finite or denumerable. It is easy to show that a set is countable if and only if there is a one-to-one function from X onto some subset K of the set of integers $\mathbb{N}$ or, equivalently of $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. Therefore, if a set X is countable, its elements can be enumerated as[1] $\{x_i : i \in K \subseteq \mathbb{N}_+\}$.

A set that is not countable will be said to be *uncountable.* The set $\mathbb{R}$ is a well-known example of an uncountable set.

It is well-known that the sets $\mathbb{R}^k$ and $[0, 1]$ have the same cardinality as $\mathbb{R}$. There are (infinitely many) sets that have a strictly larger cardinality than $\mathbb{R}$, i.e., for which there is no function from $\mathbb{R}$ onto them. For instance,

[1] Throughout this text, $\subseteq$ denotes *reflexive* inclusion between sets. The asymmetric part of $\subseteq$ is denoted by $\subsetneq$.

$2^{\mathbb{R}}$ (the set of all subsets of $\mathbb{R}$) has a strictly larger cardinality than $\mathbb{R}$ and $2^{2^{\mathbb{R}}}$ has a strictly larger cardinality than $2^{\mathbb{R}}$. For more details the reader is referred to Kelley (1955) or Munkress (1975).

1.2.2 Binary relations

A binary relation R on a set X is a subset of $X \times X = X^2$. We often write $x\ R\ y$ instead of $(x, y) \in R$. We define:

- the *complement* R^c of R letting,

$$R^c = \{(x, y) \in X^2 : (x, y) \notin R\},$$

- the *dual* R^d of R letting,

$$R^d = \{(x, y) \in X^2 : (y, x) \in R\},$$

- the *codual* R^{cd} of R letting,

$$R^{cd} = \{(x, y) \in X^2 : (y, x) \notin R\}.$$

We say that R is:

- *reflexive* if $x\ R\ x$,
- *irreflexive* if $x\ R^c\ x$,
- *symmetric* if $[x\ R\ y \Rightarrow x\ R^d\ y]$,
- *asymmetric* if $[x\ R\ y \Rightarrow x\ R^{cd}\ y]$,
- *complete* if $[x\ R\ y$ or $x\ R^d\ y]$,
- *connected* if $[x \neq y \Rightarrow x\ R\ y$ or $x\ R^d\ y]$,
- *transitive* if $[x\ R\ y$ and $y\ R\ z] \Rightarrow x\ R\ z$,
- *negatively transitive* if $[x\ R^c\ y$ and $y\ R^c\ z] \Rightarrow x\ R^c\ z$,
- *Ferrers* if $[x\ R\ y$ and $z\ R\ w] \Rightarrow [z\ R\ y$ or $x\ R\ w]$,
- *semitransitive* if $[x\ R\ y$ and $y\ R\ z] \Rightarrow [x\ R\ w$ or $w\ R\ z]$,

for all $x, y, z, w \in X$.

The *transitive closure* R^τ of R is a binary relation on X such that $x\ R^\tau\ y$ if there are $n \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ and $z_1, z_2, \ldots, z_n \in A$ such that $x = z_1\ R\ z_2\ R\ \ldots\ R\ z_{n-1}\ R\ z_n = y$. An easy proof shows that R^τ is the smallest transitive relation that contains R (there are always such binary relations since the trivial binary relation equal to $X \times X$ is transitive. Hence, $R = R^\tau$ if and only if R is transitive). We say that R is acyclic if and only if R^τ is irreflexive.

We say that R is:

- a *equivalence* if it is reflexive, symmetric and transitive,
- a *linear order* if it is connected, asymmetric and negatively transitive,
- a *weak order* if it is asymmetric and negatively transitive,
- a *biorder* if it is Ferrers,
- an *interval order* if it is irreflexive and Ferrers,
- a *semiorder* if it is irreflexive, Ferrers and semitransitive,
- a *partial order* if it is irreflexive and transitive,
- an *suborder* if it is acyclic.

The symmetric complement of R is a binary relation on a set X is the binary relation I_R defined by $I_R = R^c \cap R^{cd}$. We write I instead of I_R when there is no risk of ambiguity.

2 Linear orders

Let P be a binary relation on a set A. Suppose that there is a real-valued function u on A such that, for all $x, y \in A$,

$$x\ P\ y \Leftrightarrow u(x) > u(y), \tag{1}$$

$$x \neq y \Rightarrow u(x) \neq u(y). \tag{2}$$

It is not difficult to see that the existence of representation (1–2) implies that P is connected, asymmetric and negatively transitive, i.e., is a linear order.

When A is countable, the converse is true. We have:

Theorem 1 (Representation of linear orders on countable sets)
Let P be a binary relation on a countable set A. There is a function $u : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) > u(y),$$
$$x \neq y \Rightarrow u(x) \neq u(y).$$

iff P is a linear order.

Proof. Necessity is immediate. We show sufficiency. Since A is countable, it can be enumerated as $\{z_i : i \in K \subseteq \mathbb{N}_+\}$. For all $x \in A$, let $N(x) = \{i \in K : x \; P \; z_i\}$. It is clear that, for all $x \in A$, $N(x)$ is countable. Elementary algebra shows that the series:

$$u(x) = \sum_{i \in N(x)} \frac{1}{2^i}$$

converges (we use the common convention that summing over an empty set leads to a null sum). Let us show that such a function satisfies (1–2).

Suppose that $x \; P \; y$. Because P is transitive and asymmetric (and, hence, irreflexive), we have $xP \supsetneq yP$ so that $N(y) \subsetneq N(x)$ and $u(x) > u(y)$. Conversely, suppose that $u(x) > u(y)$ and $x \; P^c \; y$. Using the negative transitivity of P, $y \; P^c \; z$ implies $x \; P^c \; z$. Therefore, $xP \subseteq yP$, contradicting $u(x) > u(y)$.

To complete the proof, observe that, because P is connected, $x \neq y$ implies either $x \; P \; y$ or $y \; P \; x$ so that either $Px \supsetneq Py$ or $Px \subsetneq Py$.This clearly implies $u(x) \neq u(y)$. □

In view of the fact that there are sets having a cardinality strictly larger than that of $\mathbb{R}$, it should be no surprise that Theorem 1 does not generalize to arbitrary sets: any linear order on $2^{\mathbb{R}}$ cannot have a representation satisfying (1–2).

A more subtle question is the following: is the above theorem true if one restricts attention to sets that have at most the cardinality of $\mathbb{R}$? This question is all the more important that such sets are the ones of interest in most applications (think of an agent consuming bundles of goods available on a market or of a subject comparing various tones differing in duration and pitch, etc.). The following example, dating back at least to Debreu (1954), shows that the answer to that question is also negative.

Example 2 (Lexicographic preferences)
Let $A = \mathbb{R} \times \{0, 1\}$. Let P on A be the lexicographic order on A, i.e.,

$$(a, \alpha) \; P \; (b, \beta) \Leftrightarrow \begin{cases} a > b \text{ or,} \\ a = b \text{ and } \alpha > \beta. \end{cases}$$

It is easy to see that P is a linear order on A.

Suppose that there is a real-valued function u satisfying (1–2). Because $(a, 1)\ P\ (a, 0)$, we must have $u(a, 1) > u(a, 0)$. Take $b > a$. We clearly have $u(b, 1) > u(b, 0) > u(a, 1) > u(a, 0)$. Therefore u defines an uncountable collection of disjoint non-degenerate intervals of $\mathbb{R}$. This is easily seen to be contradictory since each of these intervals must contain an element of $\mathbb{Q}$. $\diamond$

Necessary and sufficient conditions for the existence of a representation (1–2) are well-known. We introduce them below.

When P is asymmetric, it is easy to see that $x\ P^{cd}\ y$ iff $x\ P\ y$ or $x\ I\ y$, where, as before, I is the symmetric complement of P ($x\ I\ y$ iff $[x\ P^c\ y$ and $y\ P^c\ x]$). Hence, when P is a linear order, $x\ P^{cd}\ y$ iff $x\ P\ y$ or $x\ D\ y$ (where, $x\ D\ y$ iff $x = y$), since, for an asymmetric and connected relation, we have $I = D$.

The following definition of a dense subset will be central in the rest of this text.

Definition 3 (Denseness)
Let P be an asymmetric binary relation on A. Then $B \subseteq A$ is said to be dense *in A for P if, for all $x, y \in A$, $x\ P\ y$ implies $x\ P^{cd}\ z$ and $z\ P^{cd}\ y$, for some $z \in B$.*

The existence of a *countable* set that is dense in A for P will turn out to be a necessary condition for (1–2).

Remark 4
The intuition behind this condition can be explained as follows. Let P be an asymmetric binary relation on A. Then $B \subseteq A$ is said to be *s-dense* in A for P if, for all $x, y \in A$, $x\ P\ y$ implies $x\ P\ z$ and $z\ P\ y$, for some $z \in B$ (This notion of "s-denseness" is what is often simply called "denseness" in the mathematical literature; because we will make little use of "s-denseness", we have reserved the term "denseness" for the condition that we will most often use below).

It is well-known that the denumerable set $\mathbb{Q}$ is s-dense in $\mathbb{R}$ for the relation $>$. The existence of a denumerable set B that is s-dense in A for P is central to the characterization in Cantor (1915) of linearly ordered sets isomorphic to $\mathbb{R}$ linearly ordered by $>$.

But the s-denseness condition is too strong if what is sought is simply an isomorphism into some subset of $\mathbb{R}$. For instance, the set $\mathbb{N}$ is linearly ordered by $>$ and has a trivial numerical representation but has no s-dense subset. The idea behind the above denseness condition is to weaken s-denseness so that it becomes necessary for the existence of an isomorphism to some subset of $\mathbb{R}$. With such a weakening, $\mathbb{N}$ becomes dense in itself for the relation $>$.

Bridges and Mehta (1995) give several conditions that are equivalent to requiring that P has a countable dense subset. •

Let $x, y \in A$ and P be a linear order on A. We say that the ordered pair (y, x) is a *jump* for P if $x\ P\ y$ and for all $z \in A$, we have either $z\ P^{cd}\ x$ or $y\ P^{cd}\ z$, i.e., if the set $\{z \in A : x\ P\ z\ P\ y\}$ is empty. If (y, x) is a jump, we say that x is the upper endpoint and y the lower endpoint of the jump. Let A_1^* (respectively, A_2^*) be the set of all upper (respectively, lower) endpoints of jumps. Define $A^* = A_1^* \cup A_2^*$. It is easy to see that the linear order in Example 2 has uncountably many jumps $((x, 0), (x, 1))$. This is impossible if a numerical representation exists.

The following two lemmas explore the properties of the set A^* of all endpoints of jumps.

Lemma 5
Let P be a linear order on A. If there is a real-valued function u satisfying (1–2), then the set A^* of all endpoints of jumps is countable.

PROOF. If (y, x) is a jump, (1) implies $u(y) < u(x)$, so that there is $\rho \in \mathbb{Q}$ such that $u(y) < \rho < u(x)$. Hence, there is a one-to-one mapping from A_1^* (respectively, A_2^*) onto some subset of $\mathbb{Q}$. This implies that $A^* = A_1^* \cup A_2^*$ must be countable. □

Lemma 6
Let P be a linear order on A. If there is a countable subset B that is dense in A for P, then the set A^* of all endpoints of jumps is countable.

PROOF. Let A_1^* (respectively, A_2^*) be the set of all upper (respectively, lower) endpoints of jumps.

Suppose that (y, x) is a jump. By construction, we must have either $y \in B$ or $x \in B$. Hence, $A_1^* \setminus B$ is in one-to-one correspondence with some subset of B, since given a jump (y, x), if $x \notin B$ then $y \in B$. Similarly, $A_2^* \setminus B$ is in one-to-one correspondence with some subset of B, since given a jump (y, x), if $y \notin B$ then $x \in B$.

Hence, $A_1^* \setminus B$ and $A_2^* \setminus B$ are both countable. Because B is countable, $B \cap A^*$ is clearly countable. Hence,

$$A^* = (A_1^* \setminus B) \cup (A_2^* \setminus B) \cup (B \cap A^*),$$

is countable. □

The following theorem gives necessary and sufficient conditions for the existence of a representation (1–2) on arbitrary sets.

Theorem 7 (Representation of linear orders)
Let P be a linear order on a set A. The following statements are equivalent.

