Chapter 3
Modelling Preferences

Denis Bouyssou, Thierry Marchant, Marc Pirlot, Alexis Tsoukiàs, and Philippe Vincke

Abstract This chapter deals with a crucial step in the decision aiding process: the aggregation of the alternatives' performances on each criterion in order to faithfully model the overall preference of the decision maker. The approach we follow is that of conjoint measurement, which aims at determining under which conditions a preference can be represented in a particular aggregation model. This approach is first illustrated with the classical additive value function model. Then, we describe two broad families of preference models, which constitute a framework encompassing many aggregation models used in practice. The aggregation rules that fit with the second family of models rely on the aggregation of preference differences. Among this family we find, in particular, models for the outranking relations (concordance relations with vetoes) that are used in several case studies in this book.

D. Bouyssou • A. Tsoukiàs
CNRS-LAMSADE UMR 7243, Université Paris Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
e-mail: bouyssou@lamsade.dauphine.fr; tsoukias@lamsade.dauphine.fr

T. Marchant (✉)
Department of Data analysis, Ghent University, H. Dunantlaan 1, 9000 Gent, Belgium
e-mail: Thierry.Marchant@UGent.be

M. Pirlot
Université de Mons, Faculté Polytechnique, Mons, Belgium
e-mail: marc.pirlot@umons.ac.be

P. Vincke
Université libre de Bruxelles, Ecole Polytechnique, Campus de la Plaine, CP210/01, boulevard du Triomphe, 1050 Bruxelles, Belgium
e-mail: Philippe.Vincke@ulb.ac.be

R. Bisdorff et al. (eds.), *Evaluation and Decision Models with Multiple Criteria*, International Handbooks on Information Systems,
DOI 10.1007/978-3-662-46816-6_3

3.1 Introduction

In this chapter, we address a very peculiar point in a particular step of the decision aiding process as it has been described in Chap. 2. We try to deal in a general way with the operation of aggregating descriptions on various dimensions into a global object, called *preference*, that synthesises all relevant features of the alternatives and incorporates the preference of the client in a given problem situation $\mathcal{P}$. To this end, we follow the tradition of *conjoint measurement theory*, first developed in Economics by Debreu (1960) and in Psychology by Luce and Tukey (1964), and then adopted in decision analysis by Edwards (1971) and Raiffa (1969). It provides us with families of models that decompose the preference into elements related to the description of the alternatives along the various dimensions. Besides these families of models, conjoint measurement theory provides us with very powerful tools: the axiomatic characterizations of these models. The characterisations take the following form: if a preference satisfies some conditions (called axioms), then it admits a description within a particular model. Characterising a model amounts to finding the properties of all the preferences that fit into the model.

Knowing the axioms characterizing a model can help the analyst to determine whether that model is adequate in the given problem situation. He can for instance ask the client how he feels the preference behaves in the situations evoked in the axioms. Depending on the answers, he can then decide to work further with that model or to reject it and examine another one. A deep understanding of a model can also help the analyst to elicit the parameters involved in that model.

Another possible framework for the analysis of aggregation techniques is *social choice theory* (see Chapter 5 in Bouyssou et al. 2006). In spite of the interest of this framework, we will not present it here because of size constraints.

Before analysing very general and abstract families of models, we will start with a somewhat easier section (Sect. 3.2), focussing on a specific and well-known model: the additive value function model. This will give us the opportunity to introduce some notation, to define many concepts and to discuss many aspects of conjoint measurement theory.

In Sects. 3.3 and 3.4, we will mainly analyse two types of models. In the first one, the comparison of two alternatives results from the comparison of the description of each of them on the various dimensions. In the second type of models, for each pair of alternatives and each dimension, the differences of preference between these alternative on that dimension is assessed and the model makes the balance between all these differences in order to determine which of the two alternatives is the preferred one. Each type of model has its own logic and suggests a corresponding strategy of elicitation. Section 3.5 is devoted to concordance relations.

3.2 The Additive Value Function Model

Suppose that, within a certain problem formulation, we have started to build an evaluation model: we have determined a set of alternatives A and n dimensions that can describe all the aspects relevant to the decision problem at hand. We shall assume that the set of functions g_i used to describe the alternatives on each dimension is exhaustive, so that any alternative a can be identified with the vector $(g_1(a), \dots, g_i(a), \dots, g_n(a))$. We may work with the set of vectors representing the alternatives instead of the alternatives themselves. These vectors form a subset $\{(g_1(a), \dots, g_n(a)),\ a \in A\}$ of the Cartesian product $X = X_1 \times X_2 \times \dots X_i \times \dots X_n$ of the various scales. We assume further that each vector of X corresponds to an alternative and that the client's preferences, denoted by $\succsim$, is a relation on the whole[1] set X. Hence, any alternative will be identified with a vector $x = (x_1, \dots, x_n)$ of X where $x_1, \dots, x_i, \dots, x_n$ denote the evaluations of the alternative x on the n criteria. And any vector $x = (x_1, \dots, x_n)$ of X represents an alternative.

Conjoint measurement theory studies the links that may exist—depending on the properties of $\succsim$—between any pair (x, y) of vectors of X and the fact that $x \succsim y$ or not.

In the most popular model of this theory, it can be determined that x is preferred to y by comparing the values that a function u, defined on X, assigns to x and y; u is called a *multi-attribute value function* (MAV function). A very particular case for u, but also by far the most frequent in practice, is when u decomposes into a sum of n functions u_i each of a single variable, i.e. $u(x) = u(x_1, \dots, x_n) = \sum_{i=1}^{n} u_i(x_i)$. The main model of conjoint measurement—called *additive value function model*—thus deals with preferences on X such that for all $x, y \in X$:

$$x \succsim y \Leftrightarrow u(x) = \sum_{i=1}^{n} u_i(x_i) \geq u(y) = \sum_{i=1}^{n} u_i(y_i), \tag{3.1}$$

where u_i is a function from X_i into $\mathbb{R}$ for all i. In this representation, the relative importance of the criteria is reflected in the magnitude of the functions u_i.

There is an alternative way of representing the same model, which makes more explicit the importance of the criteria:

$$x \succsim y \Leftrightarrow v(x) = \sum_{i=1}^{n} k_i v_i(x_i) \geq v(y) = \sum_{i=1}^{n} k_i v_i(y_i), \tag{3.2}$$

[1] This postulates the extension to all the Cartesian product X of the preference relation that is perceived on $\overline{g}(A) = \{(g_1(a), \dots, g_n(a)),\ a \in A\}$. In practice, such an extension could force the client to compare alternatives that appear artificial or unrealistic to him. Despite possible unwanted practical consequences and provided that the range X_i is not unrealistic, we consider that the extension of $\succsim$ to X is not an outrageous assumption.

in which k_i are nonnegative "weighting factors" summing up to 1. Representations (3.1) and (3.2) are perfectly equivalent; indeed, it suffices to set $u_i = kv_i$ to find that any relation representable in (3.1) is also representable in (3.2). Depending on the context, one or another formulation of the model may offer an advantage.

3.2.1 Additive Value Function and Conjoint Measurement

The above model, in either of its forms (3.1) or (3.2), will be referred to as the *additive value function model*; u is called an additive MAV function. Conjoint measurement theory is concerned with establishing conditions on $\succsim$ under which a representation according to model (3.1) (or (3.2)) exists. Conditions of uniqueness of the representation are also looked for.

Why is this interesting? Clearly, if we have reasons to believe that a preference might obey model (3.1), we can try to determine the preference—which is usually not known explicitly—by constructing the functions u_i; alternatively, for eliciting model (3.2), we should construct the functions v_i and estimate the coefficients k_i. Each model suggests a strategy (or several ones) for eliciting preferences that are representable in the model. Of course, not all preferences satisfy model (3.1); we shall not specify here the necessary and sufficient conditions but just mention the following two important and obvious requirements on the preference:

- $\succsim$ must be a weak order, i.e. a transitive and complete preference, in other words a complete ranking, possibly with ties. This is clearly a necessary requirement since model (3.1) exactly says that the order $\succsim$ on X is obtained by transporting the natural order on $\mathbb{R}$ onto X by means of the function u.
- $\succsim$ must satisfy (strong) preference independence. The decomposition of u into a sum of functions each of a single variable reveals that if $x \succsim y$ while x and y have received the same assessment on dimension i, then, if we change that common level into another level still keeping it common, the transformed x and y will compare in the same way as before. More formally, let x and y be such that $x_i = y_i = a_i$; let x' be equal to x except that $x'_i = b_i \neq x_i$ and let y' be equal to y except that $y'_i = b_i \neq y_i$, then:

$$x \succsim y \Leftrightarrow x' \succsim y'$$

since

$$u_i(a_i) + \sum_{j \neq i} u_j(x_j) \geq u_i(a_i) + \sum_{j \neq i} u_j(y_j) \quad \Leftrightarrow$$

$$u_i(b_i) + \sum_{j \neq i} u_j(x_j) \geq u_i(b_i) + \sum_{j \neq i} u_j(y_j)$$

The independence property of the preference has far-reaching consequences; it allows in particular for ceteris paribus reasoning, i.e. comparing alternatives the evaluations of which differ only on a few attributes without specifying the common level of their evaluations on the remaining attributes; the independence property guarantees that the result of such a comparison is not altered when changing the common level on the attributes that do not discriminate the alternatives. We shall further discuss this property below in Sect. 3.2.5.

The two conditions stated above are not sufficient for ensuring that $\succsim$ satisfies (3.1). In case the evaluation space X is infinite, various sets of sufficient conditions have been provided in the literature; they are often categorised in two branches, the algebraic and the topological theories, respectively (see e.g. Fishburn, 1970, ch. 5). We give a schematic outline of the algebraic approach in Sect. 3.2.6. In case the set of possible levels X_i on each dimension is finite, the situation is rather unpleasant: the sufficient conditions (Fishburn, 1970, ch. 4) are quite complex and not very insightful. We therefore do not present them.

3.2.2 *Uniqueness Issues*

If the model is to be used in order to elicit preferences through the construction of functions u_i, it may also be important to know whether these u_i are uniquely determined. If they are not and provided we find a way of eliciting them independently of one another, at the end, it will remain to make sure that the obtained versions of the u_i's are compatible, i.e. that they can be used directly in (3.1).

Actually, the u_i's are not unique. For a preference $\succsim$ that fits in the additive value model, there is a family of value functions u that both

- decompose additively as $u(x) = \sum_{i=1}^{n} u_i(x_i)$
- and represent the preference i.e. satisfy $x \succsim y \Leftrightarrow u(x) \geq u(y)$.

Suppose indeed that we start with a particular representation of $\succsim$, $u(x) = \sum_{i=1}^{n} u_i(x_i)$ and we transform u_i into u'_i by a *positive affine transformation*

$$u'_i = \alpha u_i + \beta_i, \tag{3.3}$$

with $\alpha > 0$ and β_i a real number (that may vary with i). By using u'_i instead of u_i in the additive model, we get

$$u'(x) = \sum_{i=1}^{n} u'_i(x_i) = \alpha \sum_{i=1}^{n} u_i(x_i) + \sum_{i=1}^{n} \beta_i = \alpha u(x) + \sum_{i=1}^{n} \beta_i .$$

Clearly, u' is an alternative representation of the preference $\succsim$ since $x \succsim y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow u'(x) \geq u'(y)$. So, the u_i's to be used in an additive representation are at best determined up to a positive affine transformation.

In case X is infinite, it is possible to prove that the u_i's are actually unique up to a positive affine transformation (with the same α and possibly different β). This requires the use of a non-necessary condition ensuring that each set X_i is sufficiently rich (see Sect. 3.2.6 for more details).

Assuming that the u_i's are determined up to a positive affine transformation, we shall briefly explain in Sect. 3.2.4 how we can take advantage of this to construct an additive representation of the preference.

3.2.3 Marginal Preferences Within the Additive Value Model

Under the hypothesis that $\succsim$ fits with model (3.1), the model suggests that functions u_i could be elicited. Going one step further, it is readily seen that $u_i(x_i)$ must be compatible with the marginal preference relation $\succsim_i$ defined as:

$$x_i \succsim_i y_i \Leftrightarrow \forall\, a_{-i} \in X_i, (x_i, a_{-i}) \succsim (y_i, a_{-i}), \tag{3.4}$$

where (x_i, a_{-i}) represents an alternative that has x_i as ith component while the other components are those of vector a. So, (x_i, a_{-i}) and (y_i, a_{-i}) are two alternatives that may only differ on attribute i; they have common evaluations a_j on all attributes j but for $j = i$. If the client states $(x_i, a_{-i}) \succsim (y_i, a_{-i})$, this means, in terms of the marginal preference relation $\succsim_i$, that $x_i \succsim_i y_i$ and it translates in model (3.1) into:

$$u_i(x_i) + \sum_{j \neq i} u_j(a_j) \geq u_i(y_i) + \sum_{j \neq i} u_j(a_j),$$

from which we deduce $u_i(x_i) \geq u_i(y_i)$. Thus, for all levels x_i, y_i in X_i, we have $x_i \succsim_i y_i$ iff $u_i(x_i) \geq u_i(y_i)$. Therefore, in model (3.1), the function u_i interprets as a numerical representation of the marginal preference $\succsim_i$, which is a weak order.

