Preference Modelling

A brief introduction

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Introduction

Lemma

- if you have no preference...
- then there is no need to worry about decisions!

Aims

- present the standard model of preference modelling
    - analyze a few classical questions within the standard model
- introduce some extensions of the standard model

Preference Modelling

Variety of disciplines

- Economics
  - consumer theory
- Psychology
  - experiments
- Political Science
  - voters
- Marketing
  - consumers
- Operations Research
  - objective function
- Multiple Criteria Decision Making
  - criteria

Preference Modelling

Variety of perspectives

- Normative
  - link between preference models and “rational behavior”
- Descriptive
  - preference models as compatible with experimental results
- Prescriptive
  - help someone structure a preference model

Preference Modelling

Variety of objects to compare

- vectors in $\mathbb{R}^k$
  - Operations Research: Linear Programming (LP)
  - Economics: consumer theory
- finite list defined implicitly
  - Operations Research: combinatorial optimization
- finite list defined explicitly
  - candidates in an election
  - investment projects

Consequence

- huge literature
- aim: brief introduction
  - vocabulary
  - main structures
  - main questions

Remarks

Position in the decision process

- a set of objects has been identified: X
- one and only one element of X will be finally implemented

Ignored

- experiments
  - framing & presentation effects (Kahneman & Tversky, 1981)
- sophisticated recent models
  - reference points (Kahneman & Tversky, 1979)

Standard model

How to compare the objects in X?

- simple procedure
  - build a mapping $f : X \to \mathbb{R}$
  - compare objects using f

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Interpretation

- f "measures" the desirability of the objects
- $\succsim$ is a binary relation on the set X
  - $\succsim$ reads "at least as good as"

Hypotheses

1. f is "known" with precision
2. f is given for all objects
3. there is only one f

Remark

- model ignores "cardinal properties" of f

Examples

Example 1: grading students

- X is a set of students
- f is the grade (0–20 scale) obtained at a written exam

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Example 2: choosing investment projects

- X is a set of projects
- f gives the NPV (€) of the projects

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Example 3: Linear Programming

- X is a convex polytope in $\mathbb{R}^k$, $\mathbf{x} \in X$
- $f(\mathbf{x}) = \sum_{i=1}^{k} c_i x_i$: value of the objective function

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Quick reminder

Binary relation

- a binary relation T on a set A is a subset of $A \times A$
- we write $a\ T\ b$ instead of $(a, b) \in T$

Operations on binary relations

- binary relations are sets
- we may use standard set operations on them: $T \cap R$, $T \cup R$, $T \subseteq R$
- product of two binary relation T and R on A
  - $a\ T{\cdot}R\ b \Leftrightarrow [a\ T\ c$ and $c\ R\ b$, for some $c \in A]$

Quick reminder

Properties of binary relation

A binary relation T on a set A is:

- reflexive if $a\ T\ a$
- irreflexive if $Not[a\ T\ a]$
- symmetric if $a\ T\ b \Rightarrow b\ T\ a$
- antisymmetric if $a\ T\ b$ and $b\ T\ a \Rightarrow a = b$
- asymmetric if $a\ T\ b \Rightarrow Not[b\ T\ a]$
- weakly complete if $a \neq b \Rightarrow a\ T\ b$ or $b\ T\ a$
- complete if $a\ T\ b$ or $b\ T\ a$
- transitive if $a\ T\ b$ and $b\ T\ c \Rightarrow a\ T\ c$
- negatively transitive if $Not[a\ T\ b]$ and $Not[b\ T\ c] \Rightarrow Not[a\ T\ c]$

for all $a, b, c \in A$

- there are many links between these properties

Quick reminder

Links

- transitivity $T^2 \subseteq T$
- asymmetry $\Rightarrow$ irreflexivity
- complete $\Leftrightarrow$ reflexive and weakly complete
- [asymmetry and negatively transitivity] $\Rightarrow$ transitivity
- [completeness and transitivity] $\Rightarrow$ negatively transitivity

Symmetric and asymmetric parts

- asymmetric part of T: $x\ T^\alpha\ y \Leftrightarrow [x\ T\ y$ and $Not[y\ T\ x]]$
- symmetric part of T: $x\ T^\sigma\ y \Leftrightarrow [x\ T\ y$ and $y\ T\ x]$

Representation of a binary relation

Matrix representation

Let $A = \{a, b, c, d, e\}$. Consider the binary relation $\succsim = \{(a, b), (b, a), (b, c), (d, b), (d, d)\}$.

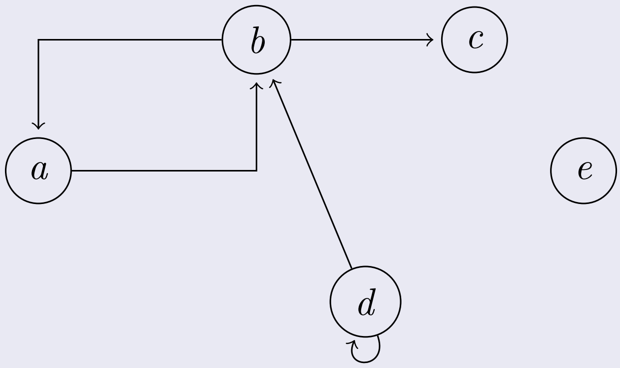
$\circlearrowleft$	a	b	c	d	e
a	0	1	0	0	0
b	1	0	1	0	0
c	0	0	0	0	0
d	0	1	0	1	0
e	0	0	0	0	0

$$M_{ab}^T = \begin{cases} 1 & \text{if } a\ T\ b \\ 0 & \text{otherwise} \end{cases}$$

Representation of a binary relation

Graph representation

Let $A = \{a, b, c, d, e\}$. Consider the binary relation $\succsim = \{(a, b), (b, a), (b, c), (d, b), (d, d)\}$.