1. *There is a countable subset B that is dense in A for P.*
2. *There is a real-valued function u on A such that, for all $x, y \in A$,*

$$x \; P \; y \Leftrightarrow u(x) > u(y),$$
$$x \neq y \Rightarrow u(x) \neq u(y).$$

PROOF. $[(2) \Rightarrow (1)]$. Let u satisfies (1–2). We define J as the set of all ordered pairs (r, r') of rational numbers such that $r' > u(a) > r$, for some $a \in A$. Using the Axiom of Choice, for each $(r, r') \in J$, choose one $a \in A$ such that $r' > u(a) > r$ holds. Let C be the set of all elements chosen in this process. By construction, C has at most the cardinality of $\mathbb{Q} \times \mathbb{Q}$ and is therefore countable. Lemma 5 has shown that A^* is countable. Therefore $B = C \cup A^*$ is countable. Let us show that B is dense in A for P.

Let $x, y \in A$ be such that $x \; P \; y$. If (y, x) is a jump, we have, by construction, $x \; P^{cd} \; x \; P^{cd} \; y$ and $x \in B$. Suppose that (y, x) is not a jump, so that for some $z \in A$, we have $x \; P \; z \; P \; y$. Since $u(x) > u(z) > u(y)$, there are rational numbers r, r' such that $u(x) > r' > u(z) > r > u(y)$. By construction of C, this implies that, for some $w \in C$, we have $u(x) > r' > u(w) > r > u(y)$. This implies $x \; P \; w \; P \; y$, so that $x \; P^{cd} \; w \; P^{cd} \; y$. Hence, the set B is dense in A for P.

$[(1) \Rightarrow (2)]$. Let D be a countable set that is dense in A. If there is a maximal element in A for P (i.e., an element $a^* \in A$ such that $a^* \; P^{cd} \; x$, for all $x \in A$), we adjoin it to D. We do the same if there is a minimal element A for P (i.e., an element $a_* \in A$ such that $x \; P^{cd} \; a_*$, for all $x \in A$). Let D' be the set obtained from D by these two possible additions. Using Lemma 6, we know that A^* is countable. Hence, $B = D' \cup A^*$ is countable.

Since B is countable, we use the construction of Theorem 1 to obtain a real-valued function v on B satisfying (1–2). For any $x \in A$ consider the set $V(x) = \{v(a) : x \; P^{cd} \; a \text{ and } a \in B\}$ and let $u(x)$ be the least upper bound (l.u.b.) of this set. This l.u.b. obviously exists for $x \in B$, and we have in this case $u(x) = v(x)$. Let us show that this l.u.b. exists for $x \notin B$. Because x is neither minimal nor maximal, there are $x_1, x_2 \in A$ such that $x_1 \; P \; x \; P \; x_2$. Using the density of D and the fact that $x \notin D$, there are $y_1, y_2 \in D$ such that $x_1 \; P^{cd} \; y_1 \; P \; x \; P \; y_2 \; P^{cd} \; x_2$. This shows that the set $V(x)$ is nonempty (it contains $v(y_2)$) and is bounded above (by $v(y_1)$). Hence it has a l.u.b.

Let us show that u defined above satisfies (1). Suppose that $x \; P \; y$.

We claim that there are $z, w \in B$ such that $x \; P^{cd} \; z \; P \; w \; P^{cd} \; y$. Taking $x = z$ and $y = w$, the claim is trivial if $x, y \in B$. Suppose that $y \notin B$.

Using the density of D and $y \notin D$, there is a $z \in D$ such that $x\ P^{cd}\ z\ P\ y$. Because $y \notin B$, it cannot be the endpoint of a jump and, hence, (y, z) is not a jump. Therefore, we know that $x\ P^{cd}\ z\ P\ z'\ P\ y$, for some $z' \in A$. Using the density of D and the fact that $y \notin D$, we obtain that, for some $w \in D$, $x\ P^{cd}\ z\ P\ z'\ P^{cd}\ w\ P\ y$, so that $x\ P^{cd}\ z\ P\ w\ P^{cd}\ y$. The argument is similar if $x \notin B$.

Using the above claim, $x\ P\ y$ implies that there are $z, w \in B$ such that $x\ P^{cd}\ z\ P\ w\ P^{cd}\ y$. This implies $u(x) \geq v(z) > v(w) \geq u(y)$, so that $u(x) > u(y)$. We have shown that, when P is a linear order, $x\ P\ y$ implies $u(x) > u(y)$. Since P is a linear order, this implies that (1) holds. Now, $x \neq y$ implies either $x\ P\ y$ or $y\ P\ x$, so that $u(x) \neq u(y)$ and (2) holds. □

Theorem 7 will be our central tool for the construction of numerical representations in this text.

Remark 8
It is easy to see that if u is a function satisfying (1–2), then $\phi \circ u$ will be another acceptable representation if ϕ is a strictly increasing function from $u(A)$ into $\mathbb{R}$. It is not difficult to prove that only such transformations are acceptable. In other terms, u is an *ordinal scale*. •

3 Weak orders

3.1 Preliminaries

We recall below a number of elementary facts concerning the set of equivalence classes generated by a weak order.

Let P be a weak order on a set A. Define the binary relation I on A, setting $I = P^c \cap P^{cd}$, i.e., for all $x, y \in A$, $x\ I\ y$ iff $[(x, y) \notin P$ and $(y, x) \notin P]$.

Lemma 9
Let P be a weak order on A. Then $I = P^c \cap P^{cd}$ is an equivalence on A (i.e., is reflexive, symmetric and transitive).

Proof. Since P is asymmetric, it is irreflexive, implying that I is reflexive. By construction, I is symmetric. Its transitivity follows from the negative transitivity of P. □

Let $x \in A$. The *equivalence class* for I generated by $x \in A$ is the subset of A defined by:

$$xI = Ix = \{y \in A : x\ I\ y\}.$$

By construction, for all $x \in A$, we have $x \in xI$. Using the fact that I is an equivalence, it is easy to prove that, for all $x, y \in A$, either $xI = yI$ or $xI \cap yI = \varnothing$. This means that the set of equivalence classes defines a *partition* of the set A. We define $A/I = \{xI : x \in A\}$ to be the set of all these equivalence classes.

Define the binary relation $\rhd_P$ on A/I setting, for all $X, Y \in A/I$,

$$X \rhd_P Y \Leftrightarrow [x \; P \; y, \text{ for some } x \in X \text{ and some } y \in Y],$$

when there will be no risk of ambiguity, we write $\rhd$ instead of $\rhd_P$. Using the fact that P is a weak order, it is easy to see that we have, for all $X, Y \in A/I$,

$$X \rhd Y \Leftrightarrow [x \; P \; y, \text{ for all } x \in X \text{ and all } y \in Y].$$

Lemma 10
Let P be a weak order on a set A. Then the relation $\rhd$ on A/I is a linear order (i.e., is connected, asymmetric and negatively transitive).

Proof. Let $X, Y \in A/I$ with $X \neq Y$. We have to show that either $X \rhd Y$ or $Y \rhd X$. Suppose that $(X, Y) \notin \rhd$, so that, for all $x \in X$ and all $y \in Y$, we have $y \; P^{cd} \; x$. We have either $y \; P \; x$ or $y \; I \; x$. Because $X \neq Y$, $y \; I \; x$ is impossible. Hence $y \; P \; x$, so that $Y \rhd X$.

Suppose that $X \rhd Y$ and $Y \rhd X$. This implies that, for some $x, x' \in X$ and some $y, y' \in Y$, $x \; P \; y$, $y' \; P \; x'$, $x \; I \; x'$ and $y \; I \; y'$. Using the fact that P is a weak order, we obtain $x \; P \; x$, a contradiction.

It remains to show that $\rhd$ is negatively transitive. Suppose that $(X, Y) \notin \rhd$ so that, for all $x \in X$ and all $y \in Y$, $x \; P^c \; y$. Similarly, $(Y, Z) \notin \rhd$ implies that, for all $y' \in Y$ and all $z \in Z$, $y' \; P^c \; z$. Since $y \; I \; y'$, the negative transitivity of P implies, for all $x \in X$ and all $z \in Z$, $x \; P^c \; z$, so that $(X, Z) \notin \rhd$. □

3.2 Numerical representation

Let P be a binary relation on a set A. Suppose that there is a real-valued function u on A such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) > u(y). \tag{3}$$

It is not difficult to see that this implies that P is a weak order on A, i.e., an asymmetric and negatively transitive relation.

Lemma 11
Let P be a weak order on A. Let $\rhd$ be the linear order on A/I induced by P. There is a real-valued function u on A such that (3) *holds iff there is a real-valued function U on A/I such that (1–2) hold.*

PROOF. *Necessity.* Suppose that U on A/I satisfies (1–2). Let $X \in A/I$. For all $x \in X$, let $u(x) = U(X)$. The function u is well defined since $x \in X$ and $x \in Y$ imply $X = Y$. Suppose that $x\ P\ y$. Let $X, Y \in A/I$ be such that $x \in X$ and $y \in Y$. Because $x\ P\ y$, we have $X \rhd Y$ so that $U(X) > U(Y)$ and $u(x) > u(y)$. Conversely, suppose that $u(x) > u(y)$, so that $X \rhd Y$, $x \in X$ and $y \in Y$. Hence, there are $z \in X$, $w \in Y$ such that $z\ P\ w$, $z\ I\ x$ and $w\ I\ y$. This implies $x\ P\ y$.

Sufficiency. Suppose that u on A satisfies (3). Let $U(X) = u(x)$ if $x \in X$. The function U is well defined since $x \in X$ and $y \in X$ imply $x\ I\ y$, so that $u(x) = u(y)$. Suppose that $X \rhd Y$. This implies that, for some $x \in X$ and some $y \in Y$, $x\ P\ y$ so that $u(x) > u(y)$ and $U(X) > U(Y)$. Suppose now that $U(X) > U(Y)$. This implies that $x\ P\ y$, for some $x \in X$ and some $y \in Y$. Hence, $X \rhd Y$. If $X \neq Y$, $x \in X$ and $y \in Y$ imply $x\ P\ y$ or $y\ P\ x$, so that $U(X) \neq U(Y)$. □

With Lemma 11 at hand, results for weak orders become simple corollaries of the results in Section 2. We first tackle the situation in which the weak order P is such that A/I is countable. We have:

Theorem 12 (Representation of weak orders on countable sets)
Let P be a weak order on a set A. If A/I is countable, there is a real-valued function u on A such that, for all $x, y \in A$,

$$x\ P\ y \Leftrightarrow u(x) > u(y).$$

PROOF. Necessity is obvious. We show sufficiency. Since P is a weak order, $\rhd$ is a linear order on A/I. Since A/I is countable, we use Theorem 1 to obtain a numerical representation of $\rhd$. The conclusion follows from Lemma 11. □

We now turn to the general case. In view of Lemma 11, for a numerical representation of P to exist, it is necessary and sufficient that $\rhd$ has one. Hence, the linear order $\rhd$ on A/I must have a countable dense subset. As soon as this is true, Lemma 11 allows to obtain the numerical representation of P. This is detailed in the following theorem.

Theorem 13 (Representation of weak orders)
Let P be a weak order on a set A. The following statements are equivalent.

1. *There is countable subset of A/I that is dense in A/I for $\rhd$.*

2. *There is a real-valued function u on A such that, for all $x, y \in A$,*

$$x \; P \; y \Leftrightarrow u(x) > u(y).$$

Proof. The equivalence follows from combining Lemma 11 with Theorem 7. □

Three remarks about the above results are in order.

Remark 14
Notice that instead of asking for a countable subset of A/I that is dense in A/I for $\rhd$ we could have simply asked for the existence of a countable subset B such that, for all $x, y \in A$, $x \; P \; y$ implies that $x \; P^{cd} \; z$ and $z \; P^{cd} \; y$ for some $z \in B$. Abusing terminology, we will say that such a set B is dense in A for the weak order P. •

Remark 15
Using a simple duality argument, the above result can be reformulated for the case of a complete and transitive binary relation S, this time asking for a representation such that, for all $x, y \in A$,

$$x \; S \; y \Leftrightarrow u(x) \geq u(y).$$

Under this form, it is clear that Theorem 13 is of central interest in Economics: it gives conditions under which the preferences of consumers can be represented by utility functions. •

Remark 16
Suppose that P and P' are two weak orders on A such that $P \subseteq P'$. Suppose that B is dense in A for P', so that P' has a numerical representation. It is easy to see that B is also dense in A for P. Hence, if $P \subseteq P'$ and P' has a numerical representation, P also has one. •

4 Biorders

4.1 Preliminaries

Let P be a binary relation on A. The relation P induces three new binary relations on A, denoted T_ℓ, T_r and T respectively called the left trace, the right trace and the trace of P. They are defined setting, for all $x, y, z \in A$,

$$\begin{gathered} x\ T_\ell\ y \Leftrightarrow yP \subseteq xP \Leftrightarrow [y\ P\ z \Rightarrow x\ P\ z], \\ x\ T_r\ y \Leftrightarrow Px \subseteq Py \Leftrightarrow [z\ P\ x \Rightarrow z\ P\ y], \\ x\ T\ y \Leftrightarrow [yP \subseteq xP \text{ and } Px \subseteq Py] \Leftrightarrow \\ \big[[y\ P\ z \Rightarrow x\ P\ z] \text{ and } [z\ P\ x \Rightarrow z\ P\ y]\big]. \end{gathered}$$

By construction, it is clear that T_ℓ, T_r and T are always reflexive and transitive and that $T_\ell P \subseteq P$ and $PT_r \subseteq P$. We denote by W_ℓ (respectively W_r and W) the asymmetric part of T_ℓ (respectively, T_r and T). The symmetric part of T_ℓ (respectively, T_r and T) is denoted by E_ℓ (respectively E_r and E). It is easy to see that E_ℓ, E_r and E are equivalence relations.