The fact that the marginal preference is a weak order has strong links with the independence property of the preference $\succsim$ (see Sect. 3.3.5). There remains however a difficulty; the u_i functions that we need for using in the additive representation of the preference are not just *any* numerical representation of the marginal preference relations $\succsim_i$. Among the whole set of possible representations of the weak order $\succsim_i$, we have to select the right one (determined up to a positive affine transformation), the one that is needed for a representation of the global preference in the additive model.

Example 3.1 (Buying a Sports Car) Let us consider an example extensively discussed in chapter 6 of Bouyssou et al. (2000). We recall briefly the context. A student, Thierry, who is also passionate about sports cars but earns little money, assesses fourteen cars among which he considers to buy one, on the five dimensions that are of importance to him, namely cost, acceleration, pick up, brakes and

Table 3.1 Ranges of the five dimensions in the "Buying a sports car example"

Attribute	i	X_i	Unit	To be
Cost	1	[13; 21]	1,000 €	Minimised
Acceleration	2	[28; 31]	Second	Minimised
Pick up	3	[34; 42]	Second	Minimised
Brakes	4	[1; 3]	Qualitative	Maximised
Roadholding	5	[1; 4]	Qualitative	Maximised

roadholding. Assume that his preference fits with the additive value model (3.1) and let us help Thierry to build a value function u that represents his preference according with the additive model.

We first settle the ranges X_i in which the attributes will reasonably vary (in view of the evaluations of the fourteen selected cars). These ranges are shown in Table 3.1. The evaluations on the first three attributes are expressed in "physical" units (thousands of €, and, twice, seconds, respectively); the latter two belong to a qualitative scale. On the first three attributes scales, the less is the better, while on the latter two, the more is the better. What is the relationship between the evaluations and the value function u? There are two main features that we want to emphasise:

- the information contained in the evaluations is transferred to the value function through the marginal preferences;
- the marginal preferences—which are weak orders in the additive model (3.1)—cannot be identified with the natural ordering of the evaluations although these weak orders are not unrelated.

Take for example the cost attribute. Clearly, a car, say x, that costs $15,000$ € is not preferred over a car y that costs $14,000$ € if both cars are tied on all other dimensions. And the conclusion will be the same when comparing the former car with any other one that costs less and has the same evaluations on all other attributes. More formally, the car x can be described by the vector $(15, a_2, a_3, a_4, a_5)$ and y by $(14, a_2, a_3, a_4, a_5)$; the first dimension of these vectors represent the cost (in thousands of €) and a_i, for $i = 2, \ldots, 5$, designates any level on the other attributes. The car y is certainly at least as preferred as x ($y \succsim x$) since y is cheaper than x and all other evaluations are identical for both cars. It is a typical case in which "ceteris paribus" reasoning applies; the property of the preference we use here is *weak preference independence* (see Definition 3.1, p. 45); it is implied by strong preference independence which is a necessary condition for a preference being represented in the additive value model (3.1).

The fact that car y is preferred over x, independently of the value of a_j, can be translated into a statement involving the marginal preference $\succsim_1$ on the Cost attribute, namely $14 \succsim_1 15$. For all pairs of costs x_1, y_1 in the range 13; 21, we would similarly have $y_1 \succsim_1 x_1$ as soon as $x_1 \geq y_1$. But $x_1 > y_1$ does not necessarily implies $y_1 \succsim_1 x_1$ because a small difference between x_1 and y_1 could be considered as negligible with respect to the imprecision in the evaluation of the costs.

3.2.4 Leaning on the Additive Value Model for Eliciting Preferences

The additive value model suggests a general strategy for the elicitation of a preference that fits with the model. We assume here that the conditions of uniqueness of the additive representation are fulfilled (see Sect. 3.2.2). The strategy consists in eliciting the functions u_i, relying upon the fact that the u_i's are numerical representations of the marginal preferences. The main problem is to find among the many representations of the marginal preferences, the essentially unique ones that can be summed up and yield an additive representation u of the preference. This can be done in many different ways, which have been well-studied (see, e.g., Fishburn, 1967; Keeney and Raiffa, 1976; von Winterfeldt and Edwards, 1986). For the reader's convenience, we briefly illustrate the method of *standard sequences* on the example of ranking sports cars evoked in the previous section; we refer the reader to Bouyssou et al. (2000, ch. 6) for more detail and for the illustration of other elicitation methods applied to the same example.

We start with considering two hypothetical cars that differ only on cost and acceleration attributes, their performance levels on the other dimensions being tied. We assume that the two cars differ in cost by a noticeable amount, say for instance $1,000$€; we locate an interval of cost of that amplitude, for example, in the middle of the cost range, say $[16,500; 17,500]$€. Then we fix a value of the acceleration, also in the middle of the acceleration range, say, 29.5 s in the middle of $[28; 31]$. We ask the client to consider a car costing $16,500$€ and accelerating in 29.5 s, the evaluations on the other dimensions being fixed at an arbitrary (say mid-range) value. We ask the client to assess a value x_2 of the acceleration such that he would be indifferent between the cars $(16.5; 29.5)$ and the car $(17.5; x_2)$. This question amounts to determining which improvement on the performance on the acceleration attribute (starting from a value of 29.5 s) would be worth a cost increase of $1,000$€ (starting from $16,500$€), all other performance levels remaining constant. Since the client is supposed to be fond of sports cars, he could say for instance that $x_2 = 29.2$ s, which would result in the following indifference judgement: $(16.5; 29.5) \sim (17.5; 29.2)$. In view of the hypothesis that the client's preference fits into the additive value model, this indifference judgement can be translated into the following equality:

$$u_1(16.5) + u_2(29.5) + \sum_{j=3}^{5} u_j(x_j) = u_1(17.5) + u_2(29.2) + \sum_{j=3}^{5} u_j(x_j) \qquad (3.5)$$

Since the performance of both cars on attributes $j = 3, 4, 5$ are equal, the corresponding terms of the sum cancel and we are left with $u_1(16.5) + u_2(29.5) = u_1(17.5) + u_2(29.2)$ or:

$$u_1(16.5) - u_1(17.5) = u_2(29.2) - u_2(29.5). \qquad (3.6)$$

The second question to the client uses his answer to the first question; we ask him to assess the value x_2 of the acceleration that would leave him indifferent between the two cars $(16.5; 29.2)$ and $(17.5; x_2)$. Suppose the answer is $x_2 = 28.9$; we would then infer that:

$$u_1(16.5) - u_1(17.5) = u_2(28.9) - u_2(29.2). \tag{3.7}$$

Note that the lefthand side has remained unchanged: we always ask for acceleration intervals that are considered as equivalent to the same cost interval.

The next question asks for a value x_2 such that $(16.5; 28.9) \sim (17.5; x_2)$ and so on. We may imagine that the sequence of answers could be e.g.: 29.5; 29.2; 28.9; 28.7; 28.5; 28.3; 28.1. In view of (3.6), this amounts to saying that this sequence of levels on the marginal value scale of the acceleration attribute are equally spaced and that all differences of value between consecutive pairs of levels in the list are worth the same difference of cost, namely a difference of 1,000 € placed between 16,500 and 17,500 €. In other words, the client values 1,000 € an improvement of

0.3 s w.r.t. a performance level of 29.5 s or 29.2 s
0.2 s w.r.t. a performance level of 28.9 s, 28.7 s, 28.5 s or 28.3 s

on the acceleration attribute. He thus praises more improvements in the lower range of the scale. Similar questions are asked for the upper half of the range of the acceleration attribute, i.e. from 29.5 to 31 s. We ask the client to assess x_2 such that he would be indifferent between $(16.5; x_2)$ and $(17.5; 29.5)$. Assume the client's answer is $x_2 = 30.0$. Then we go on asking for x_2 such that $(16.5; x_2) \sim (17.5; 30.0)$ and suppose we get $x_2 = 31$. From all these answers, one understands that the client values in the same way a gain in acceleration performance of 1 s between 31 and 30 and a gain of 0.2 s between e.g. between 28.9 and 28.7, a ratio of 1 to 5.

What can we do with this piece of information? We can build a piecewise linear approximation of the function u_2 (defined on the range going from 28 to 31 s). Using an arbitrary unit length on the vertical axis [the unit length represents 1,000 € or more precisely the difference $u_1(16.5) - u_1(17.5)$], we get the function u_2 represented on Fig. 3.1; it is in fact a linear interpolation of nine points the first coordinate of which correspond to the answers made by the client to seven indifference judgments; the second coordinate of these points have just to be equally spaced (by one unit length). The position of the origin is arbitrary. We have extrapolated the line from 28.1 to 28 (thinner piece of line). Note that the function u_2 is decreasing since smaller is better with the measure chosen for evaluating the acceleration.

For determining u_3, u_4 and u_5, we search successively, in the same way as for acceleration, for intervals on the pick up, brakes and roadholding scales that would compensate exactly the cost interval (16.5; 17.5) in terms of preference.

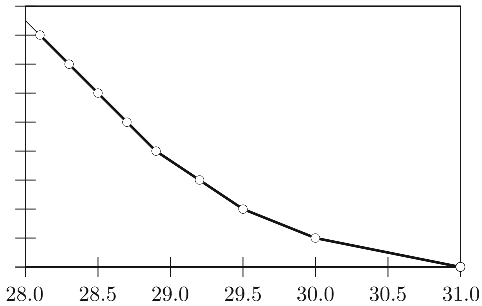


Fig. 3.1 Piecewise linear interpolation of the marginal value function u_2 on the acceleration attribute

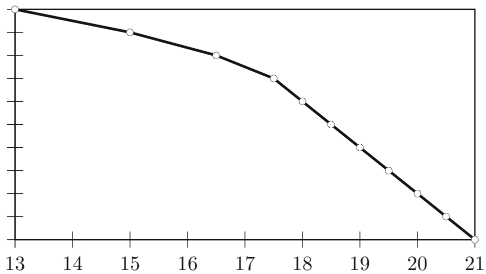


Fig. 3.2 Piecewise linear interpolation of the marginal value function u_2 on the cost attribute

Finally, we have to do the same recoding for the cost itself. We fix an interval for instance on the acceleration scale, say [29.2; 29.5]. We already know the answer to one question: (17.5; 29.2) is indifferent with $(x_1, 29.5)$ when $x_1 = 16.5$. We then ask the client, which level x_1 on the cost scale would leave him indifferent between (16.5; 29.2) and $(x_1, 29.5)$. A cost lower than 16, 500 € is expected and we use it in the next question, and so on. We might end up for instance with the curve shown on Fig. 3.2. Looking at that curves indicates that the client is inclined to pay more for the same improvement on the acceleration attribute for a car priced in the lower part of the cost range than in the upper part. Plausibly, with a limited budget as a student, Thierry can reasonably spend up to 17, 500 € on buying a car; paying more would imply restrictions on other expenses.

Suppose we have built that way piecewise linear approximations of u_1 to u_5. If we have chosen the same unit on all vertical axes to represent intervals equivalent to $u_1(16.5) - u_1(17.5)$, it only remains to add up these functions to obtain a piecewise linear approximation of u; ranking in turn the alternatives according with their decreasing value of u [formula (3.1)] yields the preference $\succsim$ (or an approximation of it). For the sake of illustration, we show in Table 3.2 the additive value function[2] computed for the five best cars among the 14 cars selected as alternatives by Thierry.

[2]In reality, these values have been determined by means of another elicitation method; details are provided in Bouyssou et al. (2000, ch. 6).

Table 3.2 Ranking of the cars in decreasing order of the value function u

Cars	Value u	Rank
Peugeot 309/16	0.85	1
Nissan Sunny	0.75	2
Honda Civic	0.66	3
Peugeot 309	0.65	4
Renault 19	0.61	5

3.2.5 Independence and Marginal Preferences

We have seen in Sect. 3.2.3 how it is possible to use the preference relation $\succsim$ in order to define a preference relation on a single dimension (i.e., on the set X_i). We now extend this concept of marginal preferences to subsets of dimensions. We denote by N the set of integers $\{1, 2, \dots, n\}$. For any nonempty subset J of N, x_J is the product set $\prod_{i \in J} X_i$ and we define the *marginal relation* $\succsim_J$ induced on X_J by $\succsim$ letting, for all $x_J, y_J \in X_J$:

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with asymmetric (resp. symmetric) part $\succ_J$ (resp. $\sim_J$). When $J = \{i\}$, we often abuse notation and write $\succsim_i$ instead of $\succsim_{\{i\}}$ (see the Definition (3.4) of $\succsim_i$ on p. 40). Note that if $\succsim$ is reflexive (resp. transitive), the same will be true for $\succsim_J$. This is clearly not true for completeness however.

Definition 3.1 (Independence) Consider a binary relation $\succsim$ on a set $X = \prod_{i=1}^{n} X_i$ and let $J \subseteq N$ be a nonempty subset of dimensions. We say that $\succsim$ is independent for J if, for all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If $\succsim$ is independent for all nonempty subsets of N, we say that $\succsim$ is *independent* (or strongly independent). If $\succsim$ is independent for all subsets containing a single dimension, we say that $\succsim$ is *weakly independent*.

In view of (3.1), it is clear that the additive value model will require that $\succsim$ is independent. This crucial condition says that common evaluations on some dimensions do not influence preference. Whereas independence implies weak independence, it is well-know that the converse is not true (Wakker, 1989).

Independence, or at least weak independence, is an almost universally accepted hypothesis in multiple criteria decision making. It cannot be overemphasised that it is possible to find examples in which it is inadequate. Yet, many authors (Keeney, 1992; Roy, 1996; von Winterfeldt and Edwards, 1986) have argued that such failures of independence were almost always due to a poor structuring of dimensions.