Graph

- elements of X are vertices
- elements related by T define arcs

Standard model

Model

$$x \succsim y \Leftrightarrow f(x) \geq f(y)$$

Two obvious properties of $\succsim$

- $\succsim$ is complete $x \succsim y$ or $y \succsim x$, for all $x, y \in X$
- $\succsim$ is transitive $[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$, for all $x, y, z \in X$

Definition

A complete and transitive relation is called a weak order (complete preorder, total preorder)

Remarks

- if f is injective, $\succsim$ becomes antisymmetric
- a complete, antisymmetric and transitive relation is a total order

Strict preference and Indifference

Asymmetric part of $\succsim$

$$x \succ y \Leftrightarrow [x \succsim y \text{ and } Not[y \succsim x]] \Leftrightarrow f(x) > f(y)$$

- $\succ$ is interpreted as strict preference
- $\succ$ is asymmetric, transitive, negatively transitive
  - $[Not[x \succ y]$ and $Not[y \succ z]] \Rightarrow Not[x \succ z]$, for all $x, y, z \in X$

Symmetric part of $\succsim$

$$x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x] \Leftrightarrow f(x) = f(y)$$

- $\sim$ is interpreted as indifference
- $\sim$ is reflexive, symmetric, transitive
- $\sim$ is an equivalence

Notation

$$x \succsim y \Leftrightarrow y \precsim x$$
$$x \succ y \Leftrightarrow y \prec x$$

Properties

Completeness of $\succsim$ implies that:

$$x \succsim y \Leftrightarrow Not[y \succ x]$$

Strict preference and Indifference

Summary

- with a weak order $\succsim$ we have equivalence classes of $\sim$ that are totally ordered by $\succ$

Remarks

$$x \succ y \Leftrightarrow [x \succsim y \text{ and } Not[y \succsim x]]$$
$$x \sim y \Leftrightarrow [x \succsim y \text{ and } y \succsim x]$$

- $\succ$ and $\sim$ are exhaustive: for all $x, y \in X$ we at least one among
  - $x \sim y$, $x \succ y$, $y \succ x$
- $\succ$ and $\sim$ are exclusive: for all $x, y \in X$ we at most one among
  - $x \sim y$, $x \succ y$, $y \succ x$
- there are no incomparable objects

Summary

	$y \succsim x$	$Not[y \succsim x]$
$x \succsim y$	$x \sim y$	$x \succ y$
$Not[x \succsim y]$	$y \succ x$	$\varnothing$

Alternative presentation

- question: "is x at least as good as y?"
- two exclusive answers
  - YES: $x \succsim y$
  - NO: $Not[x \succsim y]$
- these answers are such that $\succsim$ is complete and transitive

Some obvious properties

$$[x \succ y \text{ and } y \sim z] \Rightarrow x \succ z$$
$$[x \sim y \text{ and } y \succ z] \Rightarrow x \succ z$$

$$\succ \cdot \sim \subseteq \succ$$
$$\sim \cdot \succ \subseteq \succ$$

Example

$$X = \{x, y, z, w, t\}$$
$$x \succsim x,\ y \succsim y,\ z \succsim z,\ w \succsim w,\ t \succsim t$$
$$x \succsim y,\ x \succsim z,\ x \succsim w,\ x \succsim t$$
$$y \succsim x,\ y \succsim z,\ y \succsim w,\ y \succsim t$$
$$z \succsim w,\ z \succsim t$$
$$w \succsim t$$
$$t \succsim w$$

$\succsim$	x	y	z	w	t
x	1	1	1	1	1
y	1	1	1	1	1
z	0	0	1	1	1
w	0	0	0	1	1
t	0	0	0	1	1

rows and columns have been ordered according to degrees

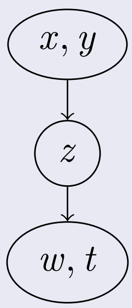
Example

$\sim$	x	y	z	w	t
x	1	1	1	1	1
y	1	1	1	1	1
z	0	0	1	1	1
w	0	0	0	1	1
t	0	0	0	1	1

$\succsim$	x	y	z	w	t		
x	1	1	1	1	1	5	3
y	1	1	1	1	1	5	3
z	0	0	1	1	1	3	0
w	0	0	0	1	1	2	-3
t	0	0	0	1	1	2	-3
	2	2	3	3	5		

Example

$$X = \{x, y, z, w, t\}$$
$$x \succsim x,\ y \succsim y,\ z \succsim z,\ w \succsim w,\ t \succsim t$$
$$x \succsim y,\ x \succsim z,\ x \succsim w,\ x \succsim t$$
$$y \succsim x,\ y \succsim z,\ y \succsim w,\ y \succsim t$$
$$z \succsim w,\ z \succsim t$$
$$w \succsim t$$
$$t \succsim w$$



Numerical representation

Problem

Let $\succsim$ be a weak order on X

Can we always build a numerical representation of $\succsim$?

Question

Given a weak order $\succsim$ on X is there a mapping $v : X \to \mathbb{R}$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow v(x) \geq v(y)$$

Obvious answer: No (thanks Georg!)

