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A NOTE ON THE RELATIONSHIPS BETWEEN UTILITY AND VALUE FUNCTIONS (*)

ABSTRACT

In order to dissipate the confusion created by the notion of "measurability", many authors have argued that, in general, a von Neumann-Morgenstern utility function does not measure preference differences between outcomes since it is only based on simple lottery comparisons. Nevertheless the relationships between a von Neumann-Morgenstern utility function and a function measuring preference differences have rarely been studied from a theoretical point of view. The aim of this paper is to provide a simple necessary and sufficient condition, relating even chance lotteries and preference differences, for the two functions to coincide. The discussion of the acceptability and reasonableness of this easily interpretable condition could shed a new light on this question through avoiding the problem, encountered in most empirical studies, of the dependence of the shape of the functions on the assessment method used.

This paper pursues previous work on this topic by M. Allais and R. Sarin.

1. INTRODUCTION AND BACKGROUND

In order to dissipate the confusion created by the notion of "measurability", many authors (e.g. Luce and Raiffa (1957) or Fishburn (1976)) have argued that, in general, a von Neumann-Morgenstern (vN-M) utility function does not measure preference differences between outcomes since it is only based on simple lottery comparisons. The purpose of this note

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is to study the relationships between these two concepts from a theoretical point of view.

This problem dates back at least to Allais (1953). In his classical attack against utility theory, he argued that if a vN-M utility function exists, it must be identical to his "cardinal utility function". Allais (1979) gives rather a complex proof of this assertion using an apparently very restrictive axiom of "cardinal isovariation", a result of "amazing strength" in the words of Machina (1981). More recently, Allais (1985) allegedly proved that, in all cases, these two functions, when they exist, must be identical.

In spite of the uncertain operational status of the very notion of preference difference, there seems to be a growing interest in this problem. Dyer and Sarin (1982) and Krzysztofowicz (1983) argue that a vN-M utility function captures both strength of preference and risk attitude, leading to the definition of a "relative risk attitude". Bell (1981) explicitly includes strength of preference among the components of risk attitude. Dyer and Sarin (1979) and Baron et al. (1984) have investigated the link between various functional decompositions of utility functions and functions measuring preference differences.

The approach taken here is similar to that of Sarin (1982). Using Savage's state of the world framework, he gives conditions that imply the existence of probabilities and a utility function representing both preferences and preference differences between acts and discusses at large the theoretical and practical interest of this problem. Within a different framework we obtain simpler and more easily interpretable conditions.

Suppose X is a nonempty set of outcomes. Expected utility is concerned with the problem of representing a preference relation R on a set of mixtures defined on the basis of X. For simplicity, we will use throughout the paper the set $P_s(X)$ of all simple probability measures on X, $p(x)$ being the probability assigned to $x \in X$ by $p \in P_x(X)$. The desired numerical representation $U : P_x(X) \to \mathbb{R}$ is such that for all $p, q \in P_s(X)$ and all $\alpha \in (0, 1)$

$$p \, R \, q \text{ iff } U(p) \geq U(q) \text{ and} \tag{1}$$

$$U(\alpha p + (1 - \alpha) q) = \alpha U(p) + (1 - \alpha) U(q). \tag{2}$$

where $\alpha p + (1 - \alpha) q$ is the probability measure assigning

the probability $\alpha\, p(x) + (1 - \alpha)\, q(x)$ to each $x \in X$.

On the basis of U defined on $P_s(X)$ it is possible to define a real-valued function u on X such that for all $p, q \in P_s(X)$

$$p \; R \; q \quad \text{iff} \quad \sum_{x \in X} p(x)\, u(x) \geq \sum_{x \in X} q(x)\, u(x)$$

by letting $u(x) = U(p_x)$ where p_x is a probability measure such that $p_x(x) = 1$.

Conditions for the existence of u can be found in Fishburn (1970). Given the richness of the structure of $P_s(X)$ they imply that u on X (or alternatively U on $P_s(X)$) defines an interval scale, i.e. is defined up to a positive affine transformation.

Throughout the paper we use capital letters to denote functions on $P_s(X)$ and lower case letters for functions on X.