We have:

Lemma 17
Let P be a binary relation on A. The following statements are equivalent.

1. *P is a biorder.*
2. *P^{cd} is a biorder.*
3. *T_ℓ is complete.*
4. *T_r is complete.*

Proof. The proof that (1) is equivalent to (2) is obvious. We show that (1) is equivalent to (3), the proof that (1) is equivalent to (4) being similar. The relation P is not a biorder iff, for some $x, y, z, w \in A$, $x\ P\ y$, $z\ P\ w$, $x\ P^c\ w$ and $z\ P^c\ y$. This is equivalent to saying that $(x, z) \notin T_\ell$ and $(z, x) \notin T_\ell$. □

The following construction of a disjoint duplication of the set A will be central in what follows. Let $A^r = \{(x, +) : x \in A\}$ and $A^\ell = \{(x, -) : x \in A\}$. We will often write x^r instead of $(x, +)$ and x^ℓ instead of $(x, -)$. By construction, $A^\ell \cap A^r = \varnothing$. Let $A^{\ell r} = A^r \cup A^\ell$.

We define a binary relation $\succsim^P$ on $A^{\ell r}$ setting, for all $x, y \in A$,

$$\begin{aligned} & x^\ell \succsim^P y^\ell \Leftrightarrow x\ T_\ell\ y, \\ & x^r \succsim^P y^r \Leftrightarrow x\ T_r\ y, \\ & x^\ell \succsim^P y^r \Leftrightarrow x\ P\ y, \\ & x^r \succsim^P y^\ell \Leftrightarrow [\text{for all } z, w \in A, z\ P\ x \text{ and } y\ P\ w \Rightarrow z\ P\ w]. \end{aligned}$$

We will use $\sim^P$ and $\succ^P$ to denote the symmetric and asymmetric parts of $\succsim^P$. The following lemma, although quite simple, is crucial.

Lemma 18
If P is a biorder, then the relation $\succsim^P$ on $A^{\ell r}$ is complete and transitive, i.e., is a complete preorder.

PROOF. *Completeness.* We know that T_ℓ and T_r are complete. Hence, the only way to violate completeness is to assume that $(x^r, y^\ell) \notin \succsim^P$ and $(y^\ell, x^r) \notin \succsim^P$. The first relation implies that, for some $z, w \in A$, $z\ P\ x$, $y\ P\ w$ and $w\ P^{cd}\ z$. The second implies $x\ P^{cd}\ y$. Therefore we obtain $z\ P\ x$, $y\ P\ w$, $x\ P^{cd}\ y$ and $w\ P^{cd}\ z$, which violates the fact that P is a biorder.

Transitivity. Since T_ℓ and T_r are transitive, there are only six cases to examine.

1. Suppose that $x^\ell \succsim^P y^\ell$ and $y^\ell \succsim^P z^r$. We have $x\ T_\ell\ y$ and $y\ P\ z$. This implies $x\ P\ z$, so that $x^\ell \succsim^P z^r$.

2. Suppose that $x^r \succsim^P y^r$ and $y^r \succsim^P z^\ell$. We have $x\ T_r\ y$ and $[a\ P\ y$ and $z\ P\ b]$ imply $a\ P\ b$. Suppose that $c\ P\ x$ and $z\ P\ d$. Using $x\ T_r\ y$, $c\ P\ x$ implies $c\ P\ y$. Now, $c\ P\ y$ and $z\ P\ d$ imply $c\ P\ d$. Hence, $x^r \succsim^P z^\ell$.

3. Suppose that $x^\ell \succsim^P y^r$ and $y^r \succsim^P z^r$. We have $x\ P\ y$ and $y\ T_r\ z$. This implies $x\ P\ z$, so that $x^\ell \succsim^P z^r$.

4. Suppose that $x^\ell \succsim^P y^r$ and $y^r \succsim^P z^\ell$. We have $x\ P\ y$ and $[a\ P\ y$ and $z\ P\ b]$ imply $a\ P\ b$. Hence, since $x\ P\ y$, $z\ P\ c$ implies $x\ P\ c$. This shows that $x\ T_\ell\ z$, so that $x^\ell \succsim^P z^\ell$.

5. Suppose that $x^r \succsim^P y^\ell$ and $y^\ell \succsim^P z^r$. We have $[a\ P\ x$ and $y\ P\ b]$ imply $a\ P\ b$ and $y\ P\ z$. Hence, since $y\ P\ z$, $c\ P\ x$ implies $c\ P\ z$. This shows that $x\ T_r\ z$, so that $x^r \succsim^P z^r$.

6. Suppose that $x^r \succsim^P y^\ell$ and $y^\ell \succsim^P z^\ell$. We have $y\ T_\ell\ z$ and $[a\ P\ x$ and $y\ P\ b]$ imply $a\ P\ b$. Suppose that $c\ P\ x$ and $z\ P\ d$. Using $y\ T_\ell\ z$, $z\ P\ d$ implies $y\ P\ d$. Now, $c\ P\ x$ and $y\ P\ d$ imply $c\ P\ d$. Hence, $x^r \succsim^P z^\ell$. □

Remark 19
Observe that in the above lemma, the Ferrers property is only used to prove the completeness of $\succsim^P$. Indeed, it is easy to see that, for any binary relation S, the relation $\succsim^S$ induced on $A^{\ell r}$ is always reflexive and transitive. Doignon, Ducamp, and Falmagne (1984) have shown that being a biorder is a necessary and sufficient condition for $\succsim^P$ to be complete. Furthermore, they have shown that, in the set of all relations $\succsim$ on $A^{\ell r}$ such that $x^\ell \succsim y^r$ iff $x\ P\ y$, $\succsim^P$ is maximal, i.e., $\succsim \subseteq \succsim^P$. •

4.2 The countable case

Let P be a binary relation on A. Suppose that there are two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) > v(y). \tag{4}$$

For the sake of completeness, let us first note the following obvious result.

Lemma 20
Let P be a binary relation on A. If P has a representation (4) *then it is a biorder.*

Proof. Suppose that $x \; P \; y$, $z \; P \; w$, $x \; P^c \; w$ and $z \; P^c \; y$. Using the representation, this implies $u(x) > v(y)$, $u(z) > v(w)$, $u(x) \leq v(w)$ and $u(z) \leq v(y)$. The first and the third relations lead to $v(w) > v(y)$. The second and the fourth imply $v(w) < v(y)$, a contradiction. □

Remark 21
Suppose that a binary relation P on A is such that there is a function $u : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) \geq v(y).$$

Paraphrasing the proof of lemma 20, it is easy to show that the existence of such a numerical representation implies that P must be a biorder. •

Suppose that P is a biorder on a countable set A. Lemma 18 shows that $\succsim^P$ on $A^{\ell r}$ is a complete and transitive. Because A is countable, the same is true for $A^{\ell r}$. Hence, we can use Theorem 12, to obtain a real-valued function V on $A^{\ell r}$ such that, for all $\alpha, \beta \in A^{\ell r}$,

$$\alpha \succsim^P \beta \Leftrightarrow V(\alpha) \geq V(\beta).$$

Define u and v on A setting, for all $x \in A$, $u(x) = V(x^\ell)$ and $v(x) = V(x^r)$. Using the definition of $\succsim^P$, we obviously have that, for all $x, y \in A$, $x \; P \; y$ iff $x^\ell \succsim^P y^r$ iff $V(x^\ell) \geq V(y^r)$, so that,

$$x \; P \; y \Leftrightarrow u(x) \geq v(y). \tag{5}$$

Furthermore, this representation is such that:

$$\begin{aligned} x \; T_\ell \; y &\Leftrightarrow u(x) \geq u(y), \\ x \; T_r \; y &\Leftrightarrow v(x) \geq v(y). \end{aligned} \tag{6}$$

This gives a first result for the numerical representation of biorders in the countable case. Observe that (5) is however different from (4), the strict inequality having been replaced by a non-strict one.

Fortunately, no further analysis is required to cover the case of (4). Indeed, remember from Lemma 17 that P is a biorder if and only if P^{cd} is a biorder. Hence, the above reasoning can be conducted starting with P^{cd} instead of P through the construction of $\succsim^{P^{cd}}$ on $A^{\ell r}$. Doing so, we would obtain a numerical representation of P^{cd} such that, for all $x, y \in A$,

$$y \; P^{cd} \; x \Leftrightarrow V(y) \geq U(x),$$

so that

$$x \; P \; y \Leftrightarrow U(x) > V(y).$$

With such a construction, we have, for all $x, y, z \in A$,

$$[x \; P^{cd} \; z \Rightarrow y \; P^{cd} \; z] \Leftrightarrow [z \; P \; y \Rightarrow z \; P \; x] \Leftrightarrow y \; T_r \; x \Leftrightarrow V(y) \geq V(x),$$
$$[z \; P^{cd} \; y \Rightarrow z \; P^{cd} \; x] \Leftrightarrow [x \; P \; z \Rightarrow y \; P \; z] \Leftrightarrow y \; T_\ell \; x \Leftrightarrow U(y) \geq U(x),$$

so that (6) holds.

Lemma 20 and Remark 21 have shown that being a biorder is a necessary condition for both (4) and (5). Hence, the above observations have proved the following result giving necessary and sufficient conditions for the existence of a strict representation (4) and of a non-strict representation (5) on a countable set.

Theorem 22 (Representation of biorders on countable sets)
Let P be a binary relation on a countable set A. The following statements are equivalent.

1. *P is a biorder.*

2. *There are two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,*
$$x \; P \; y \Leftrightarrow u(x) > v(y).$$

3. *There are two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,*
$$x \; P \; y \Leftrightarrow u(x) \geq v(y).$$

Furthermore, the functions u and v used in statements 2 or 3 above can always be chosen in such a way that, for all $x, y \in A$,

$$x \; T_\ell \; y \Leftrightarrow u(x) \geq u(y),$$
$$x \; T_r \; y \Leftrightarrow v(x) \geq v(y).$$

Remark 23

Fishburn (1985) suggested a different construction on $A^{\ell r}$ for the study of interval orders. It is not difficult to see that this construction also works for biorders. The basic idea of this construction is as follows. Define $\succsim^P_\circ$ as $\succsim^P$, except that now:

$$x^r \succsim^P_\circ y^\ell \Leftrightarrow x \; P^{cd} \; y. \tag{7}$$

It is easy to prove that $\succsim^P_\circ$ on $A^{\ell r}$ is complete and transitive when P is a biorder. An advantage of this construction is that the equivalence classes of $\sim^P_\circ$ can now be given a simple form since, obviously, $x^r \succsim^P_\circ y^\ell$ and $x^\ell \succsim^P_\circ y^r$ is now impossible. We leave to the reader the simple task of showing that, with this construction, if A/E is countable (where E is the symmetric part of T), the same will be true for $A^{\ell r}/\sim^P_\circ$. This allows to state a slightly stronger result, replacing in Theorem 22, "on a countable set A" by "such that A/E is countable" with no further modification.

Although this alternative construction allows to state a slightly more powerful result in the countable case, it leads to a more complex analysis of the general case, to which we now turn. •

4.3 The general case

The basic idea used to extend Theorem 22 to the general case is simple. If it can be ensured that $\succsim^P$ on $A^{\ell r}$ has a numerical representation, a numerical representation for P on A will easily follow. The difficulty here is to formulate a condition that will guarantee that the weak order $\succsim^P$ has a numerical representation and is necessary for the existence of representation of the biorder P. This has been achieved in Doignon et al. (1984).

It is first important to realize that the equivalence between the strict form of the representation (4) and the non-strict one (5) does not carry over to the general case. This is exemplified below.