When $\succsim$ is a weak order and is weakly independent, marginal preferences are also weak orders and combine in a *monotonic* manner with the preference $\succsim$. For instance, if an alternative is preferred to another on all dimensions, then the former should be globally preferred to the latter. This monotonicity property of the preference with respect to the marginal preferences has strong links with the idea of *dominance*.

It should however be kept in mind that preferences that are not weak orders may show different behaviours. For more general preferences, the marginal preferences may no longer be the adequate tool on which to lean for eliciting the preference. This will be strongly emphasised and analysed in the generalisations of the additive value model discussed in Sects. 3.3, 3.4, and 3.5.

3.2.6 *The Additive Value Model in the "Rich" Case*

The purpose of the rest of Sect. 3.2 is to present the conditions under which a preference relation on a product set may be represented by the additive value function model (3.1) and how such a model can be assessed. Some limitations of this approach will also be discussed. We begin here with the case that most closely resembles the measurement of physical dimensions such as length.

When the structure of X is supposed to be "adequately rich", conjoint measurement is an adaptation of the process that is used for the measurement of physical extensive quantities such as length. The basic idea of this type of measurement (called *extensive measurement*, see Krantz et al., 1971, ch. 3) consists in comparing the object to be measured to a standard object that can be replicated while the length of the chains of replicas is an integer number of times that of the standard "unit" object. What will be measured here is the "length" of preference intervals on a dimension using a preference interval on another dimension as a standard.

3.2.6.1 The Case of Two Dimensions

Consider first the two dimension case, where the relation $\succsim$ is defined on a set $X = X_1 \times X_2$. In Sect. 3.2.1, p. 38, we already identified necessary conditions for a relation to be representable in the additive value model, namely, we have to assume that $\succsim$ is an *independent weak order*. In such a case, $\succsim_1$ and $\succsim_2$ are weak orders, as stated in Sect. 3.2.5. Consider two levels $x_1^0, x_1^1 \in X_1$ on the first dimension such that $x_1^1 \succ_1 x_1^0$, i.e. x_1^1 is preferable to x_1^0. Note that we will have to exclude the case in which all levels on the first dimension would be marginally indifferent in order to be able to find such levels.

Choose any $x_2^0 \in X_2$. The, arbitrarily chosen, element $(x_1^0, x_2^0) \in X$ will be our "reference point". The basic idea is to use this reference point and the "unit" on the first dimension given by the reference preference interval $[x_1^0, x_1^1]$ to build a standard sequence on the preference intervals on the second dimension. Hence, we are looking for an element $x_2^1 \in X_2$ that would be such that:

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0). \tag{3.8}$$

Clearly this will require the structure of X_2 to be adequately "rich" so as to find the level $x_2^1 \in X_2$ such that the reference preference interval on the first dimension $[x_1^0, x_1^1]$ is exactly matched by a preference interval of the same "length" on the second dimension $[x_2^0, x_2^1]$. Technically, this calls for a solvability assumption or, more restrictively, for the supposition that X_2 has a (topological) structure that is close to that of an interval of $\mathbb{R}$ and that $\succsim$ is "somehow" continuous.

If such a level x_2^1 can be found, model (3.1) implies:

$$\begin{aligned} u_1(x_1^0) + u_2(x_2^1) &= u_1(x_1^1) + u_2(x_2^0) \text{ so that} \\ u_2(x_2^1) - u_2(x_2^0) &= u_1(x_1^1) - u_1(x_1^0). \end{aligned} \tag{3.9}$$

Let us fix the origin of measurement letting: $u_1(x_1^0) = u_2(x_2^0) = 0$, and our unit of measurement letting: $u_1(x_1^1) = 1$ so that $u_1(x_1^1) - u_1(x_1^0) = 1$. Using (3.9), we therefore obtain $u_2(x_2^1) = 1$. We have therefore found an interval between levels on the second dimension ($[x_2^0, x_2^1]$) that exactly matches our reference interval on the first dimension ($[x_1^0, x_1^1]$). We may proceed to build our standard sequence on the second dimension (see Fig. 3.3) asking for levels $x_2^2, x_2^3, \ldots$ such that:

$$\begin{aligned} &(x_1^0, x_2^2) \sim (x_1^1, x_2^1), \\ &(x_1^0, x_2^3) \sim (x_1^1, x_2^2), \\ &\ldots \\ &(x_1^0, x_2^k) \sim (x_1^1, x_2^{k-1}). \end{aligned}$$

As above, using (3.1) leads to:

$$\begin{aligned} u_2(x_2^2) - u_2(x_2^1) &= u_1(x_1^1) - u_1(x_1^0), \\ u_2(x_2^3) - u_2(x_2^2) &= u_1(x_1^1) - u_1(x_1^0), \\ &\ldots \\ u_2(x_2^k) - u_2(x_2^{k-1}) &= u_1(x_1^1) - u_1(x_1^0), \end{aligned}$$

so that:

$$u_2(x_2^2) = 2, u_2(x_2^3) = 3, \ldots, u_2(x_2^k) = k.$$

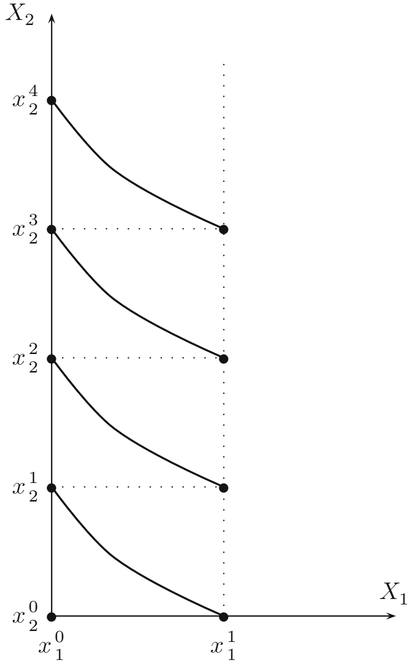


Fig. 3.3 Building a standard sequence on X_2

This process of building a standard sequence of the second dimension therefore leads to defining u_2 on a number of, carefully, selected elements of X_2. Suppose now that there is a level $y_2 \in X_2$ that can never be "reached" by our standard sequence, i.e. such that $y_2 \succ_2 x_2^k$, even for very large k. This is clearly not compatible with an additive representation as in (3.1). We therefore need to exclude this case by imposing a specific condition, called *Archimedean* because it mimics the property of the real numbers saying that for any positive real numbers x, y it is true that $nx > y$ for some integer n, i.e. y, no matter how large, may always be exceeded by taking any x, no matter how small, and adding it with itself and repeating the operation a sufficient number of times.

Now that a standard sequence is built on the second dimension, we may use any part of it to build a standard sequence on the first dimension. This will require finding levels $x_1^2, x_1^3, \ldots \in X_1$ such that (see Fig. 3.4): $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$, $(x_1^3, x_2^0) \sim (x_1^2, x_2^1)$, $\ldots (x_1^k, x_2^0) \sim (x_1^{k-1}, x_2^1)$. Using (3.1) leads to:

$$u_1(x_1^2) - u_1(x_1^1) = u_2(x_2^1) - u_2(x_2^0),$$
$$u_1(x_1^3) - u_1(x_1^2) = u_2(x_2^1) - u_2(x_2^0),$$
$$\ldots$$
$$u_1(x_1^k) - u_1(x_1^{k-1}) = u_2(x_2^1) - u_2(x_2^0),$$

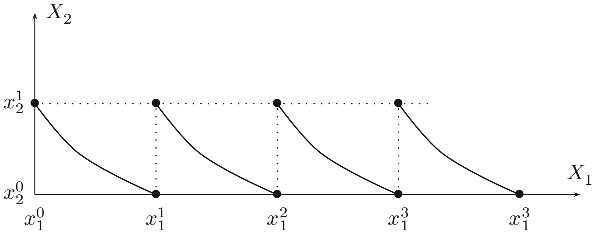


Fig. 3.4 Building a standard sequence on X_1

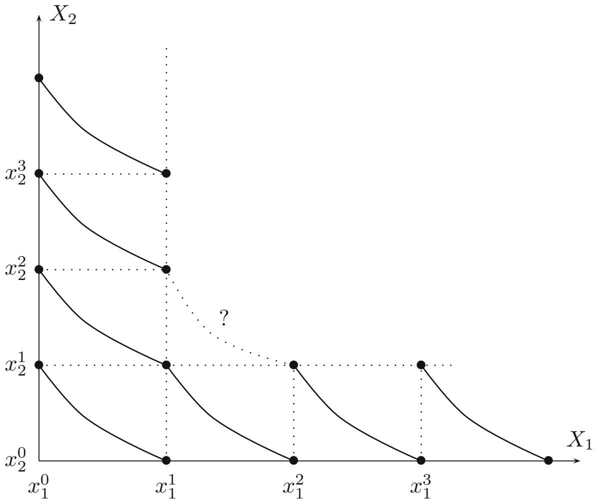


Fig. 3.5 The grid

so that: $u_1(x_1^2) = 2, u_1(x_1^3) = 3, \ldots, u_1(x_1^k) = k$. As was the case for the second dimension, the construction of such a sequence will require the structure of X_1 to be adequately rich, which calls for a solvability assumption. An Archimedean condition will also be needed in order to be sure that all levels of X_1 can be reached by the sequence.

We have defined a "grid" in X (see Fig. 3.5) and we have $u_1(x_1^k) = k$ and $u_2(x_2^k) = k$ for all elements of this grid. Intuitively such numerical assignments seem to define an adequate additive value function on the grid. We have to prove that this intuition is correct. Let us first verify that, for all integers $\alpha, \beta, \gamma, \delta$:

$$\alpha + \beta = \gamma + \delta = \epsilon \Rightarrow (x_1^\alpha, x_2^\beta) \sim (x_1^\gamma, x_2^\delta). \tag{3.10}$$

When $\epsilon = 1$, (3.10) holds by construction because we have: $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$. When $\epsilon = 2$, we know that $(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$ and $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$ and the claim is proved using the transitivity of $\sim$.

Consider the $\epsilon = 3$ case. We have $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$ and $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$. It remains to be shown that $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$ (see the dotted arc in Fig. 3.5). This does not seem to follow from the previous conditions that we more or less explicitly used: transitivity, independence, "richness", Archimedean. Indeed, it does not. Hence, we have to suppose that: $(x_1^2, x_2^0) \sim (x_1^0, x_2^2)$ and $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$ imply $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$. This condition, called the *Thomsen condition*, is clearly necessary for (3.1). The above reasoning easily extends to all points on the grid, using weak ordering, independence and the Thomsen condition. Hence, (3.10) holds on the grid.

It remains to show that:

$$\epsilon = \alpha + \beta > \epsilon' = \gamma + \delta \Rightarrow (x_1^\alpha, x_2^\beta) \succ (x_1^\gamma, x_2^\delta). \tag{3.11}$$

Using transitivity, it is sufficient to show that (3.11) holds when $\epsilon = \epsilon' + 1$. By construction, we know that $(x_1^1, x_2^0) \succ (x_1^0, x_2^0)$. Using independence this implies that $(x_1^1, x_2^k) \succ (x_1^0, x_2^k)$. Using (3.10) we have $(x_1^1, x_2^k) \sim (x_1^{k+1}, x_2^0)$ and $(x_1^0, x_2^k) \sim (x_1^k, x_2^0)$. Therefore we have $(x_1^{k+1}, x_2^0) \succ (x_1^k, x_2^0)$, the desired conclusion.

Hence, we have built an additive value function of a suitably chosen grid (see Fig. 3.6). The logic of the assessment procedure is then to assess more and more points somehow considering more finely grained standard sequences. Going to the limit then unambiguously defines the functions u_1 and u_2. Clearly such u_1 and u_2 are intimately related. Once we have chosen an arbitrary reference point (x_1^0, x_2^0) and a level x_1^1 defining the unit of measurement, the process just described entirely defines u_1 and u_2. It follows that the only possible transformations that can be applied to u_1 and u_2 is to multiply both by the same positive number α and to add to both a, possibly different, constant. This is usually summarised saying that u_1 and u_2 define interval scales with a common unit.

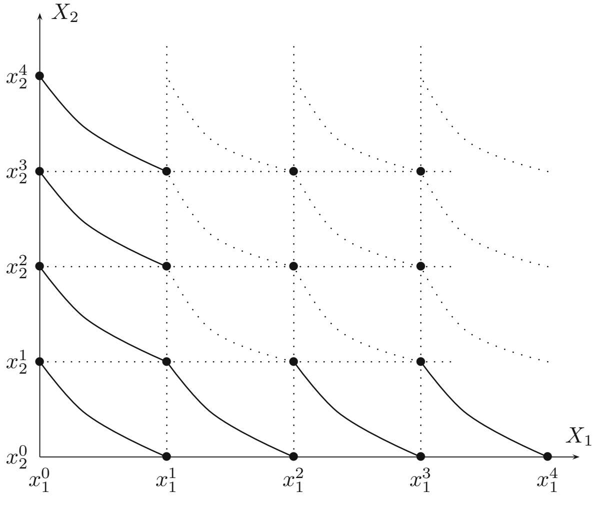


Fig. 3.6 The entire grid

The above reasoning is a rough sketch of the proof of the existence of an additive value function when $n = 2$, as well as a sketch of how it could be assessed. Careful readers will want to refer to Fishburn (1970, ch. 5), Krantz et al. (1971, ch. 6) and Wakker (1989, ch. 3).