The notion of preference differences is based upon the existence of a binary relation R^o on $X \times X$ allowing to compare "differences" between outcomes. We will be interested in a numerical representation $v : X \to \mathbb{R}$ such that for all $x, y, z, w \in X$:

$$(x, y)\; R^o\; (z, w) \quad \text{iff} \quad v(x) - v(y) \geq v(z) - v(w).$$

We call such a function v a (value) function measuring preference differences. Conditions for the existence of such a function can be found in Scott and Suppes (1958), Debreu (1960) and Krantz et al. (1971). It is well-known that unless the structure of X is sufficiently rich, v does not provide an interval scale of measurement. According to our conventions, we note V a function measuring preference differences on $P_s(X)$.

In order to explore the relationships between u and v we begin in section 2 with the study of the rather theoretical case in which R^o is defined on $P_s(X) \times P_s(X)$, i.e. when it is supposed that it is possible to compare preference differences, not only between outcomes, but also between mixtures - so that we compare U and V. Section 3 studies the case in which R is defined on $P_s(X)$ and R^o on $X \times X$ - so that we compare u and v. In section 4 we discuss

the implications of our results and their relations to previous empirical studies.

Throughout the paper I and P (resp. I^o and P^o) will respectively denote the symmetric and asymmetric part of R (resp. R^o). When needed we use the notation $p\ \alpha\ q$ instead of $\alpha p + (1 - \alpha) q$ to denote a mixture.

2. PREFERENCE DIFFERENCES ON MIXTURES

When R^o is defined on $P_s(X) \times P_s(X)$, fairly simple conditions ensure that U and V are identical up to a positive affine transformation.

We have the following :

Theorem 1 : Let

- U be a real-valued function on $P_s(X)$ satisfying (1) and (2), and
- V be a real-valued function on $P_s(X)$ satisfying the condition

For all $p, q, s, t \in P_s(X)$: $(p, q)\ R^o\ (s, t)$ iff $V(p) - V(q) \geq V(s) - V(t)$. (3)

Then there are $a, b \in \mathbb{R}$ with $a > 0$ such that :

$V(p) = a\,U(p) + b$ for all $p \in P_s(X)$

iff for all $p, q \in P_s(X)$

A1 : $p\ R\ q$ iff $(p, q)\ R^o\ (q, q)$

A2 : $(p, \frac{1}{2} p + \frac{1}{2} q)\ I^o\ (\frac{1}{2} p + \frac{1}{2} q, q)$.

Proof : Necessity is obvious. Given the well-known unicity result for functions satisfying (1) and (2), all we have to prove to establish sufficiency is that V satisfies (1) and (2). The fact that V satisfies (1) is obvious from A.1 and (3). It remains to prove that V is linear. Let us first observe that A.2 and (3) imply that, for all $p, q \in P_s(X)$, $V(p\ \frac{1}{2}\ q) = \frac{1}{2} V(p) + \frac{1}{2} V(q)$ and that repeated applications of this last result show that, for all $p, q \in P_s(X)$ and all positive integers m, n such that $0 \leq m/2^n \leq 1$, we have :

$$V(p(m/2^n)q) = (m/2^n)\ V(p) + (1 - (m/2^n))\ V(q).$$

Suppose that $p\ I\ q$. It follows from the existence of U that $p\ I\ p\ \alpha\ q\ I\ q$ for all $\alpha \in (0, 1)$. Hence $V(p\ \alpha\ q) = \alpha\ V(p) + (1 - \alpha)\ V(q)$.

Suppose now that $V(p \alpha q) > \alpha V(p) + (1 - \alpha) V(q)$ for some $\alpha \in (0, 1)$ and for some $p, q \in P_s(X)$ such that $p P q$. Given that $p P q$, we know from the existence of U that $p P p \alpha q P q$ and thus $V(p) > V(p \alpha q) > V(q)$. Hence there exists a $K \in (0, 1)$ such that $K > \alpha$ and $V(p \alpha q) = K V(p) + (1 - K) V(q)$. From the properties of rational numbers we know that there exist positive integers m^o and n^o such that $K > m^o/2^{n^o} > \alpha$. We thus have $p(m^o/2^{n^o})q P p \alpha q$ so that $V(p(m^o/2^{n^o})q) = (m^o/2^{n^o}) V(p) + (1 - (m^o/2^{n^o})) V(q) > V(p \alpha q) = K V(p) + (1 - K) V(q)$. But this is impossible since $V(p) > V(q)$ and $K > m^o/2^{n^o}$.

The case $V(p \alpha q) < \alpha V(p) + (1 - \alpha) V(q)$ is proved similarly.

Q.E.D.

Theorem 1 says that if preference differences, expressed on mixtures, are related to preferences in a consistent way through A.1 and if there exist real-valued functions U and V respectively satisfying (1)-(2) and (3), then, given A.2, every function measuring preference differences must be linear, so that such a function becomes measurable.