Example 24

It is obvious that the relation $>$ on $\mathbb{R}$ is a biorder. It clearly has a strict representation (4) with $u(x) = v(x) = x$, for all $x \in \mathbb{R}$. Suppose that $>$ on $\mathbb{R}$ has a non-strict representation (5):

$$x > y \Leftrightarrow f(x) \geq g(y).$$

Because $>$ is irreflexive, we must have $g(x) > f(x)$, for all $x \in \mathbb{R}$. We can always find a rational number $\rho(x) \in \mathbb{Q}$ such that $g(x) > \rho(x) > f(x)$. Now, $x > y$ implies $f(x) \geq g(y)$, so that we obtain $g(x) > \rho(x) > f(x) \geq g(y) > \rho(y) > f(y)$, implying that ρ is a one-to-one mapping from $\mathbb{R}$ onto $\mathbb{Q}$. This shows that $>$ cannot have a non-strict representation (5).

A similar example can be constructed to show that, on uncountable sets, P may have a non-strict representation (5) without having a strict representation (4). $\diamond$

We first tackle the case of a non-strict representation (5), following Doignon et al. (1984). To this end, we first introduce a new notion of density.

Definition 25 (Widely dense)
Let P be a biorder on A. A subset $B \subseteq A$ is said to be widely dense *in A for P if, for all, $x, y \in A$, $x\ P^c\ y$ implies that $w\ P^c\ y$ and $w\ T_\ell\ x$, for some $w \in B$.*

The central condition for our purpose will be to require that there is a *countable* subset B^w that is widely dense in A for P.

Remark 26
The version of the condition used here may be slightly weakened. In fact it could just be asked that there is a countable subset B^w such that $x\ P^c\ y$ implies either that $w\ P^c\ y$ and $w\ T_\ell\ x$, for some $w \in B^w$ or that $y\ T_r\ z$ and $x\ P^c\ z$, for some $z \in B^w$. For the sake of simplicity, we do not consider this weaker version in what follows.

The extension of Theorem 22 to arbitrary sets will go through two lemmas.

Lemma 27
Let P be a biorder on A. Suppose that there is a countable subset B^w that is widely dense in A for P. Then there is a countable subset $B^* \subseteq A^{\ell r}$ that is dense in $A^{\ell r}$ for $\succ^P$.

Proof. We have to show that there is a countable set B^* such that, for all $\alpha, \beta \in A^{\ell r}$, $(\beta, \alpha) \notin \succsim^P$ implies $\alpha \succsim^P \gamma$ and $\gamma \succsim^P \beta$, for some $\gamma \in B^*$. Since α and β can belong either to A^ℓ or to A^r, there are four cases to distinguish.

1. Suppose that $(x^\ell, y^\ell) \notin \succsim^P$ so that, for some $z \in A$, $y\ P\ z$ and $x\ P^c\ z$. Using the fact that B^w is widely dense, $x\ P^c\ z$ implies $w\ P^c\ z$ and $w\ T_\ell\ x$, for some $w \in B^w$. By construction, $w\ T_\ell\ x$ implies $w^\ell \succsim^P x^\ell$. Suppose that $w\ P\ a$. Using the Ferrers property, $y\ P\ z$ and $w\ P\ a$ implies either $y\ P\ a$ or $w\ P\ z$. Since, $w\ P^c\ z$, we have $y\ P\ a$, so that $y\ T_\ell\ w$ and $y^\ell \succsim^P w^\ell$.

2. Suppose that $(x^r, y^r) \notin \succsim^P$ so that, for some $z \in A$, $z\ P\ x$ and $z\ P^c\ y$. Using the fact that B^w is widely dense, $z\ P^c\ y$ implies $w\ P^c\ y$ and $w\ T_\ell\ z$, for some $w \in B^w$. Because $z\ P\ x$ and $w\ T_\ell\ z$, we have $w\ P\ x$, i.e., $w^\ell \succsim^P x^r$. Suppose that $a\ P\ y$ and $w\ P\ b$. Using the Ferrers property, $w\ P^c\ y$ implies $a\ P\ b$. Hence, $y^r \succsim^P w^\ell$.

3. Suppose that $(x^\ell, y^r) \notin \succsim^P$ so that $x\ P^c\ y$. Using the fact that B^w is widely dense this implies $w\ T_\ell\ x$ and $w\ P^c\ y$, for some $w \in B^w$. This implies $w^\ell \succsim^P x^\ell$. Suppose now that $a\ P\ y$ and $w\ P\ b$. Using the Ferrers property and $w\ P^c\ y$, we obtain $a\ P\ b$, so that $y^r \succsim^P w^\ell$.

4. Suppose that $(x^r, y^\ell) \notin \succsim^P$ so that, for some $a, b \in A$, $a\ P\ x$, $y\ P\ b$ and $a\ P^c\ b$. Using the fact that B^w is widely dense, $a\ P^c\ b$ implies $w\ T_\ell\ a$ and $w\ P^c\ b$, for some $w \in B^w$. Because we have $y\ P\ b$ and $w\ P^c\ b$ and T_ℓ is complete, we obtain $y\ T_\ell\ w$ so that $y^\ell \succsim^P w^\ell$. Using $w\ T_\ell\ a$ and $a\ P\ x$ leads to $w\ P\ x$, so that $w^\ell \succsim^P x^r$.

Hence, $B^* = \{w^\ell, w^r : w \in B^w\}$ is dense in A for $\succ^P$. □

Lemma 28
Suppose that P on A has a non-strict representation (5). *Then there is a countable subset B^w that is widely dense in A for P.*

Proof. Suppose that there are two real-valued functions u and v on A such that $z\ P\ w$ iff $u(z) \geq v(w)$. Let $u(A) = \{\alpha \in \mathbb{R} : u(x) = \alpha, \text{ for some } x \in A\}$ and $v(A) = \{\alpha \in \mathbb{R} : v(x) = \alpha, \text{ for some } x \in A\}$.

Suppose that $\alpha_i \in u(A)$ is such that, for some $\delta_i \in v(A)$,

$$\alpha_i < \delta_i \text{ and }]\alpha_i, \delta_i[\ \cap\ u(A) = \varnothing. \tag{8}$$

If such a real number α_i exists, we associate to it a particular number $\delta_i \in v(A)$ such that (8) holds.

Observe that, by construction, the interval $]\alpha_i, \delta_i[$ is nonempty. Consider now two intervals $]\alpha_i, \delta_i[$ and $]\alpha_j, \delta_j[$ such that (8) holds and $\alpha_j > \alpha_i$. The two intervals $]\alpha_i, \delta_i[$ and $]\alpha_j, \delta_j[$ must be disjoint because $\alpha_j < \delta_i$ would violate the fact that $]\alpha_i, \delta_i[\ \cap\ u(A) = \varnothing$. Therefore, this process leads to defining a collection of disjoint and nonempty real intervals. Each interval in this collection contains a rational number. Hence, the collection must be countable. We can therefore obtain a countable subset C of A such that C contains all the lower endpoints of these intervals.

Consider now any two distinct rational numbers $\rho < \varrho$ such that $\rho < u(w) < \varrho$, for some $w \in A$. There are only countably many such pairs. For each of them, we select a particular $w \in A$ satisfying $\rho < u(w) < \varrho$. This defines a countable subset D of A.

Clearly $B^w = C \cup D$ is countable. We claim that it is widely dense in A for P. Let $x, y \in A$ be such that $x \; P^c \; y$, i.e., $u(x) < v(y)$. We have to show that there is a $w \in B^w$ such that $w \; P^c \; y$ and $w \; T_\ell \; x$. There are two cases to consider.

1. If $]u(x), v(y)[\; \cap \; u(A) = \varnothing$, then $u(x)$ must be one of the α_i used to define the set C. Hence, for some $w \in C$, we have $\alpha_i = u(x) = u(w)$. Hence, $u(x) = u(w) < v(y)$ so that $w \; P^c \; y$. Because $u(w) = u(x)$, $u(x) \geq v(z)$ implies $u(w) \geq v(z)$, so that $w \; T_\ell \; x$.

2. If $]u(x), v(y)[\; \cap \; u(A) \neq \varnothing$, we have that $u(x) < u(z) < v(y)$, for some $z \in A$. Hence there are rational numbers $\rho < \varrho$ such that $u(x) < \rho < u(z) < \varrho < v(y)$. Hence, for some $w \in D$, we have $u(x) < \rho < u(w) < \varrho < v(y)$. Because $u(w) < v(y)$, we have $w \; P^c \; y$. Because $u(w) > u(x)$, $u(x) \geq v(z)$ implies $u(w) > v(z)$, so that $w \; T_\ell \; x$. □

Let P be a binary relation on A. If P has a non-strict representation (5), it is clear that it must be a biorder. Furthermore, Lemma 28 has shown that there must exist a countable subset B^w that is widely dense in A for P.

Conversely, suppose that P is a biorder and that there is a countable subset B^w that is widely dense in A for P. Lemma 18 implies that $\succsim^P$ on $A^{\ell r}$ is a complete preorder. Using Lemma 27, we know that there is a countable subset that is dense in $A^{\ell r}$ for $\succ^P$. Theorem 13 therefore implies that there is a real-valued function V on $A^{\ell r}$ such that, for all $\alpha, \beta \in A^{\ell r}$,

$$\alpha \succsim^P \beta \Leftrightarrow V(\alpha) \geq V(\beta).$$

Define u and v on A setting, for all $x \in A$, $u(x) = V(x^\ell)$ and $v(x) = V(x^r)$. Using the definition of $\succsim^P$, we obviously have that, for all $x, y \in A$, $x \; P \; y$ iff $x^\ell \succsim^P y^r$ iff $V(x^\ell) \geq V(y^r)$. Hence, we have:

$$x \; P \; y \Leftrightarrow u(x) \geq v(y).$$

The preceding observations prove the following theorem giving necessary and sufficient conditions for the existence of a non-strict representation (5) on an arbitrary set.

Theorem 29 (Non-strict representation of biorders)
Let P be a binary relation on a set A. There are two real-valued functions u and v such that such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) \geq v(y),$$

iff P is a biorder and there is a countable set B^w that is widely dense in A for P.

Furthermore, the functions u and v used above can always be chosen in such a way that, for all $x, y \in A$,

$$x\, T_\ell\, y \Leftrightarrow u(x) \geq u(y),$$
$$x\, T_r\, y \Leftrightarrow v(x) \geq v(y).$$

As in the countable case, it suffices to apply the above result to P^{cd} to obtain a strict representation (4). Imposing that P^{cd} is a biorder is equivalent to imposing that P is a biorder. Requiring the existence of countable subset that is widely dense in A for P^{cd} is tantamount to requiring the existence of a countable subset B^s of A such that $x\ P\ y$ implies $x\ T_\ell\ z$ and $z\ P\ y$, for some $z \in B^s$. This is formalized in the following definition of a strictly dense subset.

Definition 30 (Strictly dense)
Let P be a biorder on A. We say that B^s is strictly dense *in A for P if $x\ P\ y$ implies $x\ T_\ell\ w$ and $w\ P\ y$, for some $w \in B^s$.*

The above observations prove the following theorem giving necessary and sufficient conditions for the existence of a strict representation (4) on an arbitrary set.

Theorem 31 (Strict representation of biorders)
Let P be a binary relation on a set A. There are two real-valued functions u and v such that such that, for all $x, y \in A$,

$$x\ P\ y \Leftrightarrow u(x) > v(y),$$

iff P is a biorder and there is a countable set B^s that is strictly dense in A for P.

Furthermore, the functions u and v used above can always be chosen in such a way that, for all $x, y \in A$,

$$x\, T_\ell\, y \Leftrightarrow u(x) \geq u(y),$$
$$x\, T_r\, y \Leftrightarrow v(x) \geq v(y).$$

Remark 32
As was the case above, the formulation of the strict denseness can be weakened. In fact, all what is needed is the existence of a countable subset B^s of A such that $x\ P\ y$ implies either $x\ T_\ell\ w$ and $w\ P\ y$, for some $w \in B^s$ or $z\ T_r\ y$ and $x\ P\ z$, for some $z \in B^s$. •

The reader might well be puzzled by the fact that, in general, the strict and the non-strict representations of biorders are not equivalent. This is, in fact, a common feature of most numerical representations involving nontrivial thresholds. It is possible to envisage more general representations than the ones considered here that leave undecided what happens at the "boundary" between $u(x)$ and $v(y)$. For space reasons, we do not develop this point (see the comments in Section 8).

5 Interval orders

Let P be a binary relation on a set A.