- any total order on $2^{\mathbb{R}}$ cannot have a numerical representation
- there is no injection from $2^{\mathbb{R}}$ to $\mathbb{R}$

Quick reminder on sets

Finite sets

The set X is finite if there $n \in \mathbb{N}$ such that there is a bijection between X and $\{0, 1, 2, \ldots, n\}$

Countably infinite sets

The set X is countably infinite if there is a bijection between X and $\mathbb{N}$ or, equivalently, $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$

Denumerable sets

- the set X is denumerable if it is finite or countably infinite
- the union or the Cartesian product of two denumerable sets is denumerable
- $\mathbb{Z}$ and $\mathbb{Q}$ are denumerable

Quick reminder on sets

Cardinality

- the set X have a larger cardinality at least as large as Y is there a mapping of X onto Y
- this defines a complete and transitive relation

Infinite sets

- the set $\mathbb{R}$ have a larger cardinality than the set $\mathbb{Q}$
- the converse is false
- $\mathbb{R}^n$ and $\mathbb{R}$ have the same cardinality
- 2^X has a cardinality that is strictly larger than that of X

Results

Theorem (Cantor, 1895)

Let X be a denumerable set (i.e., finite or countably infinite). Let $\succsim$ be a binary relation on X.

There is a real valued function v on X such that

$$x \succsim y \Leftrightarrow v(x) \geq v(y)$$

for all $x, y \in X$
if and only if
$\succsim$ is a weak order

Proof

Proof.

Necessity is clear.

Let us show sufficiency. Since X is denumerable, we can number its elements in such a way that $X = \{x_i : i \in K \subseteq \mathbb{N}_+\}$.
To each $y \in X$ define $N(y) = \{i \in K : y \succsim x_i\}$.
Define u letting $u(y) = \sum_{i \in N(y)} 1/2^i$. This series obviously converges.
If $x \succsim y$ we have, using transitivity, $N(x) \supseteq N(y)$ so that $u(x) \geq u(y)$.
Conversely suppose that $u(x) \geq u(y)$ and $Not[x \succsim y]$. We have $y \succsim x$, using completeness, and $Not[x \succsim y]$. Hence $N(y) \supsetneq N(x)$, so that $u(y) > u(x)$, a contradiction. □

General case

Remark

There are weak orders on sets having at most the cardinality of $\mathbb{R}$ that do not have a numerical representation

Lexicographic preferences

Let $X = \mathbb{R} \times \{1, 0\}$. Define $\succsim$ letting

$$(x, \alpha) \succsim (y, \beta) \Leftrightarrow \begin{cases} x > y \text{ or} \\ x = y \text{ and } \alpha \geq \beta \end{cases}$$

It is clear that $\succsim$ is a weak order.
Suppose that there is a numerical representation of $\succsim$. Take any $x > y$. We have $(x, 1) \succ (x, 0) \succ (y, 1) \succ (y, 0)$ so that $v(x, 1) > v(x, 0) > v(y, 1) > v(y, 0)$.
But there is a rational number $\rho(x)$ in the interval $(v(x, 0), v(x, 1))$ and there is a there is a rational number $\rho(y)$ in the interval $(v(y, 0), v(y, 1))$. We have $x > y \Rightarrow \rho(x) > \rho(y)$. Hence ρ is an injection from $\mathbb{R}$ to $\mathbb{Q}$, which is impossible.

General case

$\mathbb{Q}$ is dense in $\mathbb{R}$

Let $x, y \in \mathbb{R}$. If $x > y$ then there is $z \in \mathbb{Q}$ such that $x > z > y$.

Denseness

Let $\succsim$ be a weak order on X. The set $Y \subseteq X$ is dense in X for $\succsim$ if, for all $x, y \in X$ such that $x \succ y$, we have $x \succsim z \succsim y$, for some $z \in Y$.

Hint

- with this definition $\mathbb{N}$ is dense in itself for $\geq$

Theorem (Debreu, 1954)

Let $\succsim$ be a binary relation on X. There is a numerical representation of $\succsim$ if and only if $\succsim$ is a weak order and there is a denumerable set $Y \subseteq X$ that is dense in X for $\succsim$.

Uniqueness

Uniqueness

Suppose that $\succsim$ on X has a numerical representation v. This numerical representation is not unique. Indeed, it is easy to see that $\Phi \circ v$ is also a numerical representation as soon as Φ is strictly increasing. It is easy to see that these are the only possible transformations that can be applied to v. Hence v is an ordinal scale.

Scales

- ordinal scale: unique up to a strictly increasing transformation $u = \Phi \circ v$
- interval scale: unique up to a positive affine transformation $u = \alpha v + \beta$ with $\alpha > 0$
- ratio scale: unique up to a positive linear transformation $u = \alpha v$ with $\alpha > 0$

Question

- how could we obtain an interval or a ratio scale?

Ordinal scales

Example

	v_1	v_2	v_3	v_4
x	0	0	0	1
y	3	9	27	3
z	4	16	64	3.5
w	5	25	125	1000

- the functions v_1, v_2, v_3, v_4 are all numerical representations of the weak order $w \succ z \succ y \succ x$
- assertion: the average desirability of x and w is larger than the average desirability of y and z
- we have $(v_1(w) + v_1(x))/2 < (v_1(z) + v_1(y))/2$
- but $(v_2(w) + v_2(x))/2 > (v_2(z) + v_2(y))/2$
- this is an example of a meaningless statement

Meaningfulness

Meaningful and meaningless statements

- I weigh twice as much as you
  - meaningful (but may be false!)
- Average temperature are twice higher in Paris than in Moscow
  - meaningless (unless you use the Kelvin scale!)
- the difference in average temperature between Paris and Moscow is twice the difference in average temperature between Rome and London
  - meaningful (but may be false!)

How do I observe $\succsim$?

