Choosing and Ranking on the Basis of Fuzzy Preference Relations with the "Min in Favour"

Denis Bouyssou[1] and Marc Pirlot[2]

[1] ESSEC, BP 105, F-95021 Cergy-Pontoise Cedex, France
[2] Faculté Polytechnique de Mons, 9 rue de Houdain, B-7000 Mons, Belgium

Abstract. In some MCDM techniques – most notably in Outranking Methods – the result of the comparison of a finite set of alternatives according to several criteria is summarised using a fuzzy preference relation. This fuzzy relation does not, in general, possess "nice properties" such as transitivity or completeness and elaborating a recommendation on the basis of such information is not an obvious task. The purpose of this paper is to study techniques exploiting fuzzy preference relations in order to choose or rank. We present a number of results concerning techniques based on the "min in Favour" score, *i.e.* the minimum level with which an alternative is "at least as good as" all other alternatives.

Keywords. MCDM, Fuzzy Preferences, Choice Rules, Ranking Rules, Outranking Methods.

1 Introduction

Consider a finite set of alternatives that are evaluated along several criteria. In order to select a subset of alternatives or to rank order them, methods related to the "Outranking Approach" (see [16] or [17]) usually proceed in two steps. The construction step consists in a pairwise comparison of the alternatives taking all criteria into account. In most methods this is done using a "concordance-discordance" principle which leads to declaring that "alternative a is at least as good as alternative b" when:

– a "sufficient" majority of criteria supports this proposition and
– the opposition of the minority is not "too strong".

In many methods, *e.g.*, in ELECTRE III (see [15]) or in PROMETHEE (see [8]), the result of these pairwise comparisons is summarised using a fuzzy relation. This means that a number between 0 and 1 is associated to each ordered pair (a, b) of alternatives indicating the credibility of the proposition "a is at least as good as b", *e.g.* the sum of the weights of the criteria favouring a in the comparison (eventually corrected to take into account the opposition of the other criteria). Such a way of modelling comparisons made along several dimensions is quite reminiscent of classical electoral techniques which summarise a ballot by an "electoral matrix" giving for each ordered pair (a, b) of candidates the number of voters having declared that "a is at least as good as b". It is well-known in Social Choice Theory that it is not easy to tell which candidate(s) should be elected on the basis of an "electoral matrix" as soon as the opinion of the voters is "sufficiently" conflictual. A similar problem occurs with fuzzy preference relations built in Outranking Methods. When the different criteria taken into account are conflictual, these relations do not, in general, possess "nice properties" such as transitivity or completeness. With the construction technique of ELECTRE III, it is shown in [6] that a stronger conclusion holds: any reflexive fuzzy preference relation may be obtained as soon as there is a "sufficient" number of "sufficiently conflicting" criteria. Therefore, it is far from being an easy task to select a subset of alternatives or to rank order them on the basis of such information. This calls for the application of specific techniques which constitute the "exploitation step" of Outranking Methods. Many such techniques have been proposed in the literature (see, *e.g.*, [16] or [17]) most often on a purely *ad hoc* basis. This has often been seen as a major weakness of Outranking Methods. The aim of this paper is to contribute to their analysis. We present a number of results concerning choice and ranking techniques that are based on the "min in Favour" (mF) score, *i.e.* the minimum level with which an alternative is "at least as good as" all other alternatives, consolidating and extending previous results appeared in [3], [5], [12], [13] and [14]. The axiomatic characterisations presented here will hopefully allow to emphasise the specific features of the techniques studied and, hence, to compare them more easily with other ones.

This paper is organised as follows. We introduce our definitions and notations in section 2. In section 3 we analyse a choice technique based on the mF score. Two ranking techniques based on the mF score are then studied in section 4. A final section, discussing the results and mentioning open problems, concludes the paper.

2 Definitions and Notations

Throughout this paper X will denote a non empty finite set of "alternatives". A *fuzzy* (binary) *relation* T on X is a function from X$\times$X to [0, 1]. With ELECTRE III in mind, we shall interpret fuzzy relations as "large" preference relations, the valuation of (a, b) indicating the credibility of the proposition "a is at least as good as b". Thus, all fuzzy relations in this paper will be supposed to be reflexive (a fuzzy relation T on X is reflexive if T(a, a) = 1, for all $a \in X$). If $Y \subseteq X$ and T is a fuzzy relation on X, we denote by T/Y the restriction of T to Y, *i.e.* the fuzzy relation on Y such that for all $a, b \in Y$, T/Y(a, b) = T(a, b). A fuzzy relation T on X such that $T(a, b) \in \{0, 1\}$, for all $a, b \in X$, is said to be *crisp*. We often write a T b instead of T(a, b) = 1 and Not(a T b) instead of T(a, b) = 0 when T is a crisp relation. We denote by $\mathcal{F}_X$ (resp. $\mathcal{U}_X$) the set of all fuzzy (resp. crisp) reflexive relations on X.