It is worth emphasising that the assessment technique using standard sequences outlined above makes no use of the vague notion of the "importance" of the various dimensions. The "importance" is captured here in the lengths of the preference intervals on the various dimensions.

A common but critical mistake is to confuse the additive value function model (3.1) with a weighted average and to try to assess weights asking whether a dimension is "more important" than another. This makes no sense.

3.2.6.2 The Case of More Than Two Dimensions

The good news is that the process is exactly the same when there are more than two dimensions. With one surprise: the Thomsen condition is no longer needed to prove that the standard sequences defined on each dimension lead to an adequate value function on the grid. A heuristic explanation of this strange result is that, when $n = 2$, there is no difference between independence and weak independence. This is no more true when $n \geq 3$ and assuming independence is much stronger than just assuming weak independence.

We use below the "algebraic approach" (Krantz, 1964; Krantz et al., 1971; Luce and Tukey, 1964). A more restrictive approach using a topological structure on X is given in Debreu (1960), Fishburn (1970, ch. 5) and Wakker (1989, ch. 3). We formalise below the conditions informally introduced in the preceding section. The reader not interested in the precise statement of the results or, better, having already written down his own statement, may skip this section.

Definition 3.2 (Thomsen Condition) Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. It is said to satisfy the Thomsen condition if

$$(x_1, x_2) \sim (y_1, y_2) \text{ and } (y_1, z_2) \sim (z_1, x_2) \Rightarrow (x_1, z_2) \sim (z_1, y_2),$$

for all $x_1, y_1, z_1 \in X_1$ and all $x_2, y_2, z_2 \in X_2$.

Figure 3.7 shows how the Thomsen condition uses two "indifference curves" (i.e. curves linking points that are indifferent) to place a constraint on a third one. This was needed above to prove that an additive value function existed on our grid. Remember that the Thomsen condition is only needed when $n = 2$; hence, we only stated it in this case.

Definition 3.3 (Standard Sequences) A standard sequence on dimension $i \in N$ is a set $\{a_i^k : a_i^k \in X_i, k \in K\}$ where K is a set of consecutive integers (positive or negative, finite or infinite) such that there are $x_{-i}, y_{-i} \in X_{-i}$ satisfying $Not[x_{-i} \sim_{-i} y_{-i}]$ and $(a_i^k, x_{-i}) \sim (a_i^{k+1}, y_{-i})$, for all $k \in K$.

Fig. 3.7 The Thomsen condition

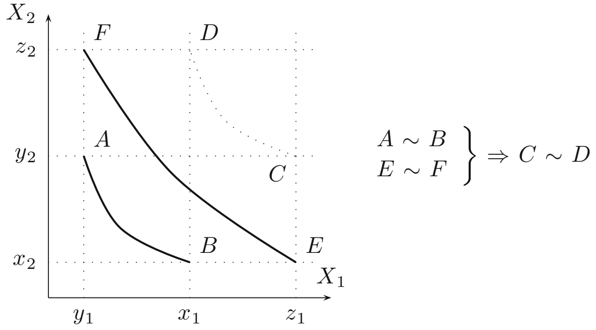
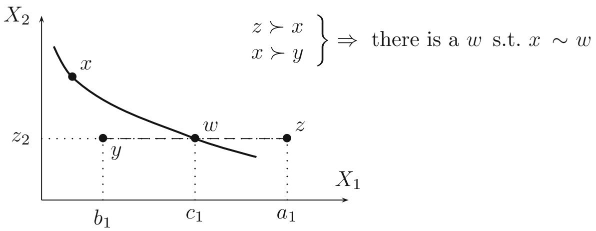


Fig. 3.8 Restricted solvability on X_1



A standard sequence on dimension $i \in N$ is said to be *strictly bounded* if there are $b_i, c_i \in X_i$ such that $b_i \succ_i a_i^k \succ_i c_i$, for all $k \in K$. It is then clear that, when model (3.1) holds, any strictly bounded standard sequence must be finite.

Definition 3.4 (Archimedean) For all $i \in N$, any strictly bounded standard sequence on $i \in N$ is finite.

The following condition rules out the case in which a standard sequence cannot be built because all levels are indifferent.

Definition 3.5 (Essentiality) Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. dimension $i \in N$ is said to be essential if $(x_i, a_{-i}) \succ (y_i, a_{-i})$, for some $x_i, y_i \in X_i$ and some $a_{-i} \in X_{-i}$.

Definition 3.6 (Restricted Solvability) Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \cdots \times X_n$. Restricted solvability is said to hold with respect to dimension $i \in N$ if, for all $x \in X$, all $z_{-i} \in X_{-i}$ and all $a_i, b_i \in X_i$, $[(a_i, z_{-i}) \succsim x \succsim (b_i, z_{-i})] \Rightarrow [x \sim (c_i, z_{-i})$, for some $c_i \in X_i]$.

Restricted solvability is illustrated in Fig. 3.8 in the case where $n = 2$. It says that, given any $x \in X$, if it is possible find two levels $a_i, b_i \in X_i$ such that when combined with a certain level $z_{-i} \in X_{-i}$ on the other dimensions, (a_i, z_{-i}) is preferred to x and x is preferred to (b_i, z_{-i}), it should be possible to find a level c_i, between a_i and b_i, such that (c_i, z_{-i}) is exactly indifferent to x.

We are now in position to state the central results concerning model (3.1). Proofs may be found in Krantz et al. (1971, ch. 6) and Wakker (1991).

Theorem 3.1 (Additive Value Function When $n = 2$) *Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. If restricted solvability holds on all dimensions and each dimension is essential then $\succsim$ has a representation in model (3.1) if and only if $\succsim$ is an independent weak order satisfying the Thomsen and the Archimedean conditions.*

Furthermore in this representation, u_1 and u_2 are interval scales with a common unit, i.e. if u_1, u_2 and w_1, w_2 are two pairs of functions satisfying (3.1), there are real numbers α, β_1, β_2 with $\alpha > 0$ such that, for all $x_1 \in X_1$ and all $x_2 \in X_2$

$$u_1(x_1) = \alpha w_1(x_1) + \beta_1 \text{ and } u_2(x_2) = \alpha w_2(x_2) + \beta_2.$$

When $n \geq 3$ and at least three dimensions are essential, the above result simplifies in that the Thomsen condition can now be omitted.

Theorem 3.2 (Additive Value Function When $n \geq 3$) *Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \ldots \times X_n$ with $n \geq 3$. If restricted solvability holds on all dimensions and at least three dimensions are essential then $\succsim$ has a representation in (3.1) if and only if $\succsim$ is an independent weak order satisfying the Archimedean condition. Furthermore in this representation $u_1, u_2, \ldots, u_n$ are interval scales with a common unit.*

3.2.6.3 Implementation: Standard Sequences and Beyond

The assessment procedure based on standard sequences is, as we have seen, rather demanding; hence, it seems to be seldom used in the practice of decision analysis (Keeney and Raiffa, 1976). Many other simplified assessment procedures have been proposed that are less firmly grounded in theory. These procedures include *(1) direct rating* techniques in which values of u_i are directly assessed with reference to two arbitrarily chosen points; *(2)* procedures based on *bisection*, the decision-maker being asked to assess a point that is "half way", in terms of preference, two reference points, *(3)* procedures trying to build *standard sequences* on each dimension in terms of "preference differences." An excellent overview of these techniques may be found in von Winterfeldt and Edwards (1986, ch. 7).

3.2.7 Insufficiency of Additive Conjoint Measurement

We now present two examples showing that there are preferences that are both reasonable and do not satisfy the hypotheses for an additive representation. We also present an example that can be represented within the additive value function model but also in a more specific model than (3.1), with special u_i functions.

Example 3.2 A solution of a Flexible Constraint Satisfaction Problem is assessed by a vector of n numbers that represent the degree to which each of the n constraints are satisfied; the satisfaction degree is usually modelled as a number between 0 and 1. For instance, in certain scheduling problems (Dubois et al., 1995; Dubois and Fortemps, 1999), there may be an ideal range of time between the end of some tasks and the starting of some other ones; if more (or less) time elapses, then the schedule is less satisfactory; for each constraint of that type, the degree of satisfaction is equal to 1 if the corresponding slack time lies in the ideal range; it decreases outside this range and, outside a larger interval corresponding to the admissible delays between the end of a task and the beginning of another, the degree of satisfaction reaches 0. Usually, one considers that the scale on which the satisfaction degrees are assessed is ordinal and the same for all constraints: one may meaningfully compare satisfaction degrees (saying for instance that one is larger than the other), but the difference between two degrees cannot be compared meaningfully to another difference; moreover, the satisfaction degrees of two different constraints are commensurate: it is meaningful to say that a constraint is satisfied at a higher level than another one. A solution to such a scheduling problem is an assignment of a starting time to each task; comparing two solutions amounts to comparing their associated vectors of satisfaction degrees. Usually in practice, a solution is evaluated by its weakest aspect, i.e. the lowest degree of satisfaction it attains on the set of constraints. In other words, vectors of satisfaction can be compared using the "min-score"; for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, where x_i and y_i respectively denote the degrees of satisfaction of constraint i by the two alternatives to be compared, one has:

$$x \succsim y \Leftrightarrow \min(x_1, \ldots, x_n) \geq \min(y_1, \ldots, y_n) \tag{3.12}$$

Clearly, the relation comparing the vectors of satisfaction degrees can be viewed as a relation $\succsim$ on the product set $X = [0, 1]^n$. It is defined by means of the "min"-score instead of an additive value function as in model (3.1). One can not exclude *a priori* that the relation defined by (3.12) could also be represented in model (3.1). This is however not the case, since this relation does not satisfy one of the necessary conditions stated above, namely the strong independence property: we can indeed transform an indifference into a strict preference by changing the common level of satisfaction that is achieved by two alternatives on the same constraint. This is shown by the following example. Suppose there are two constraints ($n = 2$) and $x = (0.6, 0.5)$, $y = (0.6, 0.7)$; one has $x \succ y$, but lowering for instance to 0.3 the common satisfaction level yields $x' \sim y'$ (with $x' = (0.3, 0.5)$ and $y' = (0.3, 0.7)$). It should be clear from this example that there are simple and well-motivated procedures the additive value function model is not able to encompass.

Example 3.3 The other necessary condition for model (3.1), namely transitivity, may also fail to be satisfied by some reasonable preferences. Let us just recall R. D. Luce's famous example (Luce, 1956) of the cup of coffee: a person who likes coffee is indifferent between two cups of coffee that differ by the addition of one grain of sugar; he normally would not be indifferent between a cup with no sugar and a cup containing one thousand grains of sugar; he would definitely prefer the latter or the former. A long sequence of indifferent alternatives may thus result in preference, contrary to the hypothesis of the additive value model, in which preferences are weak orders, hence transitive.[3]

Example 3.4 Assume $g_i(a)$ is a number. The PROMETHEE II method (Brans and Vincke, 1985) starts with comparing alternatives, in a pairwise manner, with respect to each attribute i. The intensity $S_i(a,b)$ of the preference of a over b on attribute i is a nondecreasing function P_i of the difference $g_i(a) - g_i(b)$:

$$S_i(a,b) = P_i(g_i(a) - g_i(b)). \tag{3.13}$$

When the difference $g_i(a) - g_i(b)$ is negative, it is assumed that $S_i(a,b) = 0$. The global intensity of the preference of a over b is described by means of a weighted sum of the S_i functions:

$$S(a,b) = \sum_{i=1}^{n} w_i S_i(a,b), \tag{3.14}$$

where w_i is the weight associated with attribute i. In a further step, the alternatives are evaluated by their "net flow" defined by:

$$\Phi(a) = \sum_{b \in \mathcal{A}} S(a,b) - S(b,a). \tag{3.15}$$

This score is then used to determine that a is preferred over b if $\Phi(a) \geq \Phi(b)$. This is the customary presentation of PROMETHEE II (see e.g. Vincke, 1992, p. 74).

By using Eq. (3.15), it is easy to rewrite $\Phi(a)$ as follows:

$$\Phi(a) = \sum_{i=1}^{n} w_i \sum_{b \in \mathcal{A}} [S_i(a,b) - S_i(b,a)]. \tag{3.16}$$

The latter formula can be seen as defining an additive value model in which the marginal value functions u_i have the particular form:

$$u_i(g_i(a)) = \sum_{b \in \mathcal{A}} [S_i(a,b) - S_i(b,a)]. \tag{3.17}$$

The computation of function u_i that models the influence of criterion i depends on the other alternatives (thereby violating a property called "independence of irrelevant alternatives" (Arrow, 1951)). Equation (3.17) suggests that constructing

[3] For further discussion of the transitivity of preference issue, mainly in the context of decision under risk, the reader could see Fishburn (1991a). For counter-arguments against considering intransitive preferences, see (Luce, 2000, section 2.2).

the preference can go through modelling for each dimension the value of any echelon $g_i(a)$ as the sum of its "advantages" and "disadvantages", respectively coded by $S_i(a,b)$ and $S_i(b,a)$. Model (3.1) makes no suggestion of intuitively interpretable concepts that would suggest that u_i could be viewed as a superposition (through a sum) of more elementary elements.

In the following sections, we present more general conjoint measurement models (providing more general representations of the preference); the proposed models all induce concepts that can support the construction or elicitation process.