Observability

- I cannot simply ask for $\succsim$ for epistemological reasons
- I cannot simply ask for the performance measure that is used

Samuelson (1938)

Solution: choice functions

Let $P(X)$ be the set of all nonempty subsets of X. A choice function C is a function from $P(X)$ to $P(X)$ such that $C(A) \subseteq A$, for all $A \in P(X)$.

The set $C(A)$ contains the objects that are judged "choosable" in A.

Remarks

- a choice function can be observed
- is it possible to infer preference from choices?

Revealed preferences

Rationalizable choice function

A choice function C is rationalizable if there is a binary relation $\succsim$ such that, for all $A \in P(X)$,

$$C(A) = M(A, \succ) = \{b \in A : Not[a \succ b] \text{ for all } a \in A\}$$

- when A is finite, it is clear that if $\succsim$ is a weak order on X, $M(A, \succ)$ is nonempty for all $A \in P(X)$
- the same is true as soon as $\succ$ has no circuit

Not all choice functions can be rationalized

Let $X = \{a, b, c\}$. Suppose that

$$C(\{a, b\}) = \{a\}$$
$$C(\{b, c\}) = \{b\}$$
$$C(\{a, c\}) = \{c\}$$

Then we must have $a \succ b$, $b \succ c$ and $c \succ a$. This implies that $M(X, \succ)$ is empty. Hence C cannot be rationalized.

Revealed preferences

Condition α

$$\left.\begin{array}{r} x \in B \subseteq A \\ x \in C(A) \end{array}\right\} \Rightarrow x \in C(B)$$

If the World champion is Italian, she must be the champion of Italy

Condition β

$$\left.\begin{array}{r} B \subseteq A \\ x, y \in C(B) \\ y \in C(A) \end{array}\right\} \Rightarrow x \in C(A)$$

If there are two Italian champions (tied) and one of them is a World champion, the other must also be a World champion

Revealed preferences: results

Theorem (Sen, 1970)

Let C be a choice function on a finite set X. It can be rationalized by a weak order if and only if it satisfies conditions α and β.

Numerous extensions

- C is not observed for all elements of $P(X)$
- X is not finite
- rationalization by an acyclic relation $\succ$

Revealed preferences: questions

Condition α

- if I have to choose in $\{\text{Steak}, \text{Sole Meunière}\}$
  - I choose Steak
- if I have to choose in $\{\text{Steak}, \text{Sole Meunière}, \text{Frog Legs}\}$
  - I choose Sole Meunière
- epistemic value of the menu
- violates condition α

Condition β

- if I have to choose in $\{\text{Bike}, \text{Horse}\}$
  - I am indifferent and both are choosable
- if I have to choose in $\{\text{Bike}, \text{Bike with bell}, \text{Horse}\}$
  - I am indifferent between Bike with bell and Horse (both are choosable)
- violates condition β

Aggregation

Question

- let $\succsim_1, \succsim_2, \ldots, \succsim_k$ be weak orders on X
- do "reasonable" aggregation methods of these k weak orders always lead to a weak order?

Answer

- No!!!! (thanks Marie Jean Antoine Nicolas!)

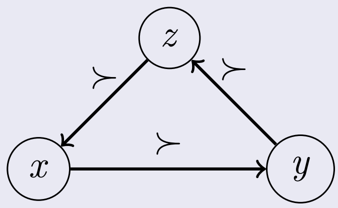
Aggregation

- $X = \{x, y, z\}$ is a set of candidates
- three voters express preferences on X as weak orders
- social preference is an aggregation of individual preferences:

$$x \succsim y \Leftrightarrow |\{i \in N : x \succsim_i y\}| \geq |\{i \in N : y \succsim_i x\}|$$

$$1 : x \succ_1 y \succ_1 z$$
$$2 : z \succ_2 x \succ_2 y$$
$$3 : y \succ_3 z \succ_3 x$$

Condorcet's paradox, 1785



Special structures for X

Structure of X

- left unspecified until now
- when X has a special structure it may be possible to take advantage of this extra structure

Examples of special structures

- decision under risk
  - X is a set of probability distribution on a set of consequences C
- decision under uncertainty
  - X has a homogeneous Cartesian product structure: Y^n is there are n states of nature
- multiple criteria decision making
  - X has a Cartesian product structure: $X_1 \times X_2 \times \cdots \times X_n$

Special structures for X

Independence properties

- independence wrt probabilistic mixtures: expected utility (von Neumann & Morgenstern, 1947)

$$f(x) = \sum_{\gamma \in C} p_x(\gamma) u(\gamma)$$

- sure thing principle: subjective expected utility (Savage, 1954)

$$f(x) = \sum_{e \in E} p(e) u(x(\gamma))$$

- independence: additive value functions (Debreu, 1960, Luce & Tukey 1964)

$$f(x) = \sum_{i=1}^{n} v_i(x_i)$$

More than one performance measure

How to compare the objects in X?

- simple procedure
  - build several mappings $f_1, f_2, \ldots, f_n$

$$x \succsim y \Leftrightarrow \begin{cases} f_1(x) \geq f_1(y) \\ f_2(x) \geq f_2(y) \\ \quad \ldots \\ f_n(x) \geq f_n(y) \end{cases}$$

- dominance (“Pareto front” and the like)

Alternative: lexicographic aggregation

$$x \succ y \Leftrightarrow \begin{cases} f_1(x) > f_1(y) \\ f_1(x) = f_1(y) \text{ and } f_2(x) > f_2(y) \\ \ldots \\ f_1(x) = f_1(y), \ldots, f_{n-1}(x) = f_{n-1}(y) \text{ and } f_n(x) > f_n(y) \end{cases}$$