Let T be a crisp relation on X. It is said to be *complete* if [a T b or b T a] and *transitive* if [a T b and b T c ⇒ a T c], for all a, b, c ∈ X. A *weak order* is a crisp, complete and transitive binary relation. Let T be a weak order on X.We denote by $U_k(X, T)$ the kth equivalence class of T, *i.e.* for k = 1, 2, 3, …, $U_k(X, T) = \{a \in X[T, k] : a \; T \; b$, for all $b \in X[T, k]\}$, where X[T, 1] = X and for k = 2, 3, ..., $X[T, k] = X[T, k\text{-}1] \backslash U_{k-1}(X, T)$. Observe that $U_1(X, T)$ is always non empty and that all non empty equivalence classes of a weak order are disjoint.

A *choice rule* C is a function associating with each finite set X and each fuzzy relation $R \in \mathcal{F}_X$ a choice set C(X, R) such that $C(X, R) \subseteq X$ and $C(X, R) \neq \emptyset$. A choice rule therefore allows to select a non empty choice set on the basis of any reflexive fuzzy relation defined on a finite set. Similarly, a *ranking rule* $\succcurlyeq$ is a function associating with each finite set X and each fuzzy relation $R \in \mathcal{F}_X$ a weak order $\succcurlyeq(X, R)$ on X. Such definitions are adapted to methods such as ELECTRE III which can lead to any reflexive fuzzy relation on a finite set.

As shown in [2], a simple way to define choice and ranking rules is to make use of a scoring function (for alternative ways of building such rules, we refer to [9]). A scoring function S is a function associating a real number S(a, R, X) with each finite set X, each $R \in \mathcal{F}_X$ and each $a \in X$. We shall interpret the number S(a, R, X) as a measure of the "attractiveness" of alternative a within the set X endowed with the fuzzy relation R. Given a scoring function, selecting the alternatives with the highest score (resp. rank ordering the alternatives according to their scores) defines a choice rule (resp. a ranking rule). Formally we define the choice rule C_S and the ranking rule $\succcurlyeq_S$ associated to the scoring function S, letting, for all finite set X, all a, b ∈ X and all $R \in \mathcal{F}_X$:

$C_S(X, R) = \{c \in X : S(c, R, X) \geq S(d, R, X)$ for all $d \in X\}$ and

$a \succcurlyeq_S(X, R) \; b \iff S(a, R, X) \geq S(b, R, X)$.

As argued in [1], an alternative way of defining a ranking rule also deserves interest. It consists in the (downward) iteration of a choice rule which leads to a weak order in the following way. The alternatives selected by the choice rule form the first equivalence class of the weak order. These alternatives are then removed from consideration. The alternatives selected in the reduced set form the second equivalence class and so on. Formally, the iteration of a choice rule C leads to a ranking rule $\succcurlyeq$ such that, for all finite set X and for all $R \in \mathcal{F}_X$, we have (T standing for $\succcurlyeq(X, R)$): $U_k(X, T) = C(X[T, k], R/X[T, k])$, for all integer k such that X[T, k] is non empty. Though the ranking rule directly based on scores is much simpler than the one defined by iterated choice, the latter deserves attention since it corresponds to a very intuitive behaviour for ranking objects: the objects ranked in first place are the "best" objects (according to a choice rule), the objects ranked in second place are the best objects between those remaining and so on. Associated with a scoring function S we have thus defined the choice rule C_S, the ranking rule $\succcurlyeq_S$ and the ranking rule $\succcurlyeq_{IS}$ corresponding to the iteration of C_S. It should be observed that $\succcurlyeq_S$ and $\succcurlyeq_{IS}$ are not identical in general.

In this paper, we shall be concerned with the "min in Favour" scoring function, *i.e.* the scoring function such that, for all finite set X, all $R \in \mathcal{F}_X$ and all $a \in X$,

$$mF(a, X, R) = \underset{b \in X \setminus \{a\}}{\text{Min}} R(a, b).$$

It indicates the credibility with which an alternative is at least as good as other alternatives. This scoring function defines the min in Favour choice rule C_{mF}, the min in Favour ranking rule $\succcurlyeq_{mF}$ and the Iterated min in Favour ranking rule $\succcurlyeq_{ImF}$. The following numerical example illustrates these three rules. Let X = {a, b, c, d} and let $R \in \mathcal{F}_X$ be defined by the following table (to be read from row to column):

R	a	b	c	d
a	1	0.9	0.4	1
b	0.5	1	0.3	0.3
c	0.7	0.6	1	0.4
d	0.2	0.8	0.5	1

The min in Favour Choice rule obviously gives $C_{mF}(X, R) = \{a, c\}$. Using $\succcurlyeq_{mF}$, we obtain the following weak order (using obvious abbreviated notations): (ac) > b > d. Using $\succcurlyeq_{ImF}$, we obtain: (ac) > d > b. This shows that $\succcurlyeq_{mF}$ and $\succcurlyeq_{ImF}$ are distinct rules in spite of the fact that $U_1(X, \succcurlyeq_{mF}(X, R)) = U_1(X, \succcurlyeq_{ImF}(X, R)) = C_{mF}(X, R)$.