Suppose that there are $u : A \to \mathbb{R}$ and a threshold function $\varepsilon : A \to \mathbb{R}_+$ such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) - u(y) > \varepsilon(y),$$

or, equivalently, of two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$\begin{aligned} x \; P \; y \Leftrightarrow u(x) > v(y), \\ u(x) \leq v(x). \end{aligned} \tag{9}$$

The representation (9) will be called a strict representation.

Similarly to what was done with biorders, we can also consider a non-strict representation such that:

$$\begin{aligned} x \; P \; y \Leftrightarrow u(x) \geq v(y), \\ u(x) < v(x). \end{aligned} \tag{10}$$

It is clear that irreflexivity and the Ferrers property are necessary for both (9) and (10).

The results for biorders in Section 4 give all what is necessary to study the numerical representation of interval orders on arbitrary sets. Adding irreflexivity to these results immediately leads to corresponding ones for interval orders.

5.1 The countable case

Suppose that (4) holds. The irreflexivity of P implies $x \; P^c \; x$ so that $u(x) \leq v(x)$, which is exactly what is needed in (9). Adding irreflexivity to (5) has a similar effect and leads to (10). This proves the following corollary of Theorem 22 giving giving necessary and sufficient conditions for the existence of a strict representation (9) or a non-strict representation (10) on a countable set.

Theorem 33 (Representation of interval orders on countable sets)
Let P be a binary relation on a countable set A. The following statements are equivalent.

1. *P is an interval order.*

2. *There are two real-valued functions u and v on A such that, for all $x, y \in A$,*
$$\begin{gathered} x \; P \; y \Leftrightarrow u(x) > v(y), \\ u(x) \leq v(x). \end{gathered}$$

3. *There are two real-valued functions u and v on A such that, for all $x, y \in A$,*
$$\begin{gathered} x \; P \; y \Leftrightarrow u(x) \geq v(y), \\ u(x) < v(x). \end{gathered}$$

Furthermore, the functions u and v used in statements 2 or 3 above can always be chosen in such a way that, for all $x, y \in A$,

$$\begin{gathered} x \; T_\ell \; y \Leftrightarrow u(x) \geq u(y), \\ x \; T_r \; y \Leftrightarrow v(x) \geq v(y). \end{gathered}$$

Remark 34
As mentioned in Remark 23, the construction of Fishburn (1970b) (see also Fishburn, 1985, ch. 2) is different from ours. It allows to state a slightly stronger result, replacing in Theorem 33, “on a countable set A” by “such that A/E is countable” with no further modification. •

5.2 The general case

Similarly to what we did in the countable case, adding irreflexivity to the conditions needed to obtain a numerical representation in Theorems 29 and 31, leads to corresponding results for the representations (9) and (10) on an arbitrary set. As before, we have to separate the case of strict and non-strict representations.

Theorem 35 (Non-strict representation of interval orders)
Let P be a binary relation on a set A. There are two real-valued functions u and v on A such that, for all $x, y \in A$,

$$\begin{gathered} x \; P \; y \Leftrightarrow u(x) \geq v(y), \\ u(x) < v(x), \end{gathered}$$

iff P is an interval order and there is a countable set B^w that is widely dense in A for P.

Furthermore, the functions u and v can always be chosen so that, for all $x, y \in A$,

$$x \; T_\ell \; y \Leftrightarrow u(x) \geq u(y),$$
$$x \; T_r \; y \Leftrightarrow v(x) \geq v(y).$$

Theorem 36 (Strict representation of interval orders)
Let P be a binary relation on a set A. There are two real-valued functions u and v on A such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) > v(y),$$
$$u(x) \leq v(x),$$

iff P is an interval order and there is a countable set B^s that is strictly dense in A for P.

Furthermore, the functions u and v can always be chosen so that, for all $x, y \in A$,

$$x \; T_\ell \; y \Leftrightarrow u(x) \geq u(y),$$
$$x \; T_r \; y \Leftrightarrow v(x) \geq v(y).$$

6 Semiorders

Let P be a semiorder on a set A, i.e., a semitransitive interval order. A famous result due to Scott and Suppes (1958) shows that, when A is finite, this is a necessary and sufficient condition for the existence of a function $u : A \to \mathbb{R}$ and a threshold $\varepsilon \in \mathbb{R}_+$ such that, for all $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) - u(y) > \varepsilon. \tag{11}$$

Clearly, in the finite case, the strict representation (11) is equivalent to the following non-strict one:

$$x \; P \; y \Leftrightarrow u(x) - u(y) \geq \sigma, \tag{12}$$

with, now, $\sigma > 0$ (see Pirlot and Vincke, 1997, Section 4.2).

An important difference between the finite and the denumerable cases is that being a semiorder is no more equivalent to having a constant threshold representation (strict or not) when A is denumerable. This is demonstrated in the following example borrowed from Fishburn (1985, p. 30).

Example 37
Let $A = \mathbb{N} \cup \{\omega\}$. For all $x, y \in \mathbb{N}$, let $x \; P \; y \Leftrightarrow x - y > 1$. Let $\omega \; P \; x$, for all $x \in \mathbb{N}$. It is not difficult to show that P on A is a semiorder. Since I on $\mathbb{N}$ is not transitive, a constant threshold representation must have $\varepsilon > 0$. Suppose that u and $\varepsilon > 0$ give such a representation. We have:

$$\omega \; P \ldots P \; (2n+1) \; P \; (2(n-1)+1) \; P \ldots P \; 3 \; P \; 1$$

for all $n \in \mathbb{N}$, implying $u(2n+1) > u(1) + n\varepsilon$, so that $u(\omega) > u(1) + n\varepsilon$, for all $n \in \mathbb{N}$. This is clearly impossible. Hence, this semiorder does not have a strict constant threshold representation. A similar reasoning shows that P does not have a non-strict constant threshold representation either. $\diamond$

This raises the question of determining what is an adequate generalization of the constant threshold representation in view of obtaining numerical representations of semiorders on arbitrary sets. There is no agreement on that point in the literature. We concentrate below on what we will call representations with no proper nesting. Besides being easily derivable from results on biorders, we believe that such representations are the more adequate ones in the infinite case.

We will consider three different types of representations for a semiorder. They are all special cases of the representations considered for interval orders (see the strict representation (9) and the non-strict one (10)).

1. In the first representation, called "representation with no proper nesting", we add either to (9) or to (10) the fact that it is impossible to have at the same time

   $$u(y) > u(x) \text{ and } v(x) > v(y).$$

   We forbid here a representation in which the interval associated to an element of A will be strictly included in another interval. Observe that with such a representation, it is possible to have

   $$u(y) = u(x) \text{ and } v(x) > v(y) \text{ or,}$$
   $$u(y) > u(x) \text{ and } v(x) = v(y).$$

2. In the second representation, called "representation with no nesting", we add either to (9) or to (10) the fact that if the lower bound of an interval is smaller than the lower bound of another interval, the same will be true for their respective upper bounds and vice versa. More precisely, we add either to (9) or to (10) the following condition:

   $$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y).$$

As we will see, although such representations may seem more attractive than representations with no proper nesting, they also have drawbacks.

3. In the third type of representation, the "constant threshold representation", all intervals must have the same length. Contrary to what happens in the finite case, this type of representations raises many problems when it turns to infinite sets, as Example 37 shows.

Clearly, every constant threshold representation is a representation with no nesting and every representation with no nesting is a representation with no proper nesting. We examine each of these representations in turn. Before that, we study a number of simple properties of semitransitive biorders.

6.1 Preliminaries

We begin by stating a simple but crucial property of semitransitive biorders. Such relations were called "coherent biorders" in Aleskerov and Monjardet (2002).

Lemma 38
Let P be a binary relation on A. The following statements are equivalent.

1. *P is a semitransitive biorder.*
2. *P^{cd} is a semitransitive biorder.*
3. *T is complete.*

Proof. [$1 \Leftrightarrow 2$]. Lemma 17 has shown that P is a biorder iff P^{cd} is a biorder. Suppose that P is not semitransitive. Therefore, we have, for some $x, y, z, w \in A$, $x\ P\ y$, $y\ P\ z$, $x\ P^c\ w$ and $w\ P^c\ z$. This is equivalent to saying that $z\ P^{cd}\ w\ w\ P^{cd}\ x$, $(z, y) \notin P^{cd}$ and $(y, x) \notin P^{cd}$. This is equivalent the fact that P^{cd} is not semitransitive.

[$1 \Leftrightarrow 3$]. Since $T = T_\ell \cap T_r$, it is easy to see that it will be complete if and only if both T_r and T_ℓ are complete and there are no $x, y \in A$ such that $(x, y) \notin T_r$ and $(y, x) \notin T_\ell$. We know from Lemma 17 that the Ferrers property is a necessary and sufficient condition for both T_r and T_ℓ to be complete. Suppose that $(x, y) \notin T_r$ and $(y, x) \notin T_\ell$. Therefore, we have, for some $z, w \in A$, $z\ P\ x$, $z\ P^c\ y$, $x\ P\ w$ and $y\ P^c\ w$. This is equivalent to violating the semitransitivity of P. □

6.2 Representations with no proper nesting

6.2.1 The countable case

Suppose that P is a biorder. If $\succsim^P$ has a numerical representation (and we now know from Section 4 necessary and sufficient conditions for that), there will be two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$, $x\ P\ y$ iff $u(x) > v(y)$. Furthermore, we know that such functions can always be chosen so that $x\ T_\ell\ y$ iff $u(x) \geq u(y)$ and $x\ T_r\ y$ iff $v(x) \geq v(y)$.

If furthermore P is semitransitive, Lemma 38 leads to conclude that T is complete, so that $x\ W_\ell\ y$ implies $x\ T_r\ y$. This shows that $u(y) > u(x)$ implies $v(y) \geq v(x)$.

Given the above observations, we say that P on A has a strict representation with no proper nesting if there are two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$\begin{aligned} x\ P\ y &\Leftrightarrow u(x) > v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x). \end{aligned} \tag{13}$$

Repeating the above steps starting with a non-strict representation of a biorder, we define a non-strict representation with no proper nesting requiring the existence of two functions $u : A \to \mathbb{R}$ and $v : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$\begin{aligned} x\ P\ y &\Leftrightarrow u(x) \geq v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x). \end{aligned} \tag{14}$$

Let us first note that if P has a strict or a non-strict representation with no proper nesting, it must be a semitransitive biorder (i.e., a coherent biorder).

Lemma 39
Let P be a binary relation on A. If P has a strict representation (13) *or a non-strict one* (14) *then P is a semitransitive biorder.*

Proof. The fact that P is a biorder follows from Theorems 35 and 36. Let us show that P is semitransitive. Suppose that $x\ P\ y$, $y\ P\ z$, $x\ P^c\ w$ and $w\ P^c\ z$. The strict representation (13), implies $u(x) > v(y)$, $u(y) > v(z)$, $u(x) \leq v(w)$ and $u(w) \leq v(z)$. The first and the third relations imply $v(w) > v(y)$. The second and the fourth imply $u(y) > u(w)$. This contradicts the fact that there in no proper nesting of intervals. The proof with (14) is similar. $\square$

It is obvious to see that the irreflexivity of P implies $u(x) \leq v(x)$ (respectively, $u(x) < v(x)$) in (13) (respectively, (14)). The preceding observations allow us to state the following corollary of Theorem 22.

Theorem 40 (Representation with no proper nesting of semiorders on countable sets) *Let P be a binary relation on a countable set A. The following statements are equivalent.*

1. *P is a semiorder.*

2. *There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) \leq v(x)$ and*

$$\begin{aligned} x \; P \; y &\Leftrightarrow u(x) > v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x). \end{aligned}$$

3. *There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) < v(x)$ and*

$$\begin{aligned} x \; P \; y &\Leftrightarrow u(x) \geq v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x). \end{aligned}$$

Furthermore, the functions u and v used in statements 2 or 3 above can always be chosen in such a way that, for all $x, y \in A$,

$$\begin{aligned} x \; T_\ell \; y &\Leftrightarrow u(x) \geq u(y), \\ x \; T_r \; y &\Leftrightarrow v(x) \geq v(y). \end{aligned}$$

Two remarks on this result are in order.

Remark 41
Removing irreflexivity and the requirement that $u(x) \leq v(x)$ (respectively, that $u(x) < v(x)$) from the above result easily leads to a representation theorem on countable sets that deals with coherent biorders. •

Remark 42
Using the construction mentioned in Remark 23, leads to a slightly stronger result. Indeed, it is possible to replace in Theorem 40 "on a countable set A" by "such that A/E is countable" with no further modification (where, as before, E is the symmetric part of T). •

6.2.2 The general case

Using the denseness condition introduced above to deal with the case of biorders, allows us, in a simple way, to generalize the above result to arbitrary sets. As before, we separate the study of strict and non-strict representations.