Quasi orders

$$x \succsim y \Leftrightarrow \begin{cases} f_1(x) \geq f_1(y) \\ f_2(x) \geq f_2(y) \\ \vdots \\ f_n(x) \geq f_n(y) \end{cases}$$

Two obvious properties of $\succsim$

- $\succsim$ is reflexive ($x \succsim x$, for all $x \in X$)
- $\succsim$ is transitive ($[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$, for all $x, y, z \in X$)

Quasi order

- a reflexive and transitive relation is called a quasi order

- if $\succsim$ is antisymmetric it is a partial order

Quasi orders

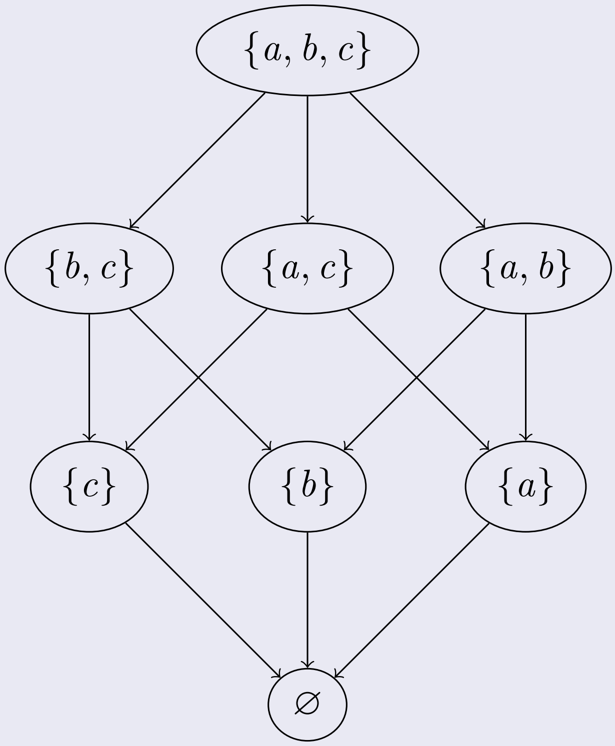
Any partial order on a set X can be obtained as the intersection of a number of total orders. When X is finite, it only takes a finite number of total orders to obtain a partial order (dimension of a partial order, Dushnik & Miller, 1941). The same is true for quasi orders and weak orders.

Theorem (Folk)

Any quasi order on a finite set has a numerical representation such that

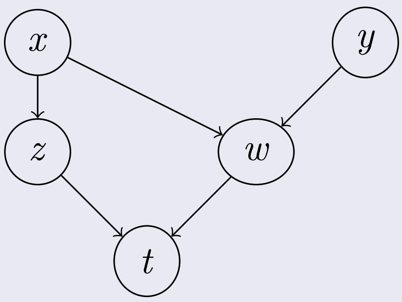
$$x \succsim y \Leftrightarrow \begin{cases} u_1(x) \geq u_1(y) \\ u_2(x) \geq u_2(y) \\ \dots \\ u_k(x) \geq u_k(y) \end{cases}$$

Example: partial order of dimension 3



Example

Let $X = \{x, y, z, w, t\}$.



Remark

- $\succ$ is asymmetric and transitive
- $M(A, \succ)$ is always nonempty when $A \neq \varnothing$ and is finite
- non-dominated solutions in MCDM

Standard model with caution

How to compare the objects in X?

- simple procedure
  - build a mapping $f : X \to \mathbb{R}$
  - compare objects using f with caution

$$x \succ y \Leftrightarrow f(x) > f(y) + q$$

$$x \sim y \Leftrightarrow |f(x) - f(y)| \geq q$$

$q \geq 0$: constant threshold

$$x \succsim y \Leftrightarrow f(x) \geq f(y) - q$$

Remark

- if $q = 0$ are back to the standard model

Standard model with caution

Model

$$x \succsim y \Leftrightarrow f(x) \geq f(y) - q$$

$q \geq 0$: constant threshold

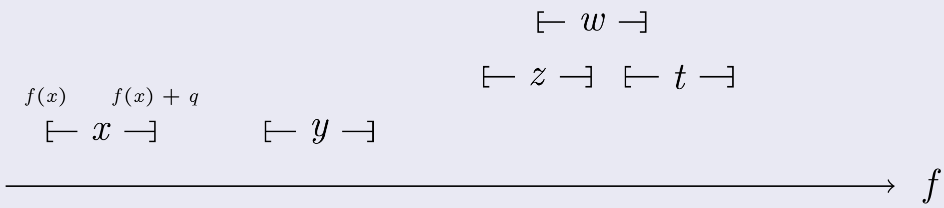
Obvious properties of $\succsim$

- $\succsim$ is complete $x \succsim y$ or $y \succsim x$, for all $x, y \in X$
- $\succsim$ is not transitive but $\succ$ is transitive

$$\succ \cdot \sim \subseteq \succ$$

$$\sim \cdot \succ \subseteq \succ$$

Both these relations are false



$y \succ x$ (the y interval does not intersect and is to the right of the x interval)
$z \sim w$ (the z interval intersect the w interval)

$z \sim w$ and $w \sim t$ but $t \succ z$
$t \succ z$ and $z \sim w$ but $t \sim w$
$w \sim t$ and $t \succ z$ but $w \sim t$

Ferrers

Model

$$x \succsim y \Leftrightarrow f(x) \geq f(y) - q$$

Ferrers

$$\left. \begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} x \succsim w \\ \text{or} \\ z \succsim y \end{array} \right.$$