Though many other scoring functions have been proposed in the literature (see [2]), a remarkable feature of the min in Favour scoring function is that it leads to choice and ranking rules that do not make use of the cardinal properties of the valuations R(a, b). Though this might be seen as too radical an interpretation of fuzziness, it is not clear from the construction technique of the numbers R(a, b) in many outranking methods and especially in ELECTRE III, whether or not they convey any information beyond the fact that $R(a, b) \geq R(c, d)$ means that the proposition "a is at least as good as b" is no less credible than the proposition "c is at least as good as d". Using only the ordinal information conveyed by the fuzzy relation may thus be seen as a principle of prudence.

3 The Min in Favour Choice Rule

The aim of this section is to provide an axiomatic characterisation of the min in Favour choice rule C_{mF}. Our first axiom is designed to capture the already-mentioned ordinal character of C_{mF}. We say that a choice rule C is *ordinal* if, for all finite set X, all $R \in \mathcal{F}_X$ and all strictly increasing and one-to-one transformation ϕ on [0, 1], $C(X, R) = C(X, \phi[R])$, where $\phi[R]$ is the element of $\mathcal{F}_X$ such that $\phi[R](a, b) = \phi(R(a, b))$ for all $a, b \in X$. It is not difficult to see that C_{mF} is indeed ordinal. There are many ordinal choice rules; the "Max in Favour" choice rule C_{MF} and the "Max Against" choice rule C_{MA} that respectively use the scores

$$MF(a, X, R) = \underset{b \in X \setminus \{a\}}{\text{Max}} R(a, b),\ \ MA(a, X, R) = -\underset{b \in X \setminus \{a\}}{\text{Max}} R(b, a),$$

are both ordinal.

Consider a crisp relation $R \in \mathcal{U}_X$. If the set $G(X, R) = \{a \in X: a R b \text{ for all } b \in X\}$, *i.e.*, the set of the greatest elements in X given R, is non empty, there exist alternatives in X that are unambiguously "at least as good as" all other alternatives. Thus, it seems that there is little point in selecting alternatives outside of G(X, R). This motivates the following axiom. We say that a choice rule C is *greatest faithful* if, for all finite set X and all $R \in \mathcal{F}_X$, $[R \in \mathcal{U}_X$ and $G(X, R) \neq \emptyset] \Rightarrow C(X, R) \subseteq G(X, R)$.

It is not difficult to see that C_{mF} is greatest faithful contrary to C_{MF} and C_{MA}. The conjunction of ordinality and greatest faithfulness does not characterize C_{mF} however. This is because greatest faithfulness imposes a constraint on the result of a choice rule when applied to crisp relations whereas ordinality imposes a constraint on the result of the choice rule for "ordinaly equivalent" fuzzy relations. Since no truly fuzzy relation (*i.e.* belonging to $\mathcal{F}_X \backslash \mathcal{U}_X$) can be ordinaly equivalent to a crisp relation, the conjunction of these two axioms imposes very few constraints on the Behavior of C when applied to truly fuzzy relations. Furthermore it should be observed that C_{mF} cannot be seen as the largest or the smallest (w.r.t inclusion) choice rule in the set of all ordinal and faithful choice rules: a choice rule discriminating among the elements of C_{mF} according to their MF score is ordinal and greatest faithful but smaller than C_{mF}; a choice rule coinciding with C_{mF} for crisp relations and selecting all alternatives otherwise is ordinal and greatest faithful but larger than C_{mF}. This calls for axioms that would relate the result of a choice rule when applied to crisp relations and truly fuzzy ones. Hence, we introduce the following continuity requirement which is obviously fulfilled by C_{mF}.

Consider a sequence of fuzzy relations ($R_i \in \mathcal{F}_X$, i = 1, 2, ...). We say that this sequence converges to $R \in \mathcal{F}_X$ if, for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there is an integer k such that, for all $j \geq k$, $|R_j(a, b) - R(a, b)| < \varepsilon$, for all $a, b \in X$. A choice rule C is said to be *continuous* if, for all finite set X, all $R \in \mathcal{F}_X$ and all sequences ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R, $[a \in C(X, R_i)$ for all R_i in the sequence$] \Rightarrow [a \in C(X, R)]$.

Considering a sequence of strictly increasing transformations converging (pointwise) to a step function, it is not difficult to see that ordinality and continuity imply that if $a \in C(X, R)$ then for any $\lambda \supseteq\in (0, 1]$, $a \in C(X, R_\lambda)$ where R_λ denotes the λ- cut of R, *i.e.* the crisp relation such that, for all $c, d \in X$, $c R_\lambda d \Leftrightarrow R(c, d) \geq \lambda$. The following result (first obtained in [5]) is based on this simple observation coupled with the fact that, since λ- cuts are crisp relations, the result of a choice rule with such relations may be constrained by greatest faithfulness.

Proposition 1. The min in Favour choice rule C_{mF} is the only ordinal, continuous and greatest faithful choice rule.