Theorem 43 (Non-strict representation with no proper nesting of semiorders) *Let P be a binary relation on a set A. There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) < v(x)$ and*

$$\begin{aligned} x\ P\ y &\Leftrightarrow u(x) \geq v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x), \end{aligned}$$

iff P is a semiorder and there is a countable set B^w that is widely dense in A for P.

Furthermore, the functions u and v can always be chosen in such a way that, for all $x, y \in A$,

$$\begin{aligned} x\ T_\ell\ y &\Leftrightarrow u(x) \geq u(y), \\ x\ T_r\ y &\Leftrightarrow v(x) \geq v(y). \end{aligned}$$

Theorem 44 (Strict representation with no proper nesting of semiorders) *Let P be a binary relation on a set A. There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) \leq v(x)$ and*

$$\begin{aligned} x\ P\ y &\Leftrightarrow u(x) > v(y), \\ u(y) > u(x) &\Rightarrow v(y) \geq v(x), \end{aligned}$$

iff P is a semiorder and there is a countable set B^s that is strictly dense in A for P.

Furthermore, the functions u and v can always be chosen in such a way that, for all $x, y \in A$,

$$\begin{aligned} x\ T_\ell\ y &\Leftrightarrow u(x) \geq u(y), \\ x\ T_r\ y &\Leftrightarrow v(x) \geq v(y). \end{aligned}$$

Remark 45

Again, removing irreflexivity and the requirement that $u(x) \leq v(x)$ in the statement of theorem 44 (respectively, the requirement that $u(x) < v(x)$ in the statement of theorem 43) easily leads to a representation theorem on arbitrary sets that deals with coherent biorders. •

There are two main arguments to support the claim that representations with no proper nesting are the adequate numerical representations of semiorders on arbitrary sets. The first one is the simplicity with which the results for this type of representation can be deduced from the ones on biorders. The second, and most important one, is that they always give a faithful representation of T_ℓ and T_r. In any more constrained representation, the knowledge of u and v will not be equivalent to the knowledge of T_ℓ and T_r any more.

6.3 Representations with no nesting

A representation with no proper nesting allows to have $u(x) = u(y)$ but $v(x) \neq v(y)$. Forbidding such a possibility gives rise to what we call representations with no nesting. More precisely, we say that a pair of real-valued functions u and v on A is a strict representation with no nesting of P if

$$\begin{aligned} x \; P \; y &\Leftrightarrow u(x) > v(y), \\ u(x) \geq u(y) &\Leftrightarrow v(x) \geq v(y). \end{aligned} \tag{15}$$

We show below that in the countable case, representations with no nesting of semiorders always exist and are, therefore, equivalent to representations with no proper nesting. However, the situation in the general case is more delicate.

Following Fishburn (1973, 1985), define a binary relation $\succsim_*^P$ on $A^{\ell r}$ setting, for all $x, y \in A$,

$$\begin{aligned} x^\ell \succsim_*^P y^\ell &\Leftrightarrow x \; T \; y, \\ x^r \succsim_*^P y^r &\Leftrightarrow x \; T \; y, \\ x^r \succsim_*^P y^\ell &\Leftrightarrow x \; P^{cd} \; y, \\ x^\ell \succsim_*^P y^r &\Leftrightarrow x \; P \; y. \end{aligned}$$

Lemma 46
If P is a semitransitive biorder, then the relation $\succsim_*^P$ on $A^{\ell r}$ is complete and transitive.

Proof. Using Lemma 38, we know that T is complete. Hence, to violate the completeness of $\succsim_*^P$, we must have $(x^r, y^\ell) \notin \succsim_*^P$ and $(y^\ell, x^r) \notin \succsim_*^P$, for some $x^r, y^\ell \in A^{\ell r}$. This is equivalent to saying that $y \; P \; x$ and $y \; P^c \; x$, a contradiction.

Because T is transitive, we have to consider six cases to show that $\succsim_*^P$ is transitive.

1. $[x^\ell \succsim_*^P y^\ell$ and $y^\ell \succsim_*^P z^r]$. We have $x \; T \; y$ and $y \; P \; z$. This implies $x \; P \; z$ so that $x^\ell \succsim_*^P z^r$.

2. $[x^\ell \succsim_*^P y^r$ and $y^r \succsim_*^P z^\ell]$. We have $x \; P \; y$ and $y \; P^{cd} \; z$. If $z \; P \; s$, the Ferrers property and $y \; P^{cd} \; z$ imply $x \; P \; s$. Similarly, suppose that $s \; P \; x$. Using semitransitivity and $y \; P^{cd} \; z$, $s \; P \; x$ and $x \; P \; y$ imply $s \; P \; z$. This shows that $x \; T \; z$ and, hence, $x^\ell \succsim_*^P z^\ell$.

3. $[x^r \succsim_*^P y^\ell$ and $y^\ell \succsim_*^P z^\ell]$. We have $x \; P^{cd} \; y$ and $y \; T \; z$. If $z \; P \; x$, $y \; T \; z$ implies $y \; P \; x$, a contradiction. Hence, we have $x \; P^{cd} \; z$ so that $x^r \succsim_*^P z^\ell$.

4. $[x^\ell \succsim_*^P y^r$ and $y^r \succsim_*^P z^r]$. We have $x\ P\ y$ and $y\ T\ z$. This implies $x\ P\ z$, so that $x^\ell \succsim_*^P z^r$.

5. $[x^r \succsim_*^P y^\ell$ and $y^\ell \succsim_*^P z^r]$. We have $x\ P^{cd}\ y$ and $y\ P\ z$. Suppose that $w\ P\ x$. Using the Ferrers property, $y\ P\ z$, $w\ P\ x$ and $x\ P^{cd}\ y$, imply $w\ P\ z$. Similarly, suppose that $z\ P\ w$. Using semitransitivity, $z\ P\ w$, $y\ P\ z$ and $x\ P^{cd}\ y$ imply $x\ P\ w$. Hence, $x\ T\ z$, so that $x^r \succsim_*^P z^r$.

6. $[x^r \succsim_*^P y^r$ and $y^r \succsim_*^P z^\ell]$. We have $x\ T\ y$ and $y\ P^{cd}\ z$ Using $x\ T\ y$, $z\ P\ x$ would imply $z\ P\ y$, a contradiction. Hence, we have $x\ P^{cd}\ z$, so that $x^r \succsim_*^P z^\ell$. □

Suppose now that $\succsim_*^P$ on $A^{\ell r}$ has a numerical representation, i.e., that there is a function $U : A^{\ell r} \to \mathbb{R}$ such that, for all $\alpha, \beta \in A^{\ell r}$, we have:

$$\alpha \succsim_*^P \beta \Leftrightarrow U(\alpha) \geq U(\beta).$$

Define u and v on A setting, for all $x \in A$, $u(x) = U(x^\ell)$ and $v(x) = U(x^r)$. Using the definition of $\succsim_*^P$, we obviously have that, for all $x, y \in A$, $x\ P^{cd}\ y \Leftrightarrow x^r \succsim_*^P y^\ell \Leftrightarrow U(x^r) \geq U(y^\ell) \Leftrightarrow v(x) \geq u(y)$, so that:

$$x\ P\ y \Leftrightarrow u(x) > v(y).$$

By construction, we have $x\ T\ y$ iff $u(x) \geq u(y)$ iff $v(x) \geq v(y)$. Observe that the equivalence classes of $\sim_*^P$ in $A^{\ell r}$ are in obvious one-to-one correspondence with the equivalence classes of E in A. Hence, if A/E is countable, the same will be true for $A^{\ell r}/\sim_*^P$. Finally, observe that if P is irreflexive, we have $x\ P^{cd}\ x$ so that $v(x) \geq u(x)$.

In view of Lemma 38, a similar analysis can be undertaken using P^{cd} instead of P. This would lead to the following definition. A pair u and v of real-valued functions on A is a non-strict representation with no nesting of P if

$$\begin{aligned} x\ P\ y &\Leftrightarrow u(x) \geq v(y), \\ u(x) \geq u(y) &\Leftrightarrow v(x) \geq v(y). \end{aligned} \tag{16}$$

Summarizing the above observations, we have in fact proved the following theorem about the existence of a representation with no nesting on a countable set.

Theorem 47 (Representation with no nesting of semiorders on countable sets) *Let P be a binary relation on a set A. If A/E is countable, the following statements are equivalent.*

1. *P is a semiorder.*

2. *There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) \leq v(x)$ and*

$$x \; P \; y \Leftrightarrow u(x) > v(y),$$
$$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y).$$

3. *There are two real-valued functions u and v on A such that, for all $x, y \in A$, $u(x) < v(x)$ and*

$$x \; P \; y \Leftrightarrow u(x) \geq v(y),$$
$$u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y).$$

Furthermore, the functions u and v used in statements 2 or 3 above can always be chosen in such a way that, for all $x, y \in A$,

$$x \; T \; y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y).$$

Remark 48
As before, omitting the irreflexivity requirement on P and the fact that $u(x) \leq v(x)$ (respectively, $u(x) < v(x)$) in the above theorem, leads to a theorem on the existence of a strict (respectively, non-strict) representation with no nesting on countable sets that deals with coherent biorders. •

The generalization of Theorem 47 to sets of arbitrary cardinality is presented in Fishburn (1985, Theorems 7.7 and 7.8). Because, our emphasis is on representations with no proper nesting, we do not detail this point (see Section 8).

6.4 Constant threshold representations

Example 37 has shown that a semiorder may not have a constant threshold representation (11) or (12) as soon as A is not finite. Hence, we do not view this representation to be especially attractive on infinite sets. Therefore, we only briefly mention a few results in the area without giving precise statements or proofs, referring the reader to the original sources.

Call a denumerable sequence $x_1, x_2, \ldots \in A$ to be increasing (respectively, decreasing) if $x_{i+1} \; P \; x_i$ (respectively, $x_i \; P \; x_{i+1}$), for $i = 1, 2, \ldots$.

Suppose that P is a semiorder for which I is not transitive, i.e., a semiorder that is not a weak order. Hence, any constant threshold representation of P will have a strictly positive threshold. Suppose that there is an increasing denumerable sequence $x_1, x_2, \ldots \in A$. If $w \; P^{cd} \; x_i$ for all x_i in this sequence, then $u(w)$ is pushed to $+\infty$, so that there cannot exist a constant threshold representation. A similar reasoning shows that if $x_1, x_2, \ldots \in A$ is a decreasing denumerable sequence, it is impossible that $x_i \; P^{cd} \; w$, for all x_i in the sequence.

This shows the necessity of the following condition applied to any semiorder that is not a weak order. For all $w \in A$ and all increasing (respectively, decreasing) denumerable sequence $x_1, x_2, \ldots \in A$, we must have $x_i \; P \; w$ (respectively, $w \; P \; x_i$) for some x_i in the sequence.

When is A countable, Beja and Gilboa (1992, Theorems 3.7 and 3.8) show that there is a strict constant threshold representation with $\varepsilon > 0$ for P iff P is a semiorder and the above condition holds. The necessity of the condition is easily understood from the above observations.

The sufficiency proof given in Beja and Gilboa (1992) is moderately complex, involving the study of the paths in the graph (A, P), extending to the infinite case the so-called potential technique used in Roy (1985) and Roubens and Vincke (1985). Furthermore, Beja and Gilboa (1992) show that the equivalence between the strict version of the constant threshold representation (11) and the non-strict one (12) carries over to the countable case. This paper also contains a very clear discussion of the issue of imposing that a *closed* interval is associated to each alternative. The issue is of no importance in the countable case but becomes crucial in the uncountable one.

Beja and Gilboa (1992, Theorem 4.4) have also studied constant threshold representations on arbitrary sets imposing that T has a numerical representation. In this case however, the constant length interval associated to each alternative is not necessarily closed, i.e., we only have that:

$$\begin{gathered} x \; P \; y \Rightarrow u(x) - u(y) \geq \varepsilon, \\ u(x) - u(y) > \varepsilon \Rightarrow x \; P \; y, \\ x \; T \; y \Leftrightarrow u(x) \geq u(y). \end{gathered}$$

Again, because our emphasis is on representations with no proper nesting, we do not develop this point (see Section 8).