Necessity

$$x \succsim y \Rightarrow f(x) \geq f(y) - q$$
$$z \succsim w \Rightarrow f(z) \geq f(w) - q$$
$$Not[x \succsim w] \Rightarrow f(x) < f(w) - q$$
$$Not[z \succsim y] \Rightarrow f(z) < f(y) - q$$

we obtain $f(y) > f(w)$ and $f(w) > f(y)$, a contradiction

Semi-transitivity

$$x \succsim y \Leftrightarrow f(x) \geq f(y) - q$$

Semi-transitivity

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ y \succsim z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \succsim w \\ \text{or} \\ w \succsim z \end{array}\right.$$

Necessity

$$x \succsim y \Rightarrow f(x) \geq f(y) - q$$
$$y \succsim z \Rightarrow f(y) \geq f(z) - q$$
$$Not[x \succsim w] \Rightarrow f(x) < f(w) - q$$
$$Not[w \succsim z] \Rightarrow f(w) < f(z) - q$$

we obtain $f(y) > f(w)$ and $f(w) > f(y)$, a contradiction

Semiorder

Definition

A semiorder is a complete, Ferrers and semi-transitive binary relation

Theorem (Luce, 1956)

A binary relation on a finite set X is a semiorder if and only if there is a real valued function u on X and a threshold $q \geq 0$ such that:

$$x \succsim y \Leftrightarrow u(x) \geq u(y) - q$$

Remarks

- not true if X is denumerable
- can be extended to countable set with a variable but consistent threshold

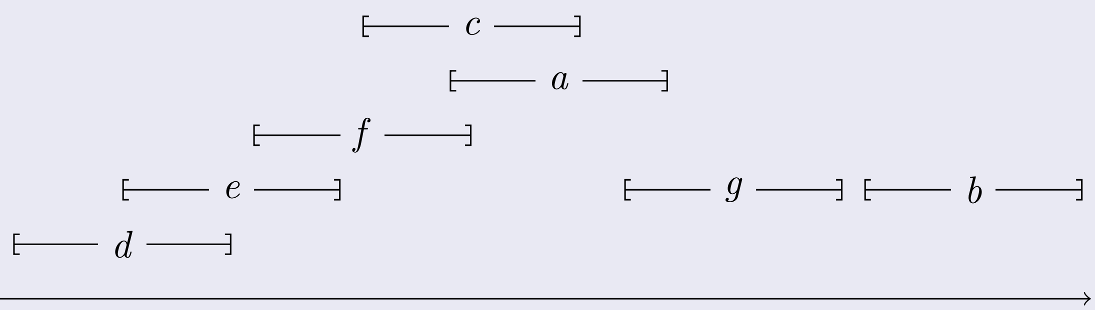
Example

	b	g	a	c	f	e	d
b	1	1	1	1	1	1	1
g	0	1	1	1	1	1	1
a	0	1	1	1	1	1	1
c	0	0	1	1	1	1	1
f	0	0	1	1	1	1	1
e	0	0	0	0	1	1	1
d	0	0	0	0	0	1	1

	b	g	a	c	f	e	d
b	1	1	1	1	1	1	1
g	0	1	1	1	1	1	1
a	0	1	1	1	1	1	1
c	0	0	1	1	1	1	1
f	0	0	1	1	1	1	1
e	0	0	0	0	1	1	1
d	0	0	0	0	0	1	1

Example

	b	g	a	c	f	e	d
b	1	1	1	1	1	1	1
g	0	1	1	1	1	1	1
a	0	1	1	1	1	1	1
c	0	0	1	1	1	1	1
f	0	0	1	1	1	1	1
e	0	0	0	0	1	1	1
d	0	0	0	0	0	1	1



Traces

If $\succsim$ is a semiorder, the relation $\succsim^+$ defined by

$$x \succsim^+ y \Leftrightarrow [y \succsim z \Rightarrow x \succsim z]$$

is a weak order (note that it is always reflexive and transitive)

If $\succsim$ is a semiorder, the relation $\succsim^-$ defined by

$$x \succsim^- y \Leftrightarrow [z \succsim x \Rightarrow z \succsim y]$$

is a weak order (note that it is always reflexive and transitive)

If $\succsim$ is a semiorder, the relation $\succsim^\pm$ defined by

$$x \succsim^\pm y \Leftrightarrow [x \succsim^+ y \text{ and } x \succsim^- y]$$

is a weak order (note that it is always reflexive and transitive)

- the relation $\succsim^\pm$ is the weak order underlying the semiorder $\succsim$
- the matrix representation of a semiorder is stepped when rows and columns are arranged wrt $\succsim^\pm$

Uniqueness

Two representations of a semiorder

$$x \succ y, x \succ z, y \sim z$$

	v_1	v_2
x	2	2
y	0	0.5
z	0	0

- irregular representation

Standard model with (even more) caution

How to compare the objects in X?

- simple procedure
  - build a mapping $f : X \to \mathbb{R}$
  - compare objects using f with (even more) caution

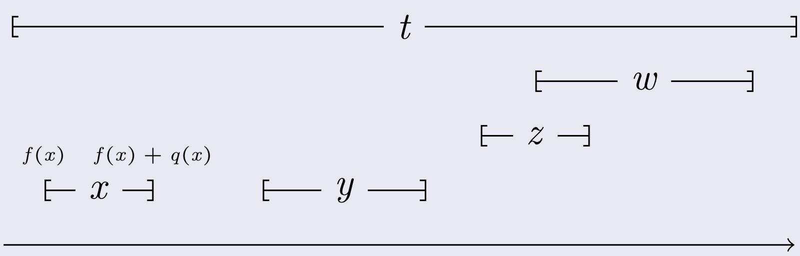
$$x \succ y \Leftrightarrow f(x) > f(y) + q(y)$$

$$x \sim y \Leftrightarrow \left\{ \begin{array}{l} f(x) \leq f(y) + q(y) \\ f(y) \leq f(x) + q(x) \end{array} \right\}$$