Proof. We already observed that C_{mF} is ordinal, continuous and greatest faithful. It remains to be shown that if a choice rule C is ordinal, continuous and greatest faithful then, for all finite set X, all R ∈ $\mathcal{F}_X$ and all a, b ∈ X:

mF(a, X, R) > mF(b, X, R) ⇒ b ∉ C(X, R) and (i)

mF(a, X, R) = mF(b, X, R) and b ∈ C(X, R) ⇒ a ∈ C(X, R). (ii)

In contradiction with (i), suppose that mF(a, X, R) > mF(b, X, R) and b ∈ C(X, R) for some ordinal, continuous and greatest faithful choice rule C. Let $\lambda \in$ (mF(b, X, R), mF(a, X, R)). Consider any sequence of strictly increasing and one-to-one transformations (ϕ_i, i = 1, 2, ...) on [0, 1] converging pointwise to the step function ϕ on [0, 1] such that $\phi(x) = 1$ iff $x \geq \lambda$ and $\phi(x) = 0$ otherwise. By construction, the sequence (ϕ_i[R], i = 1, 2, ...) converges to the λ- cut R_λ of R. Ordinality implies that b ∈ C(X, ϕ_i[R]) for all ϕ_i in the sequence and using continuity we obtain b ∈ C(X, ϕ[R]) = C(X, R_λ). The set G(X, R_λ) is non empty (since a ∈ G(X, R_λ)) and b ∉ G(X, R_λ). Using greatest faithfulness we obtain a contradiction. This proves (i).

In order to prove (ii), suppose that mF(a, X, R) = mF(b, X, R) = λ and b ∈ C(X, R), for some ordinal, continuous and greatest faithful choice rule C. Since b ∈ C(X, R), we know, using (i), that mF(b, X, R) ≥ mF(c, X, R) for all c ∈ X. Consider a sequence ($R_i \in \mathcal{F}_X$, i = 1, 2 ...) of fuzzy relations identical to R except that R_i(b, c) = Max(0 ; R(b, c) – 1/i) for all c ∈ X\{b} such that R(b, c) = λ and R_i(a, d) = Min(1 ; R(a, d) + 1/i) for all d ∈ X\{a} such that R(a, d) = λ. This sequence converges to R. For all R_i in the sequence, we have, by construction, mF(a, X, R_i) > mF(c, X, R_i) for all c ∈ X\{a}. From (i) we know that C(X, R_i) = {a} for all R_i in the sequence. Continuity implies a ∈ C(X, R) which proves (ii) and completes the proof. □

We conclude this section with some remarks.

a) As shown by the following examples, ordinality, continuity and greatest faithfulness are independent properties.

i- The Sum in Favour choice rule C_{SF} based on the following score:

$$SF(a, X, R) = \sum_{b \in X \setminus \{a\}} R(a, b)$$

is greatest faithful and continuous but not ordinal.

ii- C_{MF} is continuous and ordinal but not greatest faithful.

iii- Define C_L as:

C_L(X, R) = {a ∈ C_{mF}(X, R): MF(a, R) ≥ MF(b, R) for all b ∈ C_{mF}(X, R)},

i.e. as the choice rule discriminating among the elements selected with C_{mF} according to their MF score. It is easy to see that this choice rule is ordinal and greatest faithful but not continuous.

b) The proof of proposition 1 shows that, in presence of ordinality and greatest faithfulness, the full power of continuity is not needed to prove that C(X, R) ⊆ C_{mF}(X, R) for all finite set X and all R ∈ $\mathcal{F}_X$. Requiring continuity only for sequences of fuzzy relations converging to a crisp relation suffices. This leads to an alternative characterisation of C_{mF} as the largest choice rule among the ones that are ordinal, greatest faithful and weakly continuous in the above sense.

c) Retaining ordinality and continuity, an obvious modification of greatest faithfulness, requiring for crisp relations that the choice set should always be included in the set of "unbeaten" alternatives when this set is non empty, allows to characterize C_{MA}.

d) Our definition of continuity uses a distance between fuzzy relations that makes use of the cardinal properties of the numbers R(a, b). Though ordinality and continuity are not contradictory, as shown by proposition 1, coupling these two axioms is somewhat awkward since ordinality implies that the cardinal properties of the numbers R(a, b) should not be used. The following proposition, adapted from [14], shows that the conjunction of ordinality and continuity is equivalent to a "strong ordinality" requirement involving non-decreasing transformations (at the cost of a more complex proof, it is possible to consider only non-decreasing and continuous transformations). The use of strong ordinality avoids making explicit reference to a distance on the set of fuzzy relations.

Proposition 2. A choice rule C is ordinal and continuous if and only if it is strongly ordinal, *i.e.*, $C(X, R) \subseteq C(X, \phi[R])$, for all finite set X, all $R \in \mathcal{F}_X$ and all non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$.

Proof. [Ordinality and continuity $\Rightarrow$ strong ordinality]. Let X be a finite set and $R \in \mathcal{F}_X$. Consider any non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$. We can find a sequence of strictly increasing and one-to-one transformations (ϕ_i, i = 1, 2, ...) on [0, 1] that converges pointwise to ϕ and it can be supposed w.l.o.g. that ϕ_1 is the identity function on [0, 1]. By construction, the sequence ($\phi_i[R]$, i = 1, 2, ...) converges to $\phi[R]$. Let $a \in C(X, R)$. Since $\phi_1[R] = R$, ordinality implies that $a \in C(X, \phi_i[R])$ for all ϕ_i in the sequence. Using continuity, we obtain $a \in C(X, \phi[R])$, which shows that C is strongly ordinal.