7 Acyclic relations

7.1 Preliminaries

Let P be a binary relation on set A. Suppose that there is a function $u : A \to \mathbb{R}$ and a threshold $\varepsilon : A^2 \to \mathbb{R}$ such that, for all $x, y \in A$,

$$\begin{gathered} x \; P \; y \Leftrightarrow u(x) - u(y) > \varepsilon(x, y), \\ \varepsilon(x, y) = \varepsilon(y, x) \geq 0. \end{gathered} \tag{17}$$

Observe that acyclicity is a necessary condition for a representation (17) to exist, independently of the cardinality of A. Indeed, suppose that P has a cycle $x_1 \; P \; x_2 \; P \ldots P \; x_k \; P \; x_1$. Using (17), we obtain:

$$\begin{aligned} u(x_1) &> u(x_2) + \varepsilon(x_1, x_2), \\ u(x_2) &> u(x_3) + \varepsilon(x_2, x_3), \\ &\ldots \\ u(x_{k-1}) &> u(x_k) + \varepsilon(x_{k-1}, x_k), \\ u(x_k) &> u(x_1) + \varepsilon(x_k, x_1), \end{aligned}$$

implying that $\varepsilon(x_k, x_1) + \sum_{i=1}^{k} \varepsilon(x_i, x_{i+1}) < 0$, which is contradictory since ε only takes nonnegative values.

As far as numerical representation is concerned, most of the literature has concentrated on "one way" representations (also often called "weak utility functions") asking for the existence of a function $V : A \to \mathbb{R}$ such that, for all $x, y \in A$,

$$x \; P \; y \Rightarrow V(x) > V(y). \tag{18}$$

Remark 49

Clearly with a representation satisfying (18), the knowledge of V does not allow to entirely recover the relation P. Such representations may be useful however since it is easy to see that if the problem:

$$\arg\max_{x \in B} V(x)$$

has a solution, it must be a maximal element of P in B. •

The following simple lemma connects the representations (17) and (18).

Lemma 50

Let P be a binary relation on A. There is a function function $V : A \to \mathbb{R}$ satisfying (18) *iff there are functions $u : A \to \mathbb{R}$ and a threshold $\varepsilon : A^2 \to \mathbb{R}$ such that* (17) *holds.*

Proof. Observe first that if u and ε allow to represent P using (17), then u is a representation of P using (18).

Suppose now that V is a representation of P using (18). Let us show that it is possible to use V to build a threshold function such that (17) will hold. Suppose that $x \; P \; y$ so that $V(x) > V(y)$. In this case, take $\varepsilon(x, y) = \varepsilon(y, x) = \nu$ where $\nu \in [0, V(x) - V(y)[$. If $x \; I \; y$, take $\varepsilon(x, y) = \varepsilon(y, x) = \nu$ where $\nu \geq |V(x) - V(y)|$. It is clear that ε is nonnegative and such that $\varepsilon(x, y) = \varepsilon(y, x)$, for all $x, y \in A$. Using such a construction, we clearly have:

$$x \; P \; y \Rightarrow V(x) > V(y) + \varepsilon(x, y).$$

But $V(x) > V(y) + \varepsilon(x, y)$ excludes, by construction, to have $x\ I\ y$ or $y\ P\ x$. Hence, (17) holds. □

Remark 51
Observe that the proof of the above lemma shows that a stronger conclusion holds. Indeed, for any functions u and ε satisfying (17), (18) holds with $V = u$. Conversely, for any function V satisfying (18), it is possible to find a threshold function ε such that (17) holds with $u = V$. •

7.2 The countable case

We first show that, when A is countable, there is a real-valued function satisfying (18) as soon as P is acyclic.

Lemma 52
Let P be a binary relation on a countable set A. The following statements are equivalent.

1. *P is acyclic.*
2. *There is a function $V : A \to \mathbb{R}$ such that* (18) *holds.*

Proof. The necessity of acyclicity is obvious. We show sufficiency. Enumerate the elements in A as $\{z_i : i \in K\}$ with $K \subseteq \mathbb{N}_+$. For all $x \in A$, let $M(x) = \{i \in K : x\ P^\tau\ z_i\}$, where P^τ is the transitive closure of P. Define $V : A \to \mathbb{R}$ setting, for all $x \in A$,

$$V(x) = \sum_{i \in M(x)} \frac{1}{2^i},$$

using the convention that if $M(x) = \varnothing$ then $V(x) = 0$.

Suppose that $x\ P\ y$. By construction, we have $M(y) \subseteq M(x)$ so that $V(y) \leq V(x)$. But $x \in M(y)$ and $x\ P\ y$ implies that P has a cycle. Hence, we must have $M(y) \subsetneq M(x)$ so that $V(x) > V(y)$. □

Combining Lemmas 50 and 52 proves the following theorem giving necessary and sufficient conditions for the existence of representations (17) and (18) on a countable set.

Theorem 53 (Representation of acyclic relations on countable sets)
Let P be a binary relation on a countable set A. The following statements are equivalent.

1. *P is acyclic.*

2. *There is a function* $V : A \to \mathbb{R}$ *such that, for all* $x, y \in A$,

$$x \; P \; y \Rightarrow V(x) > V(y).$$

3. *There are two functions* $u : A \to \mathbb{R}$ *and* $\varepsilon : A^2 \to \mathbb{R}$ *such that, for all* $x, y \in A$,

$$x \; P \; y \Leftrightarrow u(x) - u(y) > \varepsilon(x, y),$$
$$\varepsilon(x, y) = \varepsilon(y, x) \geq 0.$$

7.3 The general case

In view of Lemma 50, a generalization of Theorem 53 will be at hand, if we can find necessary and sufficient conditions for (18) to hold.

Let $>$ be a partial order on a set A, i.e. an irreflexive and transitive binary relation. Let $E^{>}$ be the equivalence relation associated to the trace of $>$, i.e., the binary relation on A such that $xE^{>}y$ iff $[>x = >y]$ and $[x> = y>]$. Let $>^*$ be a weak order on A with associated equivalence $E^{>^*}$. We say that the weak order $>^*$ *extends* the partial order $>$ if, for all $x, y \in A$,

$$\begin{aligned} &xE^{>}y \Rightarrow xE^{>^*}y, \\ &x > y \Rightarrow x >^* y. \end{aligned} \tag{19}$$

Jaffray (1975b, Theorem 1) shows that a partial order can always be extended to a weak order (this is an easy generalization of Szpilrajn, 1930) and, most importantly, that this extension can always be made without increasing the cardinality of a dense subset. Since the proof of this result is long and not especially instructive, we refer to the original paper for a proof.

Lemma 54 (Jaffray 1975b, Theorem 1)
Let $>$ *be a partial order on* A *and suppose that* B *is dense in* A *for* $>$. *There is a weak order* $>^*$ *on* A *such that* (19) *holds. Furthermore, there is* $C \subseteq A$ *being either finite or having at most the cardinality of* B *such that* C *is dense in* A *for* $>^*$.

With the above lemma at hand, the generalization of Theorem 53 to arbitrary sets is easy.

Theorem 55 (Representation of acyclic relations)
Let P *be a binary relation on a set* A. *The following statements are equivalent.*

1. P *is acyclic and there is a countable set* B *that is dense in* A *for* P^{τ}.

2. *There is a function* $V : A \to \mathbb{R}$ *such that, for all* $x, y \in A$,

$$x \; P \; y \Rightarrow V(x) > V(y).$$

3. *There are two functions* $u : A \to \mathbb{R}$ *and* $\varepsilon : A^2 \to \mathbb{R}$ *such that, for all* $x, y \in A$,

$$\begin{gathered} x \; P \; y \Leftrightarrow u(x) - u(y) > \varepsilon(x, y), \\ \varepsilon(x, y) = \varepsilon(y, x) \geq 0. \end{gathered}$$

PROOF. The equivalence between statements (3) and (2) results from Lemma 50. We show below the equivalence between statements (1) and (3).

[(3) $\Rightarrow$ (1)]. Suppose that P has a representation satisfying (17). Define the binary relation $\widehat{P}$ on A setting, for all $x, y \in A$, $x \; \widehat{P} \; y$ iff $u(x) > u(y)$. By construction, $\widehat{P}$ is a weak order. Since it has a numerical representation, it must have a countable dense subset. It is clear that $x \; P^\tau \; y$ implies $x \; \widehat{P} \; y$. Hence, (see Remark 16), the countable subset dense in A for $\widehat{P}$ is also dense for P^τ.

[(1) $\Rightarrow$ (3)]. By construction, P^τ is a partial order, being irreflexive and transitive. It has a countable dense subset. Hence, using Theorem 1 in Jaffray (1975b), P^τ can be extended to a weak order having a numerical representation. Hence, (18) holds implying that (17) holds as well. □

8 Concluding remarks and guide to the literature

8.1 Linear orders

The study of the numerical representation of linear order on countable sets is classical (see Cantor, 1895, 1915). Our proof of Theorem 1 follows Bridges and Mehta (1995).

The importance of density conditions for studying numerical representations dates back to Cantor (1895, 1915). Theorem 7 was first formulated in Debreu (1954). Other early contributions include Birkhoff (1948, p. 31), Eilenberg (1941), Fleischer (1960) and Milgram (1939). Our proof of Theorem 7 closely follows Krantz, Luce, Suppes, and Tversky (1971, Theorem 2.2).

Debreu (1954, 1964) (for a simple proof, see Jaffray, 1975a) has shown that the function u in Theorem 7 can always be made continuous in all topologies that are finer than the natural topology induced on A by P (i.e., the coarsest topology in which, for all $x \in A$, the sets xP and Px are open).

This is clearly useful if one intends to use u in order to find the maximal elements of P in some (compact) subset of A. The existence of continuous numerical representations has generated numerous results. They are well summarized in Bridges and Mehta (1995).

Beardon, Candeal, Herden, Induráin, and Mehta (2002b) have studied in depth the various types of linear orders that do *not* admit a numerical representation. It turns out that there only a few different types of such linear orders. We have presented in Section 2 two examples (i.e., a linear order on set of a cardinality greater than that of $\mathbb{R}$ and lexicographic preferences on $\mathbb{R} \times \{0, 1\}$) that are instances of the two most common types of such linear orders: long chains and planar chains.

Chipman (1960) and Beardon, Candeal, Herden, Induráin, and Mehta (2002a) study numerical representations of linear orders involving several real-valued functions compared in a lexicographic way.

8.2 Weak orders

Our presentation has closely followed Krantz et al. (1971, Chapter 2).

The special case in which $A = \mathbb{R}^k$ is of particular importance in Economics (it is often supposed that agents consumes bundles consisting of quantities of perfectly divisible goods among a finite number of goods available on a market). In this case, it is possible to find conditions implying the denseness condition that have a clear intuitive content: continuity and non-satiation (see Arrow and Hahn, 1971, Theorem 1, p. 87 or Fishburn, 1970d, Section 3.3).

8.3 Biorders

Our presentation has followed Doignon et al. (1984), Doignon, Ducamp, and Falmagne (1987). The analysis of biorders shows the power of studying the traces of a binary relation and imposing conditions ensuring that they are complete (this was already stressed in Monjardet, 1978). We have followed a similar path for the study of interval orders and semiorders.

Remark 56
The study of traces has numerous applications. It is, for instance, central for the study of the "uncovered set" in Social Choice Theory (see Laslier, 1997). Recently, Bouyssou and Pirlot (2002, 2004a,b) have studied traces in the area of decision making with multiple attributes. In this case, traces are

defined on each attribute. For instance, and using obvious notation, if

$$(a_1, a_2, \dots, a_{i-1}, y_i, a_{i+1}, \dots, a_n) \; P \; (w_1, w_2, \dots, w_n) \Rightarrow$$
$$(a_1, a_2, \dots, a_{i-1}, z_i, a_{i+1}, \dots, a_n) \; P \; (w_1, w_2, \dots, w_n),$$

for all choices of $w_1, w_2, \dots, w_n$ and $a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n$, then, we have good reasons to believe that the level z_i on attribute i is surely not worse than the level y_i. •

Instead of asking that either $x \; P \; y$ iff $u(x) > v(y)$ or that $x \; P \; y$ iff $u(x) \geq v(y)$, for *all* $x, y \in A$, we could leave undecided what happens at the "boundary" between $u(x)$ and $v(y)$. This leads to more general numerical representations. For space reasons, we will not tackle this more general problem in detail. It has received a thorough treatment in Nakamura (2002a). Let us simply observe here that the Ferrers property remains a central condition with this more general representation.