3.3 A First Line of Generalisation: Models Based on Marginal Traces or Preferences

In this section we discuss a generalisation of the additive value function model while preserving the possibility of using the fundamental construction tool suggested by the model, namely marginal preferences that are weak orders represented by the functions u_i in (3.1). Interestingly, the generalised model admits a full characterisation through fairly simple and intuitive axioms, which was not the case with model (3.1) as we have just seen.

3.3.1 Decomposable Preferences

The so-called decomposable model has been introduced in Krantz et al. (1971, ch. 7) as a natural generalisation of model (3.1). The preference $\succsim$ is supposed to be a weak order and can thus be represented by a rule of the type

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \tag{3.18}$$

with u, a real-valued function defined on X. Instead of specifying u as a sum of functions u_i of the variables x_i, u is just supposed to be decomposable in the form

$$u(x) \;=\; U(u_1(x_1), \ldots, u_n(x_n)) \tag{3.19}$$

where u_i is a function from X_i to $\mathbb{R}$ (the set of real numbers) and U is increasing in all its arguments.

The interesting point with this model is that it admits an intuitively appealing characterisation. The basic axiom for characterising the above decomposable model (with increasing U) is the weak independence condition (see Definition 3.1).

For preferences that are weak orders, it is possible to prove that the weak independence property is equivalent to the fact that the marginal preferences $\succsim_i$ are weak orders (Proposition 6.1 in Bouyssou et al. 2006). Moreover, it is easy to see that u_i in (3.19) is necessarily a numerical representation of $\succsim_i$, i.e. $x_i \succsim_i y_i$ iff $u_i(x_i) \geq u_i(y_i)$. This is an important result since it opens the door to the elicitation of the u_i's by questioning in terms of the marginal preferences $\succsim_i$ like was done in the additive utility model.

The following theorem states a simple and important characterisation of the decomposable model. This result was first proved in Krantz et al. (1971, ch. 7).

Theorem 3.3 (Representation in the Decomposable Model) *A preference relation $\succsim$ on X admits a representation in the decomposable model:*

$$x \succsim y \Leftrightarrow U(u_1(x_1), \dots, u_n(x_n)) \geq U(u_i(y_1), \dots, u_i(y_n))$$

with U increasing in all its arguments iff $\succsim$ is a weak order and satisfies weak independence.

If one intended to apply this model, one would go through specifying the type of function U, possibly by verifying further conditions on the preference that impose that U belongs to some parameterised family of functions (e.g. polynomials of bounded degree). Although decomposable preferences form a large family of preferences, it is not large enough to encompass all useful cases. A major restriction is that not all preferences may be assumed to be weak orders, as illustrated in Example 3.3 by the example of the cups of coffee.

3.3.2 Insufficiency of Marginal Analysis: Marginal Traces

In the decomposable model, the preference may be reconstructed on the basis of the marginal preferences $\succsim_i$ since it is represented by a function of the u_i's, themselves representing $\succsim_i$ (at least in the strict decomposable model).

This is no longer the case when $\succsim$ is not a weak order because the relation $\succsim_i$ on X_i is not very discriminating.

Example 3.5 To fix the ideas, suppose a decision-maker has preferences that can be represented by $a \succsim b$ iff $\sum_{i=1}^{n} u_i(a_i) \geq \sum_{i=1}^{n} u_i(b_i) - \delta$ for some positive and real δ. The reason for adding δ is that the decision-maker considers that small differences between $\sum_{i=1}^{n} u_i(a_i)$ and $\sum_{i=1}^{n} u_i(b_i)$ are not significant. In particular, $a \succ b$ iff $\sum_{i=1}^{n} u_i(a_i) > \sum_{i=1}^{n} u_i(b_i) + \delta$. Suppose also $n = 10$ and the range of each mapping u_i is $[0, 1]$. Then the range of $\sum_{i=1}^{n} u_i(\cdot)$ is $[0, 10]$ and it seems plausible to use $\delta = 1$. Let us now consider objects differing only on one attribute. We have $a_i \succ_i b_i$ iff $(a_i, a_{-i}) \succ (b_i, a_{-i})$ iff $\sum_{j=1}^{n} u_j(a_j) > u_i(b_i) + \sum_{j \neq i} u_j(a_j) + 1$ iff $u_i(a_i) > u_i(b_i) + 1$. Since, the range of u_i is $[0, 1]$, it will never be the case that $u_i(a_i) > u_i(b_i) + 1$ and, hence $a_i \sim_i b_i$ for all $a_i, b_i \in X_i$. In other words, the marginal preference $\succsim_i$ is completely uninformative: it does not discriminate any level of X_i. This case is obviously extreme but it is not uncommon that $\succsim_i$ discriminates only few levels.

Is there a relation on X_i that has stronger links with the global preference $\succsim$ than the marginal preference $\succsim_i$? The answer is the *marginal trace* $\succsim_i^{\pm}$ that is defined below.

Definition 3.7 (Marginal Trace)
The marginal trace $\succsim_i^{\pm}$ of relation $\succsim$ on the product set $X = \prod X_i$ is the relation on X_i defined by:

$$a_i \succsim_i^{\pm} b_i \text{ iff } \begin{cases} \text{for all } c, d \in X, \\ [(b_i, c_{-i}) \succsim d] \Rightarrow [(a_i, c_{-i}) \succsim d] \text{ and} \\ [c \succsim (a_i, d_{-i})] \Rightarrow [c \succsim (b_i, d_{-i})] \end{cases} \tag{3.20}$$

In other words, $a_i \succsim_i^{\pm} b_i$ iff substituting b_i by a_i in an alternative does not change the way this alternative compares to others.

In the case of Example 3.5, one has $a_i \succsim_i^{\pm} b_i$ iff $u_i(a_i) \geq u_i(b_i)$, which is easily verified. Suppose indeed that $(b_i, c_{-i}) \succsim d$ for some $c_{-i} \in X_{-i}$ and $d \in X$; this means that

$$u_i(b_i) + \sum_{j \neq i} u_j(c_j)) \geq \sum_{j=1}^{n} u_j(d_j) + 1. \tag{3.21}$$

Substituting b_i by a_i : $u_i(a_i) \geq u_i(b_i)$ preserves the inequality.

In models in which $\succsim$ is not supposed to be a weak order, the information conveyed in the marginal preferences may be insufficient to reconstruct the preference. As we shall see, the marginal traces, when they are weak orders, always convey enough information. The reason why the insufficiency of marginal preferences did not show up in the decomposable model is a consequence of the following result.

Proposition 3.1 (Marginal Preferences and Marginal Traces) *If a preference relation $\succsim$ on X is reflexive and transitive, its marginal preferences $\succsim_i$ and its marginal traces $\succsim_i^{\pm}$ are confounded for all i.*

The proposition almost immediately results from the definitions of marginal preferences and traces. It makes clear that there is no need worrying about marginal traces unless $\succsim$ is not transitive. More exactly, as we shall see below, the notion that conveys all the information needed to reconstruct the global preference from relations on each scale X_i is always the marginal traces; but when $\succsim$ is reflexive and transitive, you may equivalently use marginal preferences instead. The converse of the proposition is not true however: there are cases where $\succsim$ is not transitive (e.g. when $\succsim$ is a semiorder) and $\succsim_i = \succsim_i^{\pm}$ (see Bouyssou and Pirlot, 2004, Example 4).Instead of generalising again the decomposable model in order to encompass preferences that are for instance semiorders, we propose and study a much more general model. It is so general that it encompasses all relations on X. Considering this model as a framework, we introduce successive specialisations that will bring us back to the decomposable model, but "from above", i.e.in a movement from the general to the particular. In this specialisation process, it is the marginal trace—not the marginal preference—that is the central tool.

3.3.3 Generalising Decomposable Models Using Marginal Traces

Consider the very general representation of a relation $\succsim$ described by:

$$x \succsim y \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n), u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \geq 0 \qquad (L0)$$

The main difference w.r.t. the decomposable model is that the evaluations of the two alternatives are not dealt with separately.

If no property is imposed on function F, the model is trivial since any relation can be represented within it. It obviously generalises the decomposable model and encompasses as a special case the representation involving a threshold described in Example 3.5 (in which the preference is a semiorder).

It is easy to obtain representations that guarantee simple properties of $\succsim$. For instance, $\succsim$ is reflexive iff it has a representation in model ($L0$) with $F([u_i(x_i)]; [u_i(x_i)]) \geq 0$; $\succsim$ is complete iff it has a representation in model ($L0$) with $F([u_i(x_i)]; [u_i(y_i)]) = -F([u_i(y_i)]; [u_i(x_i)])$. What if we impose monotonicity conditions on F? The natural ones in view of the decomposable model are (1) F increasing in its first n arguments and decreasing in its last n arguments and (2) F non-decreasing in its first n arguments and non-increasing in its last n arguments. The following axioms are closely linked with imposing monotonicity properties to F and, as we shall see, with properties of the marginal traces.

Definition 3.8 (Axioms $AC1, AC2, AC3, AC4$) We say that $\succsim$ satisfies:

$$AC1_i \quad \text{if} \quad \left.\begin{array}{r}(x_i, a_{-i}) \succsim y\\ \text{and}\\ (z_i, b_{-i}) \succsim w\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim y\\ \text{or}\\ (x_i, b_{-i}) \succsim w,\end{array}\right.$$

$$AC2_i \quad \text{if} \quad \left.\begin{array}{r}y \succsim (x_i, a_{-i})\\ \text{and}\\ w \succsim (z_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}y \succsim (z_i, a_{-i})\\ \text{or}\\ w \succsim (x_i, b_{-i}),\end{array}\right.$$

$$AC3_i \quad \text{if} \quad \left.\begin{array}{r}(x_i, a_{-i}) \succsim y\\ \text{and}\\ w \succsim (x_i, b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim y\\ \text{or}\\ w \succsim (z_i, b_{-i}),\end{array}\right.$$

for all $x_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $y, w \in X$.

It satisfies $AC4_i$ if $\succsim$ satisfies $AC3_i$ and, whenever one of the conclusions of $AC3_i$ is false, then the other one holds with $\succ$ instead of $\succsim$.

We say that $\succsim$ satisfies $AC1$ (resp. $AC2$, $AC3$, $AC4$) if it satisfies $AC1_i$ (resp. $AC2_i$, $AC3_i$, $AC4_i$) for all $i \in N$. We also use $AC123_i$ (resp. $AC123$) as shorthand for $AC1_i$, $AC2_i$ and $AC3_i$ (resp. $AC1$, $AC2$ and $AC3$).

The intuition behind these axioms is the following. Take axiom $AC1_i$. It suggests that x_i and z_i can be compared: either x_i corresponds to a "level" on a "scale" on X_i that is "above" z_i or the other way around. Suppose indeed that x_i is involved in an alternative that is preferred to another one ($(x_i, x_{-i}) \succsim y$); suppose further that substituting z_i to x_i would not allow to preserve the preference ($Not[(z_i, x_{-i}) \succsim y]$). Then $AC1_i$ says that substituting z_i by x_i when z_i is involved in an alternative that is preferred to another ($(z_i, z_{-i}) \succsim w$) will always preserve the preference (i.e. we have: $(x_i, z_{-i}) \succsim w$). One can interpret such a situation by saying that x_i is "above" z_i. The "being above" relation on X_i is what we call the *left marginal trace* of $\succsim$ and we denote it by $\succsim_i^+$; it is defined as follows:

$$x_i \succsim_i^+ z_i \Leftrightarrow [(z_i, z_{-i}) \succsim w \Rightarrow (x_i, z_{-i}) \succsim w]. \tag{3.22}$$

We explained above that $AC1_i$ meant that x_i and z_i can always be compared, which, in terms of the left trace, interprets as: "We may not have at the same time $Not[x_i \succsim_i^+ z_i]$ and $Not[z_i \succsim_i^+ x_i]$". It is easy to see that supposing the latter would amount to have some z_{-i} and some w such that:

$$(z_i, z_{-i}) \succsim w \text{ and } Not[(x_i, z_{-i}) \succsim w]$$

and at the same time, for some x_{-i} and some y,

$$(x_i, x_{-i}) \succsim y \text{ and } Not[(z_i, x_{-i}) \succsim y],$$

which is exactly the negation of $AC1_i$. Axiom $AC1_i$ thus says that the left marginal trace $\succsim_i^+$ is complete; since it is transitive by definition, $\succsim_i^+$ is a weak order.

$AC1_i$ deals with levels involved in alternatives that are preferred to other ones, thus in the strong (lefthand side) position in the comparison of two alternatives; in contrast, $AC2_i$ rules the behaviour of $\succsim$ when changing levels in alternatives in the weak position (another alternative is preferred to them). Clearly, $AC2_i$ is concerned with a *right marginal trace* $\succsim_i^-$ that is defined as follows:

$$y_i \succsim_i^- w_i \Leftrightarrow [x \succsim (y_i, y_{-i}) \Rightarrow x \succsim (w_i, y_{-i})]. \tag{3.23}$$

Through reasoning as above, one sees that $AC2_i$ is equivalent to requiring that $\succsim_i^-$ is a complete relation and thus a weak order (since it is transitive by definition).