$q(\cdot) \geq 0$: variable threshold

$$x \succsim y \Leftrightarrow f(x) + q(x) \geq f(y)$$

Example



$y \succ x$ (the y interval does not intersect and is to the right of the x interval)
$z \sim w$ (the z interval intersect the w interval)
t is indifferent to all other alternatives

Standard model with (even more) caution

$$x \succsim y \Leftrightarrow f(x) + q(x) \geq f(y)$$

$q(\cdot) \geq 0$: variable threshold

Obvious properties of $\succsim$

- $\succsim$ is complete $x \succsim y$ or $y \succsim x$, for all $x, y \in X$
- $\succsim$ is not transitive but $\succ$ is transitive

Ferrers

$$x \succsim y \Leftrightarrow f(x) + q(x) \geq f(y)$$

Ferrers

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x \succsim w \\ \text{or} \\ z \succsim y \end{array}\right.$$

Necessity

$$x \succsim y \Rightarrow f(x) + q(x) \geq f(y)$$
$$z \succsim w \Rightarrow f(z) + q(z) \geq f(w)$$
$$Not[x \succsim w] \Rightarrow f(x) + q(x) < f(w)$$
$$Not[z \succsim y] \Rightarrow f(z) + q(z) < f(y)$$

we obtain $f(x) + q(x) > f(z) + q(z)$ and $f(z) + q(z) > f(x) + q(x)$, a contradiction.

Interval order

Definition

An interval order is a complete and Ferrers binary relation

Theorem (Fishburn, 1970)

A binary relation on a finite set X is an interval order if and only if there is a real valued function u on X and a nonnegative threshold function q such that:

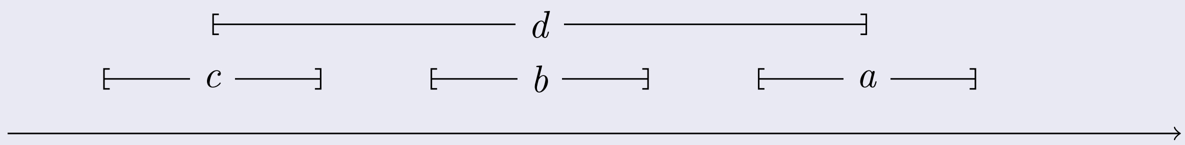
$$x \succsim y \Leftrightarrow u(x) + q(x) \geq u(y)$$

Remarks

- remains true if X is denumerable
- add order denseness condition in the general case

Example

	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	1
d	1	1	1	1



Semi-transitivity can be violated

$$\left.\begin{array}{c} c \succsim d \\ \text{and} \\ d \succsim a \end{array}\right\} \left\{\begin{array}{c} Not[c \succsim b] \\ \text{and} \\ Not[b \succsim a] \end{array}\right.$$

Example

	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	1
d	1	1	1	1

	a	b	d	c
a	1	1	1	1
d	1	1	1	1
b	0	1	1	1
c	0	0	1	1

- rows are arranged according to outdegrees
- columns are arranged according to indegrees

Traces

If $\succsim$ is an interval order, the relation $\succsim^+$ defined by

$$x \succsim^+ y \Leftrightarrow [y \succsim z \Rightarrow x \succsim z]$$

is a weak order (note that it is always reflexive and transitive)
It governs the order of the left side of intervals (outdegrees)

If $\succsim$ is an interval order, the relation $\succsim^-$ defined by

$$x \succsim^- y \Leftrightarrow [z \succsim x \Rightarrow z \succsim y]$$

is a weak order (note that it is always reflexive and transitive)
It governs the order of the right side of intervals (indegrees)

If $\succsim$ is an interval order, the relation $\succsim^\pm$ defined by

$$x \succsim^\pm y \Leftrightarrow [x \succsim^+ y \text{ and } x \succsim^- y]$$

may not be complete.

Standard model with (even even more) caution

How to compare the objects in X?

- simple procedure
  - build a mapping $f : X \to \mathbb{R}$
  - compare objects using f with (even even more) caution

$$x \succ y \Leftrightarrow f(x) > f(y) + q(x, y)$$

$$x \sim y \Leftrightarrow \left\{ \begin{array}{l} f(x) \leq f(y) + q(x, y) \\ f(y) \leq f(x) + q(y, x) \end{array} \right\}$$

$q(x, y) = q(y, x) \geq 0$: symmetric threshold depending on both alternatives

$$x \succsim y \Leftrightarrow f(x) + q(x, y) \geq f(y)$$

Standard model with (even even more) caution

Model

$$x \succ y \Leftrightarrow f(x) > f(y) + q(x, y)$$

$$x \succsim y \Leftrightarrow f(x) + q(x, y) \geq f(y)$$

$q(x, y) = q(y, x) \geq 0$: symmetric threshold depending on both alternatives

Obvious properties of $\succsim$

- $\succsim$ is complete $x \succsim y$ or $y \succsim x$, for all $x, y \in X$
- $\succsim$ is not transitive, $\succ$ is not transitive

Example

Model

$$x \succ y \Leftrightarrow f(x) > f(y) + q(x,y)$$
$$x \succsim y \Leftrightarrow f(x) + q(x,y) \geq f(y)$$

$q(x,y) = q(y,x) \geq 0$: symmetric threshold depending on both alternatives

Example

$x \succ y,\ y \succ z,\ Not[x \succ z],\ Not[z \succ x]\ (x \sim z)$

$v(x) = 10,\ v(y) = 6,\ v(z) = 2$
$q(x,y) = q(y,x) = 1,\ q(y,z) = q(z,y) = 1,\ q(x,z) = q(z,x) = 9$