Indeed, let P be a binary relation on A. Suppose that there are two real-valued functions u and v on A such that, for all $x, y \in A$,

$$x \; P \; y \Rightarrow u(x) \geq v(y),$$
$$u(x) > v(y) \Rightarrow x \; P \; y,$$
$$u(x) = u(y) \Rightarrow x \; E_\ell \; y,$$

where E_ℓ is the symmetric part of T_ℓ. Observe that these conditions do not impose either that $x \; P \; y$ iff $u(x) > v(y)$ or that $x \; P \; y$ iff $u(x) \geq v(y)$, for *all* $x, y \in A$. Nevertheless, such a representation already implies that P is a biorder. Indeed, suppose that $x \; P \; y$ and $z \; P \; w$. This implies $u(x) \geq v(y)$ and $u(z) \geq v(w)$. We have to show that either $x \; P \; w$ or $z \; P \; y$. If $u(x) > u(z)$, we have $u(x) > v(w)$, so that $x \; P \; w$. If $u(z) > u(x)$, we have $u(z) > v(y)$, so that $z \; P \; y$. Otherwise we have $u(z) = u(x)$ so that $z \; E_\ell \; x$ and $x \; P \; y$ iff $z \; P \; y$.

8.4 Interval orders

The numerical representation of interval orders on countable sets (Theorem 33) was first established by Fishburn (1970b). A similar result, using a different method of proof, can be found in Bridges (1983a).

For the study of the numerical representation of interval order in the general case (Theorems 35 and 36) we have followed Doignon et al. (1984, Proposition 10). Fishburn (1973, Theorem 3) proposed the first results giving necessary and sufficient conditions for the existence of a strict representation (9). On top of asking that P is an interval order, he adds three requirements.

The first two ensures that T_ℓ and T_r have a numerical representation. The third one is more involved and amounts to forbidding that there are "too many" jumps such that the interval $]u(x), v(y)[$ is empty. Similar results can be found in Fishburn (1985, Chapter 7). Our feeling is that the strict denseness condition is at the same time more compact and more intuitive.

The importance of the strict denseness condition was also re-discovered independently by Oloriz, Candeal, and Induráin (1998). The recent paper by Bosi, Candeal, Induráin, Oloriz, and Zudaire (2001) gives a thorough review of various conditions that are equivalent to this condition. Knoblauch (1998) studies an alternative numerical representation for an interval order involving sequences of 0 and 1 ordered by Pareto dominance.

Another line of research consists in asking for a representation (9) with both u and v being continuous in some topology defined on A. The first and major advance in this direction was done by Chateauneuf (1987). He supposes that A is a connected topological space (meaning A cannot be partitioned into two nontrivial open subsets) and suggests to study interval orders P such that there is a countable subset B that is *strongly dense* in A for P, i.e., $x \; P \; y$ implies that $x \; P \; z \; P^{cd} \; w \; P \; y$, for some $z, w \in B$. He shows that the existence of a countable strongly dense subset is a necessary and sufficient condition to obtain a representation (9) with both u and v continuous. This result is thoroughly surveyed in Bridges and Mehta (1995, Chapter 6). Other results dealing with continuous representations of interval orders can be found in Bosi (2002), Bosi, Candeal, Induráin, and Zudaire (2005), Bosi and Isler (1995), Bridges (1983b, 1985, 1986), Estévez Toranzo, García-Cutrín, and López López (1995) and Gensemer (1987b). A recent survey is Candeal, Induráin, and Zudaire (2005).

As with biorders, our choice in (9) to associate a *closed* interval to each alternative is rather arbitrary. In a more general representation, we could associate an arbitrary interval to each alternative and still compare intervals saying that $I > J$ if $x_I > y_J$, for all $x_I \in I$ and all $y_J \in J$. Such models were advocated in Beja and Gilboa (1992) and Suppes, Krantz, Luce, and Tversky (1989, Section 16.2). They were first studied in Fishburn (1973) who proposed sufficient conditions ensuring a representation of an interval order by means of arbitrary intervals (closely related results appear in Fishburn, 1985, Section 7.6). He showed that requiring that both T_ℓ and T_r have a numerical representation is sufficient to obtain such a representation; unfortunately these two conditions are not necessary (see Fishburn, 1985, Example 2, p. 140). Necessary and sufficient conditions were obtained by Nakamura (2002a). Since they are rather complex, we do not introduce them here. Related results were obtained by Lück (2004).

Intervals orders on infinite sets have been used in several more specific contexts, e.g., in the study of expected utility theory (see Nakamura, 1988) and in the measurement of probabilities (see Fishburn, 1986, Nakamura, 2000).

8.5 Semiorders

8.5.1 Representations with no proper nesting

The study of numerical representations with no proper nesting has not attracted much attention in the literature (exceptions are Pirlot and Vincke, 1997, Roubens and Vincke, 1985, Roy, 1985). Our results are easy corollaries of the ones obtained for interval orders.

8.5.2 Representations with no nesting

The generalization of Theorem 47 to sets of arbitrary cardinality is presented in Fishburn (1985, Theorems 7.7 and 7.8). This requires two additional necessary and sufficient conditions. The first one, unsurprisingly, requires that $\succsim_*^P$ on $A^{\ell r}$ has a numerical representation. The second one, as above, excludes that there are "too many" jumps concerning empty intervals of the type $]u(x), v(y)[$. We refer the reader to Fishburn (1985) for a precise statement and proof.

Nakamura (2002a) studies the case in which the interval associated with each alternative is not necessarily closed (or open). The reasons for being interested in such representations of semiorders are similar to the ones already evoked for biorders and interval orders.

8.5.3 Constant threshold representations

The study of semiorders on countable sets that have a constant threshold representation has been pioneered by Manders (1981). He gives necessary and sufficient conditions for the existence of such a representation. This analysis was pursued (and, much simplified) in Beja and Gilboa (1992).

This question of obtaining conditions guaranteeing that (11) holds in the general case has generated several studies. As it is often the case with traces, a simple analysis is possible if it is supposed that the set A is "rich". Gensemer (1988) gives sufficient conditions on a semiorder P defined on an Euclidian space to have a *continuous* constant threshold representation (11). This analysis is extended in Gensemer (1987a) who presents necessary and sufficient condition for a semiorder P defined on a connected topological space to have a continuous constant threshold representation (11). She also shows that, in such a rich setting, having a continuous representation with no nesting (15) is, in fact, equivalent to having a constant threshold representation (11). An approach in which richness is brought by algebraic assumptions is followed in Narens (1994).

The general problem of finding necessary and sufficient condition for obtaining a constant threshold representation (11) is solved in Candeal, Induráin, and Zudaire (2002). Lacking the richness provided by topological assumptions as in Gensemer (1987a) or by algebraic ones as in Narens (1994)), these conditions involve the Dedekind completion of T and are not particularly intuitive. Clearly, the equivalence between strict (11) and non-strict (12) constant threshold representations does not hold any more. This difficulty is already central in the early analysis of constant threshold representations in the uncountable case in Ŝwistak (1980).

Remark 57
The semiorder model is very natural as soon as one is willing to take "imperfect discrimination" into account. Since measurement apparatus are always imperfect, the reader may be puzzled by the fact that there are only very few models in the area of measurement theory that use semiorders.

Replacing the hypothesis that objects are compared by a weak order by the weaker hypothesis of a semiorder (or an interval order) raises no major difficulty in the finite case (see, e.g., Adams, 1965, Domotor and Stelzer, 1971, Fishburn, 1970a, Sections 3 and 4, Fishburn, 1970d, Exercises 4.16 and 4.17) or in the abstract setting proposed in Fishburn (1992) that extends the analysis for the finite case to arbitrary sets (see Fishburn, 1999, Section 4, for an application).

Clearly, such an extension becomes problematic when one wishes to use "standard sequences" as in common in extensive, difference or conjoint measurement (see Krantz et al., 1971, Chapters 3, 4 and 6). Early attempts to use semiorders in this framework have only been moderately successful (see Krantz, 1967, Luce, 1973, Suppes et al., 1989, Section 16.6 and Vincke, 1980). More recently, a special version of this problem has been elegantly dealt with in Le Menestrel and Lemaire (2004). •

8.6 Acyclic relations

Acyclic relations are often called "suborders" (see, e.g. Fishburn, 1970a,c). They have been extensively studied in Economics since it is well known that acyclicity is a necessary and sufficient condition to guarantee the existence of maximal elements in every finite subset of A, (see, e.g., Sen, 1970 and the proof of Theorem 2.5).

The representation (17) was independently formulated by Aizerman and Aleskerov (1991), Agaev and Aleskerov (1993) and Abbas and Vincke (1993) (see also Abbas, 1994, 1995) for finite sets. It was also studied by Diaye (1999) and Rodríguez-Palmero (1997) in a more general setting. Our analysis mainly follows Diaye (1999). A formally similar model has been considered in Manzini and Mariotti (2003) in the study of the possible "incomparability of psychological preferences".

The representation (18) has been much more investigated than the more recent model (17). Adams (1965, Theorem 1) shows that acyclicity implies (18) for finite sets. The generalization of this result to countable sets may be found in Fishburn (1970a, Theorem 7) or in Bridges (1983a, Theorem 1). Our proof of Lemma 52 follows Bridges (1983a).

Necessary and sufficient conditions for (18) were proposed in Alcantud and Rodríguez-Palmero (1999) and Rodríguez-Palmero (1997), following previous work in the area by Alcantud (1999), Fishburn (1970d, Theorems 2.3 and 3.2), Herden (1989), Jaffray (1975b), Peleg (1970), Peris and Subiza (1995), Richter (1966), Sondermann (1980) and Subiza and Peris (1997) (some of this works dealing with the existence of weak utilities for partial orders, a question that is close from the problem of devising conditions for the existence of a representation (18)).

We have already mentioned that solutions of the problem

$$\arg\max_{x \in B} V(x)$$

are always maximal elements of P in B. With this kind of applications in mind, it is clearly of much interest to investigate conditions that will guarantee that, in a representation (18), the function V is upper semicontinuous. This is investigated in most of the above references.

Subiza (1994) has studied a numerical representation for acyclic relation that offers an alternative to both (17) and (18). Like (17), it allows to completely recover the information contained in P. In this model, a nonempty bounded subset of $\mathbb{R}$, $\mu(x)$, is associated with each alternative $x \in A$ so that

$$x \; P \; y \Leftrightarrow \mu(x) \cap \mu(y) = \varnothing \text{ and } \sup \mu(x) > \sup \mu(y).$$

For countable sets, it is shown that all acyclic relations can be represented in this way. Provided that some countable subset of A is "adequately" dense for the transitive closure of P (we refer to Subiza, 1994, for a precise definition) a similar result holds for all sets that have at most the cardinality of $\mathbb{R}$. An extension to arbitrary sets is also proposed.

Remark 58
Following Agaev and Aleskerov (1993), Nakamura (2002b) has shown that, on finite sets, partial orders can always be represented using (17) with the additional "triangle inequality" constraint, for all $x, y, z, w \in A$, $\varepsilon(x, y) \leq \varepsilon(x, z) + \varepsilon(z, y)$. The proof of this result, as given by Nakamura (2002b), relies on the Theorem of the Alternative giving a criterion for the existence of a solution to a system of, finitely many, linear equations or inequations (see, e.g., Fishburn, 1970d, Section 4.2). An important open problem is to know whether such a result holds for infinite sets. •

Remark 59
Other approaches have been followed for the numerical representation of partial orders. The most common is to look for a weak utility, i.e., a function satisfying (18). We already mentioned that necessary and sufficient conditions for such a representation are known (see Alcantud and Rodríguez-Palmero, 1999, Herden, 1989).

Herrero and Subiza (1999) have extended the "set-valued" representation proposed in Subiza (1994) to the case of partial orders.

A very promising approach (that is reminiscent of the work of Aleskerov, 1980 and Aleskerov, Zavalishin, and Litvakov, 1979a,b) is followed in Ok (2002) where it is suggested to represent partial orders by a family of utility functions using Pareto comparisons. In other terms, there is a function $\mathbf{u} : A \to \mathbb{R}^k$, such that $x \; P \; y$ iff $u_i(x) \geq u_i(y),$ for all $i \in \{1, 2, \ldots k\}$ and $u_j(x) > u_j(y),$ for some $j \in \{1, 2, \ldots k\}$. This generalizes the analysis in Section 2.4 to sets of arbitrary cardinality. The difficulty in this analysis is twofold. First, as shown by Dushnik and Miller (1941), whereas all partial orders can be obtained as the intersection of a family of weak orders, nothing guarantees that this family is finite when the underlying A is infinite. Ok (2002) shows that this will be the case as soon as they are not "too many" alternatives that are mutually incomparable w.r.t. P, meaning that P has finite width (for an introduction to the basic theory of posets, see Fishburn, 1985, Trotter, 1992). The other difficulty is to guarantee that these weak orders will all have a numerical representation: sufficient conditions are proposed in Ok (2002). •

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