At this stage, it is natural to wonder whether the left marginal trace is related to the right one. The role of $AC3_i$ is to ensure that $\succsim_i^+$ and $\succsim_i^-$ are not incompatible, i.e. that one cannot have at the same time $Not[x_i \succsim_i^+ y_i]$ and $Not[y_i \succsim_i^- x_i]$. If $\succsim_i^+$ and $\succsim_i^-$ are complete, this means that one cannot have $[y_i \succ_i^+ x_i]$ and $[x_i \succ_i^- y_i]$ (where $\succ_i^+$ and $\succ_i^-$ denote the asymmetric part of $\succsim_i^+$ and $\succsim_i^-$, respectively) or, in other words, that $[x_i \succ_i^+ y_i]$ implies $[x_i \succsim_i^- y_i]$ and $[x_i \succ_i^- y_i]$ implies $[x_i \succsim_i^+ y_i]$. As a consequence of $AC123_i$, the intersection of the (complete) relations $\succsim_i^+$ and $\succsim_i^-$ is a complete relation, that is nothing else than the marginal trace $\succsim_i^\pm$ since Definition (3.20) is equivalent to

$$a_i \succsim_i^\pm b_i \Leftrightarrow a_i \succsim_i^+ b_i \text{ and } a_i \succsim_i^- b_i.$$

The links between the above axioms and the marginal traces can be directly exploited in the construction of a monotone numerical representation of $\succsim$ in model ($L0$). We have the following result (Bouyssou and Pirlot, 2004, Theorem 2).

Proposition 3.2 (Representation in Models L) *A preference relation $\succsim$ on X admits a representation in model ($L0$) with F non-decreasing in its first n arguments and non-increasing in the last n arguments if and only if it is reflexive and satisfies $AC1$, $AC2$ and $AC3$.*

In order to make it clear to the reader, how the marginal traces intervene in the construction of the representation, we describe how a representation can be obtained with F monotone as indicated. Due to the fact that $\succsim$ satisfies $AC123$, we know that the marginal traces $\succsim_i^\pm$ are weak orders. Take for u_i, any numerical representation of the weak order $\succsim_i^\pm$, i.e., u_i is any real-valued function defined on X_i, such that

$$x_i \succsim_i^\pm z_i \text{ iff } u_i(x_i) \geq u_i(z_i).$$

Define then F as follows:

$$F([u_i(x_i)];[u_i(y_i)]) = \begin{cases} +\exp(\sum_{i=1}^n (u_i(x_i) - u_i(y_i))) & \text{if } x \succsim y, \\ -\exp(\sum_{i=1}^n (u_i(y_i) - u_i(x_i))) & \text{otherwise.} \end{cases} \tag{3.24}$$

It can easily be shown that this representation satisfies the requirements. Clearly, the choice of the exponential function in the definition of F is arbitrary; any other positive and non-decreasing function could have been chosen instead. Again the choice of a representation u_i of the weak orders $\succsim_i^\pm$ is highly arbitrary. We are thus far from the uniqueness results that can be obtained for the representation of preferences in the additive utility model (3.1). All these representations are however equivalent from the point of view of the description of a preference.

Table 3.3 Main models using traces on levels and their characterisation

Models	Definition	Conditions
$(L0)$	$x \succsim y \Leftrightarrow F([u_i(x_i)], [u_i(y_i)]) \geq 0$	$\varnothing$
$(L1)$	$(L0)$ with $F([u_i(x_i)], [u_i(x_i)]) = 0$	refl.
$(L2)$	$(L1)$ with $F([u_i(x_i)]; [u_i(y_i)]) = -F([u_i(y_i)]; [u_i(x_i)])$	cpl.
$(L3)$	$(L0)$ with $F(\nearrow, \searrow)$	
$\Updownarrow$		$AC123$
$(L4)$	$(L0)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	
$(L5)$	$(L1)$ with $F(\nearrow, \searrow)$	
$\Updownarrow$		refl., $AC123$
$(L6)$	$(L1)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	
$(L7)$	$(L2)$ with $F(\nearrow, \searrow)$	cpl., $AC123$
$(L8)$	$(L2)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	cpl., $AC4$

$\nearrow$ means nondecreasing, $\searrow$ means nonincreasing
$\nearrow\!\!\nearrow$ means increasing, $\searrow\!\!\searrow$ means decreasing
refl. means reflexive, *cpl.* means complete

3.3.4 Models Using Marginal Traces

At this point, it might be useful to give a full picture of the models based on marginal traces. We have identified above three variants of model $(L0)$: those corresponding respectively to reflexive or complete preference $\succsim$ or to a preference with complete marginal traces. To each variant, one can associate particular features of the numerical representation in model $(L0)$. Systematising the analysis, we may define the variants of model $(L0)$ listed in Table 3.3. This table also shows a characterisation of the models using the axioms introduced in the previous section.

3.3.5 Properties of Marginal Preferences in $(L0)$ and Variants

We briefly come back to the analysis of marginal preferences in connection with the variants of $(L0)$ characterised above. As stated before (Proposition 3.1), we know that for reflexive and transitive preferences, $\succsim_i = \succsim_i^{\pm}$. For reflexive preferences, $x_i \succsim_i^{\pm} z_i$ implies $x_i \succsim_i z_i$.

The incidence of axioms $AC1$, $AC2$, $AC3$ and $AC4$ on marginal preferences is summarised in the next proposition (Bouyssou and Pirlot, 2004, Proposition 3 and Lemma 4.3).

Proposition 3.3 (Properties of Marginal Preferences)

1. *If $\succsim$ is reflexive and either $AC1_i$ or $AC2_i$ holds then $\succsim_i$ is an interval order.*
2. *If, in addition, $\succsim$ satisfies $AC3_i$ then $\succsim_i$ is a semiorder.*
3. *If $\succsim$ is reflexive and $AC4_i$ holds then $\succsim_i$ is a weak-order and $\succsim_i = \succsim_i^{\pm}$.*

The preference $\succsim$ in Example 3.5, page 57 has marginal preferences $\succsim_i$ that are semiorders, while marginal traces are the natural weak orders on $\mathbb{R}$. From the latter, applying Proposition 3.2 (in its version for sets X of arbitrary cardinality), we deduce that $\succsim$ satisfies $AC123$. Applying the third part of Proposition 3.3, we deduce further that $\succsim$ does not satisfy $AC4$.

3.3.5.1 Separability and Independence

$AC1$, $AC2$, $AC3$ and $AC4$ also have an impact on the separability and independence properties of $\succsim$ (Bouyssou and Pirlot, 2004, Proposition 3.1 and Lemma 4.3).

Proposition 3.4 (Separability and Independence) *Let $\succsim$ be a reflexive relation on X. We have:*

1. *If $\succsim$ satisfies $AC1_i$ or $AC2_i$ then $\succsim$ is weakly separable for $i \in N$.*
2. *If $\succsim$ satisfies $AC4_i$ then $\succsim$ is independent for $\{i\}$,*

The preference $\succsim$ in Example 3.5 (p. 57) is weakly separable for all i (since $\succsim$ satisfies $AC123$ and in view of part 1 of proposition 3.4); although $\succsim$ does not satisfy $AC4$, it is easy to see, applying the definition, that $\succsim$ is also independent for all i.

3.3.5.2 The Case of Weak Orders

The case in which $\succsim$ is a weak order is quite special. We have the following result (Bouyssou and Pirlot, 2004, Lemma 5 and Lemma 4.3).

Proposition 3.5 (Case of Weakly Ordered Preferences) *Let $\succsim$ be a weak order on a set X. Then:*

1. [*$\succsim$ is weakly separable*] $\Leftrightarrow$ [*$\succsim$ satisfies $AC1$*] $\Leftrightarrow$ [*$\succsim$ satisfies $AC2$*] $\Leftrightarrow$ [*$\succsim$ satisfies $AC3$*],
2. [*$\succsim$ is weakly independent*] $\Leftrightarrow$ [*$\succsim$ satisfies $AC4$*],
3. *If $\succsim$ is weakly separable, the marginal preference $\succsim_i$ equals the marginal trace $\succsim_i^{\pm}$, for all i, and these relations are weak orders.*

This result recalls that for analysing weakly separable weak orders, marginal traces can be substituted by marginal preferences (as is classically done); it also shows that weak separability masks $AC123$.

Example 3.6 (Min and LexiMin) In Example 3.2, we have shown that comparing vectors of satisfaction degrees associated with a set of constraints could be done by comparing the lowest satisfaction degree in each vector, i.e.

$$x \succsim y \Leftrightarrow \min(x_1, \ldots, x_n) \geq \min(y_1, \ldots, y_n),$$

where x and y are n-tuples of numbers in the $[0, 1]$ interval. This method for comparing vectors is known as the "Min" or "MaxMin" method. Clearly, the preference $\succsim$ that this method yields is a weak order; it is not weakly independent as was shown in Example 3.2, but it is weakly separable since $\succsim_i^{\pm}$ is just the natural weak order on the interval $[0, 1]$; the relation $\succsim$ thus satisfies $AC123$ but not $AC4$. By Proposition 3.5.3, $\succsim_i^{\pm} = \succsim_i$, for all i.

A refinement of the "Min" or "MaxMin" method is the "LexiMin" method; the latter discriminates between alternatives that the former leaves tied. When comparing alternatives x and y, LexiMin ranks x before y if $\min x_i > \min y_i$; in case the minimal value of both profiles are equal, LexiMin looks at the second minimum and decides in favour of the alternative with the highest second minimum; if again the second minima are equal, it goes to the third and so on. Only alternatives that cannot be distinguished when their coordinates are rearranged in non-decreasing order will be indifferent for LexiMin.

The preference yielded by LexiMin is again an independent weak order and $\succsim_i^{\pm} = \succsim_i$, for all i.

3.3.6 Eliciting the Variants of Model ($L0$)

This family of models suggests an elicitation strategy similar to that for the decomposable model but based on the marginal traces instead of the marginal preferences. It is not likely however that such a general model could serve as a basis for a direct practical elicitation process; we think instead that it is a framework for conceiving more specific models associated to a method; the additive value function model could be considered in this framework. Although it may seem unrealistic to work in such a general framework, Greco et al. (1999) have proposed to do so and elicit preferences using an adapted rough sets approach (indirect approach).

3.4 Following Another Path: Models Using Marginal Traces on Differences

The generalisation of the additive value model has been pursued to its most extreme limits since, with model ($L0$), we encompass all possible binary relations on a product set. This process has relied on the marginal traces on the sets X_i. Those relations have been shown to be the stepping stones to lean on for eliciting this type of model, for relations that are not transitive. For transitive (and reflexive) relations, marginal traces reduce to the usual marginal preferences.

There is however another line of generalisation of the additive value model. Obviously, it cannot be advocated as more general than the models based on marginal traces; it nevertheless sheds another light on the picture since it is based on an entirely different fundamental notion: *traces on differences*. Instead of comparing profiles of performance of alternatives like in the additive value model or the decomposable model or even, in a more implicit form, in model ($L0$), we can see the preference of x over y as resulting from a balance made between advantages and disadvantages of x w.r.t. y on all criteria. While the approach followed in the additive value model could be described as *Aggregate then Compare*, the latter is more relevant to the opposite paradigm *Compare* (on each dimension) *then Aggregate* (Perny, 1992; Dubois et al., 2003).

3.4.1 The Additive Difference Model

In conjoint measurement as well, this paradigm is not new. It is related to the introduction of intransitivity of the preference. Tversky (1969) was one of the first to propose a model generalising the additive value one and able to encompass preferences that lack transitivity. It is known as the *additive difference model* in which,

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0, \tag{3.25}$$

where Φ_i are increasing and odd functions.

Preferences that satisfy (3.25) may be intransitive but they are complete (due to the postulated oddness of Φ_i). When attention is restricted to the comparison of objects that only differ on one dimension, (3.25) implies that the preference between these objects is independent from their common level on the remaining $n-1$ dimensions. This amounts saying that $\succsim$ is independent for all i; the marginal preferences $\succsim_i$, clearly, are complete and transitive (hence weak orders) due to the oddness and the increasingness of the Φ_i. This, in particular, excludes the possibility of any perception threshold on dimensions, which would lead to an intransitive indifference relation on those dimensions. Imposing that Φ_i are nondecreasing instead of being increasing allows for such a possibility. This gives rise to what Bouyssou (1986) called the *weak additive difference model*.

Model (3.25) adds up the differences of preference represented by the functions $\Phi_i(u_i(x_i) - u_i(y_i))$; these differences are themselves obtained by recoding, through the functions Φ_i, the algebraic difference of partial value functions u_i. Due to the presence of two algebraic operations—the sum of the Φ_i and the difference of the

u_i—one should be confronted with the same difficulties as for the additive value function model when axiomatising (3.25). In case X is infinite, as in Sect. 3.2.6, characterisations are obtained by combining necessary cancellation conditions with unnecessary structural assumptions on the set X (Krantz et al., 1971, ch. 9).

Dropping the subtractivity requirement in (3.25) (as suggested in Bouyssou, 1986; Fishburn, 1990a,b, 1991b; Vind, 1991) is a partial answer to the limitations of the additive difference model. This leads to *nontransitive additive* conjoint measurement models in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0, \tag{3.26}$$

where the p_i's are real-valued functions on X_i^2 and may have several additional properties (e.g. $p_i(x_i, x_i) = 0$, for all $i \in \{1, 2, \ldots, n\}$ and all $x_i \in X_i$).