We strongly emphasize that A.1 and A.2 cannot be weakened supposing that V on $P_s(X)$ is defined up to a positive affine transformation. While A.1 alone implies that U and V are related by a strictly increasing monotonic transformation, the full force of A.2 (combined with the richness of the structure of $P_s(X)$) is needed to ensure that U and V provide the same interval scale on $P_s(X)$.

Therefore we are bound to reject the conclusions of theorem III of Allais (1985) which states that U and V always provide the same scale on $P_s(X)$ when V defines an interval scale and A.1 holds. This is because the fact that U and V both define an interval scale do not imply that there is a function h such that $a U + b = h(a' V + b')$ for all $a, a', b, b' \in \mathbb{R}$ with $a, a' > 0$; h obviously depends on a, a', b, b'.

3. PREFERENCE ON MIXTURES AND PREFERENCE DIFFERENCES ON OUTCOMES

In the previous section we admitted the possibility of com-

paring preference differences between mixtures which is surely not realistic in many situations. In this section we only suppose that R^o is defined on $X \times X$. This complicates the situation because the simplicity of the conditions used in theorem 1 was mainly due to the richness of the structure of $P_s(X)$.

In order to obtain similar conditions, we introduce the following structural condition on X :

C.1 : For each $p \in P_s(X)$, $p \; I \; p_x$ for some $x \in X$.

We then have :

Theorem 2 : Suppose that u is a vN-M utility function, v is a function measuring preference differences on X and C.1 holds. Then there are $a, b \in \mathbb{R}$ with $a > 0$ such that $u(x) = a\, v(x) + b$ for all $x \in X$ if for all $x, y, z \in X$:

A.3 : $p_x \; R \; p_y$ iff $(x, y) \; R^o \; (y, y)$.

A.4 : $p_y \; I \; \frac{1}{2} p_x + \frac{1}{2} p_z \rightarrow (x, y) \; I^o \; (y, z)$.

Proof : Necessity is obvious. To establish sufficiency, suppose A.3 and A.4 hold. We define a binary relation R^{oo} on $P_s(X) \times P_s(X)$ by $(p, q) \; R^{oo} \; (r, t)$ iff $(x, y) \; R^o \; (z, w)$ for some $x, y, z, w \in X$ such that $p_x \; I \; p$, $p_y \; I \; q$, $p_z \; I \; r$ and $p_w \; I \; t$.

We have $p \; R \; q$ iff $(p, q) \; R^{oo} \; (q, q)$. To prove it suppose that $p \; R \; q$. It follows from our structural assumption and the existence of a vN-M utility function that $p_x \; R \; p_y$ for some $x, y \in X$ such that $p_x \; I \; p$ and $p_y \; I \; q$. Therefore the conclusion follows from A.3 and the definition of R^{oo}. If $(p, q) \; R^{oo} \; (q, q)$ then there are $x, y \in X$ such that $p \; I \; p_x$, $q \; I \; p_y$, $(x, y) \; R^o \; (y, y)$ and the conclusion follows from A.3.

We now prove that for all $p, q \in P_s(X)$, $(p, \frac{1}{2} p + \frac{1}{2} q) \; I^{oo} \; (\frac{1}{2} p + \frac{1}{2} q, q)$. Our structural assumption imply the existence of $x, y, z \in X$ such that $p_x \; I \; p$, $p_y \; I \; q$ and $p_z \; I \; (\frac{1}{2} p + \frac{1}{2} q)$. The existence of u implies that $p_z \; I \; (\frac{1}{2} p_x + \frac{1}{2} p_y)$ and from A.4 we have $(x, z) \; I^o \; (z, y)$ so

that $(p, \frac{1}{2}p + \frac{1}{2}q)\ I^{oo}\ (\frac{1}{2}p + \frac{1}{2}q, q)$ by definition. The proof will thus be complete, using theorem 1 if we show the existence of real-valued functions U and V on $P_s(X)$ respectively satisfying (1)-(2) and (3). The existence of U follows from that of u. For all $p \in P_s(X)$ we define a function V as $V(p) = v(x)$ if $p_x\ I\ p$. The fact that V is well-defined (i.e. that $v(x) = v(y)$ for all $x, y \in X$ such that $p_x\ I\ p$ and $p_y\ I\ p$) straightforwardly follows from A.3 and the existence of u.