[Strong ordinality $\Rightarrow$ ordinality and continuity]. Let ϕ be strictly increasing and one-to-one on [0, 1]. Thus ϕ^{-1} is also strictly increasing and one-to-one. Strong ordinality leads to $C(X, R) \subseteq C(X, \phi[R]) \subseteq C(X, \phi^{-1}[\phi[R]]) = C(X, R)$, which shows that C is ordinal. Let $R \in \mathcal{F}_X$ and consider a sequence of fuzzy relations ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to $R \in \mathcal{F}_X$ and such that $e \in C(X, R_i)$ for all R_i in the sequence. Let |X| = n. Observe that R can take at most n(n-1) distinct values on the set of the n(n-1) ordered pairs of distinct elements in X. Suppose that R takes exactly ℓ distinct values $k_1, k_2, \ldots, k_\ell$. Suppose w.l.o.g that $0 \le k_1 < k_2 < k_3 < \ldots < k_\ell \le 1$ and define ε as:

$$\varepsilon = \operatorname*{Min}_{j=2,\ldots,\ell} k_j - k_{j-1}.$$

Since ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converges to R, we can find an integer j such that $|R_j(a, b) - R(a, b)| < \varepsilon/2$, for all $a, b \in X$. Define a function ϕ from [0, 1] to [0, 1] such that, for all j =1, 2, , ℓ, $[|x-k_j| < \varepsilon/2] \Rightarrow \phi(x) = k_j]$ and $\phi(x) = x$ otherwise. It is clear that ϕ is non-decreasing and such that $\phi(1) = 1$ and $\phi(0) = 0$. By construction, we have $R = \phi[R_j]$. Since $e \in C(X, R_j)$, we obtain $e \in C(X, R)$ using strong ordinality, which completes the proof. □

4 The Min Ranking Rule and the Iterated Min Ranking Rule

Adapting the axioms introduced for choice rules to the case of ranking rules is straightforward. We say that a ranking rule $\succcurlyeq$ is:

– *ordinal* if, for all strictly increasing and one-to-one transformation ϕ on [0, 1], $\succcurlyeq(X, R) = \succcurlyeq(X, \phi[R])$,

– *continuous* if, for all sequences ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R and all $a, b \in X$, [a $\succcurlyeq(X, R_i)$ b for all R_i in the sequence] $\Rightarrow$ [a $\succcurlyeq(X, R)$ b].

– *greatest faithful* if $[R \in \mathcal{U}_X$ and $G(X, R) \neq \emptyset] \Rightarrow U_1[X, \succcurlyeq(X, R)] \subseteq G(X, R)$,

for all finite set X and all $R \in \mathcal{F}_X$.

It is not difficult to see that $\succcurlyeq_{mF}$ is ordinal, continuous and greatest faithful. Rephrasing the proof of proposition 1 immediately leads to the following result (adapted from [3]).

Proposition 3. The min in Favour ranking rule $\succcurlyeq_{mF}$ is the only ranking rule that is ordinal, continuous and greatest faithful.

Some remarks on this proposition are in order.

a) Contrary to the case of choice rules, greatest faithfulness is not a particularly intuitive requirement for ranking rules (some alternative axioms may be found in [3]). A much more intuitive axiom would consist in imposing that $\succcurlyeq(X, T) = T$ for all weak orders T on X. It is easy to see that $\succcurlyeq_{mF}$ is not faithful in this sense since, when T is crisp, $\succcurlyeq_{mF}(X, T)$ consists of at most two equivalence classes. When T is a weak order, it is however true that $U_1(X, \succcurlyeq(X, T)) = U_1(X, T)$. In presence of ordinality and continuity, this last condition is not sufficient to characterize C_{mF} as shown by the following example. Define the scoring function:

$$S(a, X, R) = -\max_{\{b \in X : Not(aEb)\}} R(b, a),$$

where E is the equivalence relation on X defined by:

$a E b \Leftrightarrow [R(a, b) = R(b, a)$ and $R(a, c) = R(b, c), R(c, a) = R(c, b)$ for all $c \in X \backslash \{a, b\}]$.

The ranking rule based on this score is obviously ordinal. It is not difficult to show that it is continuous and that, when T is a weak order, $U_1(X, \succcurlyeq(X, T)) = U_1(X, T)$.

b) Examples similar to the ones used in section 3 show that the axioms used in proposition 3 are independent.

c) As was the case with choice rules, the conjunction of ordinality and continuity in proposition 3 is not entirely satisfactory. Similarly to what has been done in proposition 2, it is possible to replace ordinality and continuity by the following strong ordinality requirement:

$a \succcurlyeq(X, R)\ b \Rightarrow a \succcurlyeq(X, \phi[R])\ b$,

for all finite set X, all R ∈ $\mathcal{F}_X$, all a, b ∈ X and all non-decreasing transformation ϕ on [0, 1] such that $\phi(0) = 0$ and $\phi(1) = 1$,
which is equivalent to the conjunction of ordinality and continuity.
Alternative ways out of the ordinality-continuity puzzle are described in the next remark

d) The alternative characterisation of $\succcurlyeq_{mF}$ presented in [13] – and anticipated in [12] – uses neither ordinality nor continuity. We briefly recall here the essential elements of this result. Consider two fuzzy relations R and R′ on a finite set X and let a, b be distinct elements of X. We say that R and R′ are related by translation on {a, b} if R′ is identical to R except that, for some $\varepsilon \in [-1, 1]$, $R(a, c) = R'(a, c) + \varepsilon$, for all $c \in X\setminus\{a\}$and $R(b, d) = R'(b, d) + \varepsilon$, for all $d \in X\setminus\{b\}$.