This model is an obvious generalisation of the (weak) additive difference model. It allows for intransitive and incomplete preference relations $\succsim$ as well as for intransitive and incomplete marginal preferences. An interesting specialisation of (3.26) obtains when p_i are required to be *skew symmetric* i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$. This skew symmetric nontransitive additive conjoint measurement model implies the completeness and the independence of $\succsim$. In view of the addition operation involved in the model, the difficulties for obtaining a satisfactory axiomatisation of the model remain essentially as in model (3.25). Fishburn (1990a, 1991b) axiomatises the skew symmetric version of (3.26) both in the finite and the infinite case; Vind (1991) provides axioms for (3.26) with $p_i(x_i, x_i) = 0$ when $n \geq 4$; Bouyssou (1986) gives necessary and sufficient conditions for (3.26) with and without skew symmetry in the denumerable case, when $n = 2$.

3.4.2 Comparison of Preference Differences

With the nontransitive additive model (3.26), the notion of preference "difference" becomes more abstract than it looks like in Tversky's model (3.25); we still refer to p_i as to a representation of preference differences on i even though there is no algebraic difference operation involved.

This prompts the following question: is there any intrinsic way of defining the notion of "difference of preference" by referring only to the preference relation $\succsim$? The answer is pretty much in the spirit of what we discovered in the previous section: comparing difference of preferences can be done in term of traces, here, of traces on "differences". We define a relation $\succsim_i^*$, that we shall call *marginal trace on differences* comparing any two pairs of levels (x_i, y_i) and $(z_i, w_i) \in X_i^2$ in the following way.

Definition 3.9 (Marginal Trace on Differences $\succsim_i^*$) The marginal trace on differences $\succsim_i^*$ is the relation on the pairs of levels X_i^2 defined by:

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ iff } \begin{cases} \text{for all } a_{-i}, b_{-i} \in X_{-i}, \\ (z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i}). \end{cases} \quad (3.27)$$

Intuitively, if $(x_i, y_i) \succsim_i^* (z_i, w_i)$, it seems reasonable to conclude that the preference difference between x_i and y_i is not smaller that the preference difference between z_i and w_i. Notice that, by construction, $\succsim_i^*$ is reflexive and transitive.

Contrary to our intuition concerning preference differences, the definition of $\succsim_i^*$ does not imply that there is any link between two "opposite" differences (x_i, y_i) and (y_i, x_i). Henceforth we introduce the binary relation $\succsim_i^{**}$ on X_i^2.

Definition 3.10 (Marginal Trace on Differences $\succsim_i^{**}$) The marginal trace on differences $\succsim_i^{**}$ is the relation on the pairs of levels X_i^2 defined by:

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \text{ iff } [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)]. \quad (3.28)$$

It is easy to see that $\succsim_i^{**}$ is transitive and *reversible*, i.e.

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow (w_i, z_i) \succsim_i^{**} (y_i, x_i). \quad (3.29)$$

The relations $\succsim_i^*$ and $\succsim_i^{**}$ both appear to capture the idea of comparison of preference differences between elements of X_i induced by the relation $\succsim$. Hence, they are good candidates to serve as the basis of the definition of the functions p_i. They will not serve well this purpose however unless they are complete as we shall see.

3.4.3 A General Family of Models Using Traces on Differences

In the same spirit as we generalised the decomposable model to the models based on marginal traces, we envisage here a very general model based on preference differences. It formalises the idea of measuring "preference differences" separately on each dimension and then combining these (positive or negative) differences in order to know whether the aggregation of these differences leads to an advantage for x over y. More formally, this suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \quad (D0)$$

where p_i are real-valued functions on X_i^2 and G is a real-valued function on $\prod_{i=1}^n p_i(X_i^2)$.

As already noted by Goldstein (1991), all binary relations satisfy model ($D0$) when X is finite or countably infinite. Necessary and sufficient conditions for the non-denumerable case are well-known (Bouyssou and Pirlot, 2002b).

As for the variants of model ($L0$), it is easy to impose conditions on G that will result in simple properties of $\succsim$. Assume $\succsim$ has a representation in model ($D0$); then

- $\succsim$ is reflexive iff $G([p_i(x_i, x_i)]) \geq 0,$ for all x_i;
- $\succsim$ is independent iff $p_i(x_i, x_i) = 0$ for all x_i; in addition, $\succsim$ is reflexive iff $G(\mathbf{0}) \geq 0$ and $\succsim$ is irreflexive iff $G(\mathbf{0}) < 0$.
- $\succsim$ is complete iff p_i is skew-symmetric and G is odd, i.e. $p_i(x_i, y_i) = -p_i(y_i, x_i)$ for all x_i, y_i and $G(-\mathbf{p}) = -G(\mathbf{p})$ for all $\mathbf{p} = (p_1, \ldots, p_n)$.

Again, as for the models based on marginal traces, the monotonicity of G is related to the properties of traces on differences (3.27) and (3.28). The axioms needed to guarantee the monotonicity of G are very much looking like $AC1$, $AC2$ or $AC3$ because traces are involved.

Definition 3.11 We say that relation $\succsim$ on X satisfies:
$RC1_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, c_{-i}) \succsim (y_i, d_{-i})\\ \text{or}\\ (z_i, a_{-i}) \succsim (w_i, b_{-i}),\end{array}\right.$$

$RC2_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (w_i, c_{-i}) \succsim (z_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

$RC3_i$ if $\succsim$ satisfies $RC2_i$ and when one of the conclusions of $RC2_i$ is false then the other holds with $\succ$ instead of $\succsim$.

We say that $\succsim$ satisfies $RC1$ (resp. $RC2$) if it satisfies $RC1_i$ (resp. $RC2_i$) for all $i \in N$. We also use $RC12$ as shorthand for $RC1$ and $RC2$.

Condition $RC1_i$ implies that any two ordered pairs (x_i, y_i) and (z_i, w_i) of elements of X_i are comparable in terms of the relation $\succsim_i^*$. Indeed, it is easy to see that supposing $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)]$ and $Not[(z_i, w_i) \succsim_i^* (x_i, y_i)]$ is the negation of $RC1_i$. Similarly, $RC2_i$ implies that the two opposite differences (x_i, y_i) and (y_i, x_i) are linked. In terms of the relation $\succsim_i^*$, it says that if the preference difference between x_i and y_i is not at least as large as the preference difference between z_i and w_i then the preference difference between y_i and x_i should be at least as large as the preference difference between w_i and z_i (Bouyssou and Pirlot, 2002b, Lemma 1).

Proposition 3.6 (Completeness of the Traces on Differences) *We have:*

1. $[\succsim_i^*$ *is a weak order*$] \Leftrightarrow RC1_i$,
2. $[\succsim_i^{**}$ *is a weak order*$] \Leftrightarrow [RC1_i$ *and* $RC2_i]$.

Here again (like for the models based on marginal traces, see Sect. 3.3.3) the links between $RC1$, $RC2$ and properties of $\succsim_i^*$ and $\succsim_i^{**}$ play a fundamental role in the construction of a representation of a preference in model $(D0)$ with a monotone G function. Axiom $RC2$ introduces a *mirror effect* on preference differences: under $RC2_i$, the difference of preference (y_i, x_i) is the mirror image of (x_i, y_i) (Bouyssou and Pirlot, 2002b, Theorem 1).

Proposition 3.7 (Representation in Model D**)** *A preference relation* $\succsim$ *on* X *admits a representation in model* $(D0)$ *with* G *nondecreasing in all its* n *arguments iff* $\succsim$ *satisfies* $RC1$. *It admits such a representation with, in addition,* $p_i(x_i, y_i) = -p_i(y_i, x_i)$ *iff* $\succsim$ *satisfies* $RC1$ *and* $RC2$.

The construction of a representation under the hypotheses of the theorem helps to make the theorem more intuitive. We outline this construction below.

Suppose that $\succsim$ satisfies $RC1$. We know, by Proposition 3.6.1 that $\succsim_i^*$ is a weak order on the set of pairs of levels X_i^2 for all i. Select, for all i, a real-valued function p_i that represents the weak order $\succsim_i^*$, i.e. that satisfies:

$$p_i(x_i, y_i) \geq p_i(z_i, w_i) \text{ iff } (x_i, y_i) \succsim_i^* (z_i, w_i),$$

for all $x_i, y_i, z_i, w_i \in X_i$. Then define G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} \exp \sum_{i=1}^n p_i(x_i, y_i) & \text{if } x \succsim y \\ -\exp[-\sum_{i=1}^n p_i(x_i, y_i)] & \text{otherwise.} \end{cases} \tag{3.30}$$

It can easily be shown that G is well-defined. The choice of the exponential function and the sum operator is purely arbitrary; any other increasing function defined on the real numbers and taking positive values would do as well. The role of such a function is to ensure that in each of the two sub-domains $x \succsim y$ and "otherwise", the function G is increasing in the p_i's; since the relation $\succsim$ is itself non-decreasing with respect to the relations $\succsim_i^*$ for all i, raising the value of a p_i (which represents $\succsim_i^*$) may only result in remaining in the same sub-domain or passing from the domain "otherwise" to the domain "$x \succsim y$"; the value of G is negative in the former sub-domain and positive in the latter and in each sub-domain, G is increasing. This proves that G is increasing in all its arguments p_i.

The second case, in which $\succsim$ satisfies $RC1$ and $RC2$ is dealt with similarly. Since in this case $\succsim_i^{**}$ is a weak order, we use functions p_i that represent $\succsim_i^{**}$ instead of $\succsim_i^{*}$. We may moreover exploit the reversibility property (3.29) of $\succsim_i^{**}$ to ensure that we may choose a skew-symmetric function p_i to represent $\succsim_i^{**}$. Then we define G as in (3.30). In the same case, we may also get a representation in which G is increasing (instead of non-decreasing) by defining G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} \exp \sum_{i=1}^{n} p_i(x_i, y_i) & \text{if } x \succ y \\ 0 & \text{if } x \sim y \\ -\exp\left[-\sum_{i=1}^{n} p_i(x_i, y_i)\right] & \text{otherwise.} \end{cases} \tag{3.31}$$

Combining the various additional properties that can be imposed on $\succsim$, we are lead to consider a number of variants of the basic $(D0)$ model. These models can be fully characterised using the axioms $RC1$, $RC2$ and $RC3$. The definition of the models as well as their characterisation are displayed in Table 3.4.

Table 3.4 Main models using traces on differences and their characterisation

Models	Definition	Conditions
$(D0)$	$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0$	$\varnothing$
$(D1)$	$(D0)$ with $p_i(x_i, x_i) = 0$	
$\Updownarrow$		ind.
$(D2)$	$(D0)$ with p_i skew symmetric	
$(D3)$	$(D0)$ with p_i skew symmetric and G odd	cpl., ind.
$(D4)$	$(D0)$ with $G(\nearrow)$	
$\Updownarrow$		$RC1$
$(D8)$	$(D0)$ with $G(\nearrow\nearrow)$	
$(D5)$	$(D1)$ with $G(\nearrow)$	
$\Updownarrow$		$RC1$, ind.
$(D9)$	$(D1)$ with $G(\nearrow\nearrow)$	
$(D6)$	$(D2)$ with $G(\nearrow)$	
$\Updownarrow$		$RC12$
$(D10)$	$(D2)$ with $G(\nearrow\nearrow)$	
$(D7)$	$(D3)$ with $G(\nearrow)$	cpl., $RC12$
$(D11)$	$(D3)$ with $G(\nearrow\nearrow)$	cpl., $RC3$

$\nearrow$ means nondecreasing, $\nearrow\nearrow$ means increasing
cpl. means completeness, *ind.* means independence

3.4.4 Eliciting Models Using Traces on Differences

We suppose that $\succsim$ is reflexive and satisfies $RC1$, i.e., we place ourselves in model $(D5)$ [equivalent to $(D9)$]. In that model $\succsim_i^*$ is a weak order on the "differences of preference" $(x_i, y_i) \in X_i^2$, for all i, and the functions p_i may be chosen to be numerical representations of $\succsim_i^*$. To each pair of alternatives $x, y \in X$ is henceforth associated a profile $\overline{p} = (p_1, \ldots, p_n)$ of differences of preferences ($p_i = p_i(x_i, y_i)$, for $i = 1, \ldots, n$). The function G may be conceived of as a rule that assigns a value to each profile; in model $(D5)$, G is just supposed to be nonincreasing [not necessarily increasing if we choose to represent $\succsim$ into model $(D5)$ instead of the equivalent model $(D9)$] and therefore we may choose a very simple form of G that codes profiles in the following way:

$$G(\overline{p}) = \begin{cases} +1 & \text{if } \overline{p} \text{ corresponds to } x \succ y; \\ 0 & \text{if } \overline{p} \text{ corresponds to } x \sim y; \\ -1 & \text{if } \overline{p} \text{ corresponds to } Not[\, x \succsim y \,]. \end{cases} \tag{3.32}$$

The strategy for eliciting such a model (in a direct manner) may thus be as follows:

1. for all i, elicit the weak order $\succsim_i^*$ that ranks the differences of preference; choose a representation p_i of $\succsim_i^*$
2. elicit the rule (function) G that assigns a category (coded $+1$, 0 or -1) to each profile $\overline{p}$.

The initial step however is more complex than with the decomposable model, because we have to rank-order the set X_i^2 instead of X_i. In case it may be assumed that the difference of preference is reversible [see (3.29)] almost half of the work can be saved since only the "positive" (or only the "negative") differences must be rank-ordered.[4] The difficulty, that remains even in the reversible case, may motivate the consideration of another family of models that rely both on marginal traces and on traces on differences (see Bouyssou et al., 2006, Sect. 6.4). In some of these models, $\succsim_i^*$ is reacting positively (or non-negatively) to marginal traces and therefore, the elicitation of p_i may benefit of its monotonicity w.r.t. marginal traces.