The proof that V measures preference differences on $P_s(X)$ is easy and left to reader. Q.E.D.

A.3 and A.4 are the counterparts of A.1 and A.2 in this context. Condition A.4 was proposed by Roy (1985). The structural condition we used in the previous theorem is very strong. It is nevertheless satisfied in the important case where X is a closed interval of the real-line on which is defined a continuous vN-M utility function.

In a similar context, it is possible to envisage different conditions relating lottering and preference differences. Bell (1981) mentions a condition which implies that if x is the certainty equivalent of some lottery and if a same "amount" in terms of v is added to all outcomes of the lottery, then x plus this "amount" is the certainty equivalent of the "increased" lottery. According to Bell, this condition implies that v is related to u either by an affine or an exponential transformation. This condition is very similar to the "cardinal isovariation axiom" of Allais (1979).

4. DISCUSSION

That some idea of strength of preference interferes with risky choices is hardly disputable. Our main result implies that, when X is an interval of the real line, the preference between risky alternatives of an individual having a continuous vN-M utility function and a function measuring preference differences are only governed by his strength of preference for the various outcomes, i.e. that he is necessarily "relatively risk neutral" in the sense of Dyer and Sarin (1982), if he is prepared to accept A.4 (or A.2 depen-

ding on the context). This amounts to negating any specific element due to the introduction of risk in a choice situation and to reducing the concept of risk aversion to the classical idea of "decreasing marginal utility" used in economics. Intuitively, this seems hardly acceptable and may lead us to conclude that A.2 and A.4 are overly restrictive despite their rather innocuous formulation. In fact, many authors (e.g. Fishburn (1970, p. 82)) have argued that they see no reason why lottery comparisons should coincide with preference difference comparisons.

As discussed by Bell (1981), it seems that many features do influence risky choices apart from strength of preference. Among them, we feel that regret (Bell (1982) or Loomes and Sugden (1982)), disappointment (Bell (1985)), the existence of a specific utility (or disutility) of gambling, the misperception of probabilities (especially around certainty) and the avoidance of ruin are the most important ones. They nevertheless are all incompatible with a strict interpretation of the vN-M axioms. Therefore, in our view, the crucial assumption is more the existence of a vN-M utility function than A.2 or A.4. We consider that if an individual is rational enough to possess a vN-M utility function and a function measuring preference differences then he would also be prepared to accept A.2, at least as a normative guide (see also the rationale given by Bell (1981) for the acceptance of A.2). But this argument remains weak since there is no widely accepted operational interpretation of the strength of preference notion (see however Vansnick (1984) and (1986)).

There has been a large number of empirical studies aiming at comparing utility and function measuring preference differences (see Fischer (1977), Sarin et al. (1980), Krzysztofowicz (1983), McCord and de Neufville (1983) and the partial results of the famous 1952 experiment reported in Allais (1979)). They all show that, for most individuals, there is a large difference between the vN-M utility function and the function measuring preference differences, assessed using one of the methods described in Fishburn (1967) - and that this difference cannot be reconciled since most individual explicitely argue in favour of the existence of elements specific to the risky situation (see especially Sarin et al. (1980) and McCord and de Neufville (1983)). However the interpretation of these empirical results is not as easy as it might appear at first might. In fact, many empirical studies lead to the conclusion that the shape of

the utility function is highly dependent upon the assessment method used, that this dependence exhibits definite trends and that the differences between two utility functions assessed using different methods cannot be reconciled (see McCord and de Neufville (1983), and the survey of Bouyssou (1984b)). Allais uses this argument, which is strongly supported by the results of his 1952 experiment, against the vN-M axioms, without showing, however, that a similar phenomenon cannot occur for the assessment of his "cardinal utility function". Though this has never been explicitely tested, we have no reason to consider that such a phenomenon would be absent from the assessment of functions measuring preference differences. In fact McCord and de Neufville have observed that the differences between utility and functions measuring preference differences were of the same magnitude that the ones between various utility functions assessed using different assessment techniques. Though the very careful experimental study of Krzysztofowicz (1983) leads to the conclusion that utility and functions measuring preference differences are significantly different, he rightly notes that his results may well be dependent upon the assessments methods he used for both functions.

This suggests a heavy experimental agenda in order to know whether functions measuring preference differences can be assessed in a reliable way and, if this appears to be the case, whether their predictive power to explain risky choices in presence of A.2 or A.4 is acceptable compared to that of a utility function, taking into account its possible indeterminacy due to its dependence upon assessment methods.

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