A ranking rule $\succcurlyeq$ is said to be *translation invariant* if for all finite set X, all distinct a, b ∈ X and all R, R′∈ $\mathcal{F}_X$:
[R and R′ are related by translation on {a, b}] ⇒ [a $\succcurlyeq$(X, R) b ⇔ a $\succcurlyeq$(X, R′) b].

A ranking rule $\succcurlyeq$ is said to be:
– *weakly reversible* if a $\succcurlyeq$(X, R) b ⇒ [for all $c \in X\setminus\{a\}$, there is a relation $R_c \in \mathcal{F}_X$ identical to R except that $R_c(a, c) \leq R(a, c)$ and such that b $\succcurlyeq$(X, R_c) a],
– *strictly reversible* if a $\succcurlyeq$(X, R) b ⇒ [for all $c \in X\setminus\{a\}$ such that $R(b, c) \neq 0$, there is a relation $R_c \in \mathcal{F}_X$ identical to R except that $R_c(a, c) \leq R(a, c)$ and such that b $\succ$(X, R_c) a],
for all finite set X, all R ∈ $\mathcal{F}_X$ and all a, b ∈ X.

Translation invariance means that adding a constant to all valuations leaving a and b does not alter their respective comparison. Weak reversibility implies that the comparison between any two alternatives may be reversed by sufficiently decreasing any of the valuations of the arcs leaving the best ranked alternative. Strict reversibility asserts this reversal may be strict as soon as there is no boundary problem. It is not difficult to show that these three conditions are independent and are satisfied by $\succcurlyeq_{mF}$. The proof that it is the only ranking rule satisfying these three conditions is easy once it has been observed that:
– if $\succcurlyeq$ is weakly reversible then [$S_{mF}(a, X, R) = 0$] ⇒ [b $\succcurlyeq$(X, R) a, for all b ∈ X] and
– if $\succcurlyeq$ is strictly reversible then [$S_{mF}(b, X, R) > S_{mF}(a, X, R) = 0$] ⇒ [b $\succ$(X, R) a],
where $\succ$(X, R) denotes the asymmetric part of $\succcurlyeq$(X, R).
A thorough comparison between this characterisation and the one presented in proposition 3 can be found in [13].

Using the above-mentioned consequences of weak and strict reversibility, another – new – characterisation of $\succcurlyeq_{mF}$ can easily be derived combining strong ordinality and the two reversibility conditions. We have:

Proposition 4. The min in Favour ranking rule $\succcurlyeq_{mF}$ is the only ranking rule that is strongly ordinal, weakly reversible and strictly reversible.

Proof. We have already noted that $\succcurlyeq_{mF}$ is strongly ordinal, weakly reversible and strictly reversible. The proof will be complete showing that if a ranking rule $\succcurlyeq$ satisfies these conditions then, denoting by $\sim(X, R)$ the symmetric part of $\succcurlyeq(X, R)$:

$[mF(a, X, R) > mF(b, X, R) \Rightarrow a \succ(X, R)\ b]$ and

$[mF(a, X, R) = mF(b, X, R) \Rightarrow a \sim(X, R)\ b]$,

for all finite set X, all $R \in \mathcal{F}_X$ and all $a, b \in X$.

Suppose that $mF(a, X, R) > mF(b, X, R)$ and consider a non-decreasing transformation ϕ on [0, 1] such that $\phi(x) = 0$ if $x < mF(a, X, R)$ and $\phi(x) = x$ otherwise. Since $mF(a, X, \phi[R]) > mF(b, X, \phi[R]) = 0$, strict reversibility implies that $a \succ(X, \phi[R])\ b$ and $b \succcurlyeq(X, R)\ a$ would contradict strong ordinality. The proof that $mF(a, X, R) = mF(b, X, R) \Rightarrow a \sim(X, R)\ b$ is similar using weak reversibility and strong ordinality. □

Let us now turn to the study of the Iterated min in Favour ranking rule $\succcurlyeq_{ImF}$ based on the (downward) iteration of C_{mF}. This ranking rule is clearly ordinal. The iteration process may however create discontinuities as shown by the following example. Let X = {a, b, c, d} and consider the family of relations $R_\varepsilon \in \mathcal{F}_X$ defined by the following table:

R_ε	a	b	c	d
a	1	0.7	0.5	0
b	0.6	1	0.5	0
c	0.5	0.5	1	0.4 - ε
d	0.4	0.4	0.4	1

For all $\varepsilon \in (0, 0.4)$, $\succcurlyeq_{ImF}$ leads to the weak order d > (abc). When ε reaches 0, we obtain (cd) > a > b, which violates continuity. Since $U_1(\succcurlyeq_{ImF}(X, R)) = C_{mF}(X, R)$, continuity cannot be violated for the top-ranked elements however; hence the following requirement which is obviously satisfied by $\succcurlyeq_{ImF}$. A ranking rule $\succcurlyeq$ is *top continuous* if, for all finite set X, all $R \in \mathcal{F}_X$ and all sequences ($R_i \in \mathcal{F}_X$, i = 1, 2, ...) converging to R, $[a \in U_1(X, \succcurlyeq(X, R_i))$ for all R_i in the sequence$] \Rightarrow [a \in U_1(X, \succcurlyeq(X, R))]$.

Because $U_1(\succcurlyeq_{ImF}(X, R)) = U_1(\succcurlyeq_{mF}(X, R))$, $\succcurlyeq_{ImF}$ is greatest faithful. Moreover, it should be noticed that, contrary to the situation with $\succcurlyeq_{mF}$, it is also true for $\succcurlyeq_{ImF}$ that, when T is a weak order, $\succcurlyeq_{ImF}(X, T) = T$.

Since continuity implies top continuity and $\succcurlyeq_{ImF}$ is distinct from $\succcurlyeq_{mF}$, the conjunction of ordinality, top continuity and greatest faithfulness does not characterize $\succcurlyeq_{ImF}$. Replacing greatest faithfulness by the requirement of being faithful on weak orders does not solve the problem as shown by the example in remark a) above. This calls for an axiom that would capture the iterative character of $\succcurlyeq_{ImF}$. In order to do so, we use a condition proposed in [1], lacking any less transparent way to do so. A ranking rule $\succcurlyeq$ is said to be *top decomposable* if, for all finite set X and all $R \in \mathcal{F}_X$, $\succcurlyeq(X, R)/Y = \succcurlyeq(Y, R/Y)$, where Y stands for $X \setminus U_1(X, \succcurlyeq(X, R))$.

Though this condition may look complex it admits a simple interpretation. The alternatives that are ranked first in $\succcurlyeq(X, R)$ are those of $U_1(X, \succcurlyeq(X, R))$. Top decomposability says the ranking of the remaining alternatives is unaltered if the alternatives $U_1(X, \succcurlyeq(X, R))$ are taken out of X and the same ranking rule is applied to $X \backslash U_1(X, \succcurlyeq(X, R))$, R being restricted to this set. Top decomposability may be an important property in practice; if, for some reason, the best alternatives in a certain set become unavailable, it is not necessary to apply the ranking rule to the remaining alternatives: they will be ranked exactly as in the previous ranking. The technical interest of top decomposability is obvious. Let $\succcurlyeq$ be a ranking rule. To this ranking rule we associate a choice rule $C_{\succcurlyeq}$ such that, for all finite set X and all $R \in \mathcal{F}_X$, $C_{\succcurlyeq}(X, R) = U_1(X, \succcurlyeq_X(R))$. By definition, $\succcurlyeq$ is top decomposable if and only if $\succcurlyeq$ is identical to the ranking rule defined by the iteration of $C_{\succcurlyeq}$. It is clear that $\succcurlyeq$ is greatest faithful if and only if $C_{\succcurlyeq}$ is greatest faithful. If $\succcurlyeq$ is ordinal (resp. top continuous) then $C_{\succcurlyeq}$ is ordinal (resp. continuous). Given top decomposability and proposition 1, this proves:

Proposition 5. The Iterated min in Favour ranking rule $\succcurlyeq_{ImF}$ is the only ranking rule that is ordinal, top continuous, greatest faithful and top decomposable.

Since ordinality, continuity and greatest faithfulness are independent properties for choice rules and $\succcurlyeq_{mF}$ is ordinal, top continuous and greatest faithful but not top decomposable, it is easily shown that the axioms used in proposition 5 are independent. We are not presently aware of any axioms that would allow to dispense with the, strong, top decomposability requirement.

5 Discussion and Open Problems

The various results presented in section 3 and 4 raise many questions and leave open many problems. We summarise here those that appear to be the most important ones.

This paper has been concerned with the study of choice and ranking rules operating on any reflexive fuzzy relation defined on a finite set. Though we mentioned that this is an appropriate setting for ELECTRE III, this is not true for other methods such as PROMETHEE or the ones proposed in [1]. As shown in [6], though the fuzzy relations built with these methods do not possess remarkable properties, they may not lead to any fuzzy relation. Among aggregation methods leading to fuzzy relations, ELECTRE III which may produce any of them, may be considered as an exception. Since our axioms make explicit use of the richness of the set $\mathcal{F}_X$, our results cannot be used to characterize exploitation techniques to be coupled with methods that always produce fuzzy relations belonging to a proper subset of $\mathcal{F}_X$.