Classical extensions Models with threshold

Absence of circuits

Model

$$x \succ y \Leftrightarrow f(x) > f(y) + q(x,y)$$
$$x \succsim y \Leftrightarrow f(x) + q(x,y) \geq f(y)$$

$q(x,y) = q(y,x) \geq 0$: symmetric threshold depending on both alternatives

Absence of circuit

$x_1 \succ x_2 \succ \ldots \succ x_k \Rightarrow Not[x_k \succ x_1]\ (\forall k > 1)$

$x_1 \succ x_2 \succ \ldots \succ x_k \succ x_1$

$$f(x_1) > f(x_2) + q(x_1, x_2)$$
$$f(x_2) > f(x_3) + q(x_2, x_3)$$
$$\ldots$$
$$f(x_{k-1}) > f(x_k) + q(x_{k-1}, x_k)$$
$$f(x_k) > f(x_1) + q(x_k, x_1)$$

$q(x_k, x_1) + \sum_{i=1}^{k} q(x_i, x_{i+1}) < 0$, a contradiction since $q \geq 0$

Suborders

Definition

A suborder $\succsim$ is a complete binary relation such that $\succ$ has no circuit

Theorem (Fishburn, 1970)

A binary relation on a finite set X is a suborder if and only if there is a real valued function u on X and a nonnegative symmetric threshold function q such that:

$$x \succsim y \Leftrightarrow f(x) + q(x,y) \geq f(y)$$

Remarks

- remains true if X is denumerable
- add order denseness condition in the general case

Summary: structures with threshold

suborder	variable threshold	$q(x,y) \geq 0$
interval order	variable threshold	$q(x) \geq 0$
semiorder	constant threshold	$q \geq 0$
weak order	null threshold	$q = 0$
total order	no indifference	$\sim$ is trivial

Partial structures with threshold

Dominance with semiorders

$X = \{a, b, c\}$
$c \succ_1 a, c \sim_1 b, b \sim_1 a$ (semiorder)
$a \succ_2 b, a \sim_2 c, c \sim_2 b$ (semiorder)
$b \succ_3 c, b \sim_3 a, a \sim_3 c$ (semiorder)

Cycling

$a \succ b$: $a \sim_1 b, a \succ_2 b, a \sim_3 b$
$b \succ c$: $b \sim_1 c, b \sim_2 c, b \succ_3 c$
$c \succ a$: $c \succ_1 a, c \sim_2 a, c \sim_3 a$

Structures with hesitation

Remarks

- in all models studied so far there is a sharp transition between $\succ$ and $\sim$
- we may expect that in some case there is an "hesitation zone" between these two relations

Pseudo orders

$$x \succ y \Leftrightarrow f(x) > f(y) + p(y)$$
$$x \asymp y \Leftrightarrow f(y) + p(y) \geq f(x) > f(y) + q(y)$$
$$x \sim y \Leftrightarrow [f(x) \leq f(y) + q(y) \text{ and } f(y) \leq f(x) + q(x)]$$

- conditions on $\langle \succ, \asymp, \sim \rangle$ are known (Roy & Vincke, 1987)

Extensions

Interval orders

- intervals are associated to objects

Extensions

- associate other geometrical shapes objects
- circles, trapezoids, etc.

Extensions

- special points within intervals

Fuzzyness

Remarks

- all models use crisp binary relations
- either $x \succ y$ is true or it is false

Fuzzy models

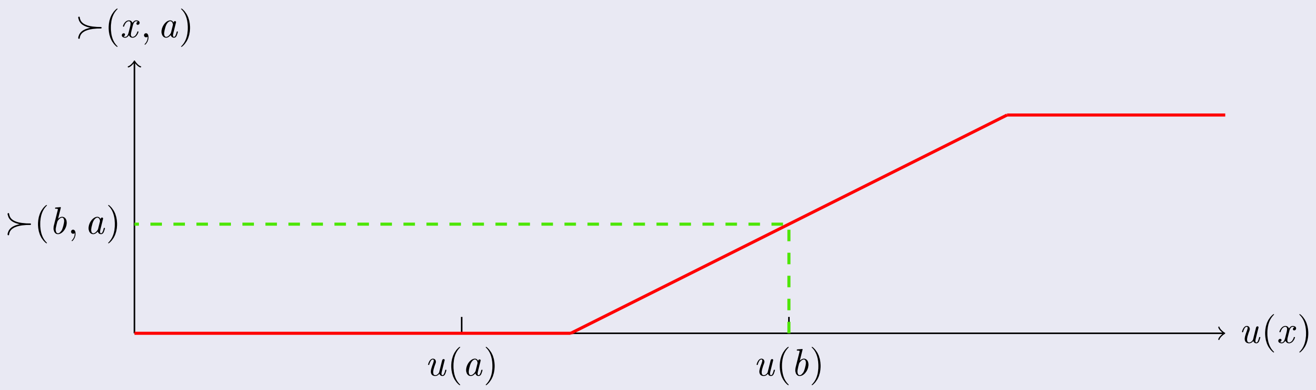
- use fuzzy binary relations
- $x \succ y$ has a degree of credibility belonging to $[0, 1]$

Questions

- how to define classical properties (completeness, transitivity, etc) for fuzzy relations?
- not obvious but the use of cut relations is useful

$$x \succ_\lambda y \Leftrightarrow \succ(x, y) \geq \lambda$$

Illustration



$\succ(b, a) \in [0, 1]$ is the credibility of the proposition "$b \succ a$"

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