In order to overcome this problem, one may try to characterize the subset of $\mathcal{F}_X$ that may be obtained with a given aggregation technique and then analyse choice and ranking rules using axioms adapted to this subset (an example of such an analysis is found in [6]). Alternatively, and perhaps more fruitfully, one may try to characterize both steps of an Outranking Method, *i.e.*, a technique starting with alternatives evaluated on several criteria and leading to a choice or a ranking, therefore ignoring the intermediate step of the construction of a fuzzy relation. Numerous examples of such characterisations can be found in the literature in Social Choice Theory (see, *e.g.*, [18]). Much work remains to be done in this direction in the area of MCDM.

We mentioned that we interpreted our fuzzy relations as "large" preference relations in accordance with their construction in ELECTRE III. Though this allows to motivate our greatest faithfulness axioms, this also raises the problem of defining the symmetric and asymmetric part of the relation, *i.e.* the indifference and the strict preference relation associated with the large preference relation. This problem is known to be difficult for fuzzy relations (on this point see [10] or [11]). It was not dealt with explicitly here. We were thus unable, for instance, to consider choice rules that would discriminate among alternatives that are "at least as good" as all other alternatives on the basis of the way they compare in terms of "indifference" or "strict preference" to these alternatives.

Let us finally mention that many scores apart from the mF score would deserve attention. If ranking and choice rules based on scores involving sums have already been well studied (see [4], [7]), the min Difference score:

$$mD(a, X, R) = \min_{b \in X \setminus \{a\}} (R(a, b) - R(b, a))$$

which seems particularly interesting (see [2]) remains to be fully explored. Similarly, a general characterisation of rules using scores based on ranks is yet to be obtained.

References

[1] Arrow, K.J. and Raynaud, H.: *Social choice and multicriterion decision making*, MIT Press, Cambridge, 1986.

[2] Barrett, C.R., Pattanaik, P.K. and Salles, M: On choosing rationally when preferences are fuzzy. *Fuzzy Sets and Systems* 34 (1990) 197-212.

[3] Bouyssou, D: "A note on the ‘min in favor’ ranking method for valued preference relations". In M. Cerny, D. Glükaufová and D. Loula (eds): *Multicriteria Decision Making. Methods - Algorithms - Applications*, Czechoslovak Academy of Sciences, Prague, 1992, pp. 16-25.

[4] Bouyssou, D.: A note on the sum of differences choice function for fuzzy preference relations. *Fuzzy Sets and Systems* 47 (1992) 197-202.

[5] Bouyssou, D.: "A note on the 'min in favor' choice procedure for fuzzy preference relations". In: P.M. Pardalos, Y. Siskos and C. Zopounidis (eds.): *Advances in Multicriteria Analysis*, Kluwer, Dordrecht, 1995, pp. 9-16.

[6] Bouyssou, D.: Outranking relations: do they have special properties ?. *Journal of Multiple*

Criteria Decision Analysis 5 (1996) 99-111.
[7] Bouyssou, D. and Perny, P: Ranking methods for valued preference relations: a characterization of a method based on entering and leaving flows. *European Journal of Operational Research* 61 (1992) 186-194.
[8] Brans J.P., Mareschal, B. and Vincke, Ph.: "PROMETHEE: a new family of outranking methods in multicriteria analysis". In: J.P. Brans (ed.): *OR '84*, North Holland, Amsterdam, 1984, pp. 408-421.
[9] Kitainik, L.: *Fuzzy decision procedures with binary relations – Towards a unified theory*, Kluwer, Dordrecht, 1993.
[10] Ovchinnikov, S.V. and Roubens, M.: On fuzzy strict preference, indifference, and incomparability relations. *Fuzzy Sets and Systems* 49 (1992) 15-20.
[11] Perny, P. and Roy, B.: The use of fuzzy outranking relations in preference modelling. *Fuzzy Sets and Systems* 49 (1992) 33-53.
[12] Pirlot, M.: Ranking procedures for valued preference relations: a characterization of the 'min' procedure. *Cahiers du CERO* 34 (1992) 233-241.
[13] Pirlot, M.: A characterization of "min" as a procedure for exploiting valued preference relations and related results. *Journal of Multiple Criteria Decision Analysis* 4 (1995) 37-56.
[14] Pirlot, M.: "About ordinal procedures for exploiting valued preference relations", Communication at ORBEL 9, Brussels, 19-20 January 1995.
[15] Roy, B.: ELECTRE III : un algorithme de classement fondé sur une représentation floue des préférences en présence de critères multiples. *Cahiers du CERO* 20 (1978) 3-24.
[16] Roy, B. and Bouyssou, D.: *Aide multicritère à la décision : méthodes et cas*, Economica, Paris, 1993.
[17] Vincke, Ph.: *Multicriteria decision-aid*, John Wiley & Sons, Chichester, 1992.
[18] Young, H.P.: An axiomatization of Borda's rule. *Journal of Economic Theory* 9 (1974) 43-52.

Acknowledgments. This work was accomplished while Denis Bouyssou was visiting the Université Libre de Bruxelles, Service de Mathématiques de la Gestion, thanks to a "Research in Brussels" action grant from the Brussels-Capital Region. We wish to thank Thierry Marchant and Philippe Vincke for their helpful comments on an earlier draft of this text. The usual caveat applies.