Models $(D4)$, $(D5)$, $(D6)$ and $(D7)$, in which G is a nondecreasing function, can be elicited in a similar fashion. The situation is different when a representation with G increasing is sought, in particular for model $(D11)$. For such representations, the definition of G by (3.32) is no longer appropriate and defining G requires more care and effort. We do not enter into this point.

[4] In case of a tie, i.e. whenever $(x_i, y_i) \sim_i^* (z_i, w_i)$, one has however to look explicitly at the relation between the reverse differences (y_i, x_i) and (w_i, z_i) since all cases ($\succsim_i^*$, $\sim_i^*$ or $\precsim_i^*$) can possibly show up.

3.4.5 Examples of Models that Distinguish No More Than Three Classes of Differences

We show in this section that simple majority (or Condorcet method), weighted majority, qualified majority and lexicographic method can be represented in some of the models $(D1)$ to $(D11)$. We consider in addition, a variant of the ELECTRE I procedure in which the profile of preferences on each dimension are not weak orders but semiorders. In each of these cases, the relation that orders the differences of preference on each criterion is revealed by the global preference relation.

We say that a relation $\succsim$ defined on a product set $X = \prod_{i=1}^{n} X_i$ is the result of the application of a majority or a lexicographic rule if there is a relation S_i on each X_i such that $\succsim$ can be obtained by aggregating the n relations S_i using that rule. Those S_i's will usually be *weak orders* but we shall also consider more general structures like semiorders. In the sequel, we refer to S_i as to the *a priori preference relation* on X_i.

Take the example of the simple majority rule. We say that $\succsim$ is a simple majority preference relation if there are relations S_i that are weak orders on the corresponding X_i such that:

$$x \succsim y \text{ iff } \begin{cases} \text{the number of criteria on which } x_i \, S_i \, y_i \\ \text{is at least as large as} \\ \text{the number of criteria such that } y_i \, S_i \, x_i. \end{cases} \tag{3.33}$$

In the rest of this section, P_i will denote the asymmetric part of a relation S_i defined on X_i and its symmetric part will be denoted by I_i. In the first five examples, the S_i's are supposed to be weak orders.

3.4.5.1 Simple Majority or Condorcet Method

A relation $\succsim$ on X is a *simple majority relation* if there is a weak order S_i on each X_i such that

$$x \succsim y \text{ iff } |\{i \in N : x_i \, S_i \, y_i\}| \geq |\{i \in N : y_i \, S_i \, x_i\}|. \tag{3.34}$$

In other terms, $x \succsim y$ if the "coalition" of criteria on which x is at least as good as y is at least as large as the "opposite coalition", i.e. the set of criteria on which y is at least as good as x. The term "coalition" is used here for "set", in reference with social choice. We do not, apparently, distinguish between the case in which x_i is better than y_i ($x_i \, P_i \, y_i$) and that in which they are indifferent ($x_i \, I_i \, y_i$). Note that the criteria for which x_i is indifferent with y_i appear in both coalitions and hence cancel. We could thus define a simple majority relation in an equivalent fashion by $x \succsim y$ iff $|\{i \in N : x_i \, P_i \, y_i\}| \geq |\{i \in N : y_i \, P_i \, x_i\}|$.

Such a relation can be represented in model ($D11$) by defining

$$p_i(x_i, y_i) = \begin{cases} 1 \text{ if } x_i P_i y_i \\ 0 \text{ if } x_i I_i y_i \\ -1 \text{ if } y_i P_i x_i \end{cases} \tag{3.35}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{3.36}$$

We have indeed that $x \succsim y$ iff $G([p_i(x_i, y_i)]) = |\{i \in N : x_i P_i y_i\}| - |\{i \in N : y_i P_i x_i\}| \geq 0$, which is clearly equivalent to Definition (3.34).

This representation of a simple majority relation can furthermore be called *regular* in the sense that the functions p_i are numerical representations of the weak orders $\succsim_i^{**}$; the latter has exactly three equivalence classes, namely, the set of pairs (x_i, y_i) such that $x_i P_i y_i$, the set of pairs for which $x_i I_i y_i$ and the set of those such that $y_i P_i x_i$. Observe that the relation $\succsim_i^*$ distinguishes the same three classes; hence $\succsim_i^* = \succsim_i^{**}$.

3.4.5.2 Weighted Simple Majority or Weighted Condorcet Method

A relation $\succsim$ on X is a *weighted simple majority relation* if there is a vector of normalised weights $[w_i]$ (with $w_i \geq 0$ and $\sum_{i \in N} w_i = 1$) and a weak order S_i on each X_i such that

$$x \succsim y \text{ iff } \sum_{i \in N : x_i S_i y_i} w_i \geq \sum_{j \in N : y_j S_j x_j} w_j. \tag{3.37}$$

In this model, the coalitions of criteria are weighted: they are assigned a value that is the sum of those assigned to the criteria belonging to the coalition. The preference of x over y results from the comparison of the coalitions like in the simple majority rule: $x \succsim y$ if the coalition of criteria on which x is at least as good as y *does not weigh less* than the opposite coalition. Like for simple majority, we could have defined the relation using strict *a priori* preference, saying that $x \succsim y$ iff $\sum_{i \in N : x_i P_i y_i} w_i \geq \sum_{j \in N : y_j P_j x_j} w_j$.

A representation of a weighted majority relation in model ($D11$) is readily obtained letting:

$$p_i(x_i, y_i) = \begin{cases} w_i \text{ if } x_i P_i y_i \\ 0 \text{ if } x_i I_i y_i \\ -w_i \text{ if } y_i P_i x_i \end{cases} \tag{3.38}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{3.39}$$

We have that $x \succsim y$ iff $G([p_i(x_i, y_i)]) = \sum_{i \in N: x_i P_i y_i} w_i - \sum_{j \in N: y_j P_j x_j} w_j \geq 0$.

This representation is *regular* since p_i is a numerical representation of $\succsim_i^{**}$ and $\succsim_i^{**}$ has only three equivalence classes as in the case of simple majority.

3.4.5.3 Weighted Qualified Majority

A relation $\succsim$ on X is a *weighted qualified majority relation* if there is a vector of normalised weights $[w_i]$ (i.e. with w_i non-negative and summing up to 1), a weak order S_i on each X_i and a threshold δ between $\frac{1}{2}$ and 1 such that

$$x \succsim y \text{ iff } \sum_{i \in N: x_i S_i y_i} w_i \geq \delta. \tag{3.40}$$

In contrast with the previous models, the preference does not result here from a comparison of coalitions but from stating that the coalition in favour of an alternative is *strong enough*, i.e. that the measure of its strength reaches a certain threshold δ (typically above one half). Even when δ is set to 0.5, this method is not equivalent to weighted simple majority, with the same weighting vector $[w_i]$, due to the inclusion of the criteria on which x and y are indifferent in both the coalition in favour of x against y and that in favour of y against x. Take for example two alternatives x, y compared on five points of view; suppose that the criteria all have the same weight, i.e. $w_i = 1/5$, for $i = 1, \ldots, 5$. Assume that x is preferred to y on the first criterion ($x_1 P_1 y_1$), x is indifferent to y on the second and third criteria ($x_2 I_2 y_2$; $x_3 I_3 y_3$) and y is preferred to x on the last two criteria ($y_4 P_4 x_4$; $y_5 P_5 x_5$). Using the weighted majority rule [Eq. (3.37)], we get $y \succ x$ since the coalition in favour of x against y is composed of criteria 1, 2, 3 (weighting 0.6) and the opposite coalition contains criteria 2, 3, 4, 5 (weighting 0.8). Using the weighted qualified majority with threshold δ up to 0.6, we get that $x \sim y$ since both coalitions weigh at least 0.6.

Note that when the criteria have equal weights ($w_i = 1/n$), weighted qualified majority could be simply called *qualified majority*; the latter has the same relationship with weighted qualified majority as weighted simple majority with simple majority.

Qualified weighted majority relations constitute a basic component of the ELECTRE I and ELECTRE II methods (Roy, 1971) as long as there are no vetoes.

Any weighted qualified majority relation admits a representation in model ($D8$). Let:

$$p_i(x_i, y_i) = \begin{cases} w_i - \frac{\delta}{n} \text{ if } x_i S_i y_i \\ \\ -\frac{\delta}{n} \text{ if } Not[x_i S_i y_i] \end{cases} \tag{3.41}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{3.42}$$

We have that

$$\begin{aligned} x \succsim y \text{ iff } G([p_i(x_i, y_i)]) &= \sum_{i \in N: x_i S_i y_i} (w_i - \frac{\delta}{n}) - \sum_{j \in N: Not[x_j S_j y_j]} \frac{\delta}{n} \\ &= \sum_{i \in N: x_i S_i y_i} w_i - \delta \\ &\geq 0. \end{aligned} \tag{3.43}$$

In this representation, p_i is a numerical representation of $\succsim_i^*$ but not of $\succsim_i^{**}$. The former has two equivalence classes: the pairs (x_i, y_i) that are in S_i form the upper class of the weak order; those that are not in S_i form the lower class. Note that there are no further distinctions between pairs; all pairs in the upper class contribute the same amount $w_i - \frac{\delta}{n}$ to the value of the coalition while the pairs of the lower class all contribute the same amount $-\frac{\delta}{n}$. The comparison of preference differences in this model is thus rather poor (as is the case of course with the two previous models).

The relation $\succsim_i^{**}$ is also a weak order; it has three equivalence classes. It makes a distinction between $x_i P_i y_i$ and $x_i I_i y_i$ (a distinction that is not made by $\succsim_i^*$): both cases play the same role when comparing (x_i, y_i) to other pairs (since what counts in formula (3.40) is whether or not (x_i, y_i) belongs to S_i); it is no longer the case when comparing (y_i, x_i) to other pairs since, then, $x_i I_i y_i$ counts in the coalition in favour of y against x while $x_i P_i y_i$ does not.

3.4.5.4 Lexicographic Preference Relations

A preference relation is lexicographic if the criteria are linearly ordered and if they are considered in that order when comparing alternatives: the first criterion, in that order, that favours one alternative with respect to another determines the global preference. Denoting by $>_l$ a linear order on the set of criteria, we rank-order the criteria according to it: $1^l >_l 2^l >_l \ldots >_l n^l$. We thus have the following definition. A relation $\succsim$ on X is a *lexicographic* preference relation if there is a linear order $>_l$ on the set of criteria and a weak order (or a semiorder) S_i on each X_i such that:

$$x \succ y \text{ if } \begin{cases} x_{1^l} P_{1^l} y_{1^l} \text{ or} \\ x_{1^l} I_{1^l} y_{1^l} \text{ and } x_{2^l} P_{2^l} y_{2^l} \text{ or} \\ x_{i^l} I_{1^l} y_{i^l} \, \forall i = 1, \ldots, k-1 \text{ and } x_{k^l} P_{k^l} y_{k^l}, \\ \text{for some } k \text{ such that } 2 \le k \le n. \end{cases} \tag{3.44}$$

and $x \sim y$ if $x_{i^l} I_{i^l} y_{i^l}$, for all $i \in N$. In words, $x \sim y$ if x_i is a priori indifferent to y_i, for all i; $x \succ y$ if, for the first index k^l for which x_{i^l} is not a priori indifferent to y_{i^l}, one has x_{k^l} a priori preferred to y_{i^l}.

Such a relation can be viewed (as long as there are only finitely many criteria) as a special case of a weighted majority relation. Choose a vector of weights w_i in the following manner: for all $i \in N$, let w_{i^l} be larger than the sum of all remaining weights (in the order $>_l$), i.e.:

$$\begin{aligned} w_{1^l} &> w_{2^l} + w_{3^l} + \ldots + w_{n^l} \\ w_{2^l} &> w_{3^l} + \ldots + w_{n^l} \\ &\ldots \\ w_{(n-1)^l} &> w_{n^l} \end{aligned} \tag{3.45}$$

Using these weights in (3.38) and (3.39) which define a representation for weighted majority relations, one gets a representation for lexicographic relations in model $(D11)$.

3.4.5.5 Other Forms of Weighted Qualified Majority

Instead of imposing—in an absolute manner—a threshold above 0.5 for defining a weighted qualified majority, as is done in Sect. 3.4.5.3, we may alternatively impose a relative majority threshold, in an additive or a multiplicative form. A preference relation $\succsim$ on X is a *weighted majority relation with additive threshold* if there is vector of normalised weights $[w_i]$ (with $w_i \ge 0$ and $\sum_{i \in N} w_i = 1$), a weak order or semiorder S_i on each X_i and a non-negative threshold γ such that

$$x \succsim y \text{ iff } \sum_{i \in N : x_i S_i y_i} w_i \ge \sum_{j \in N : y_j S_j x_j} w_j - \gamma. \tag{3.46}$$

Table 3.5 Models distinguishing no more than three classes of differences of preferences

Aggregation rule	General model	Special models
Weighted simple majority (see Sect. 3.4.5.2)	$(D11)$	$(D11)$ + additive
Weighted qualified majority (see Sect. 3.4.5.3)	$(D10)$	$(D8)$ + additive
Lexicographic (see Sect. 3.4.5.4)	$(D11)$	$(D11)$ + additive
Weighted majority with add. threshold (see Sect. 3.4.5.5)	$(D7)$	$(D10)$ + additive (with constant: Eq. (3.46))
Weighted majority with mult. threshold (see Sect. 3.4.5.5)	$(D7)$	$(D6)$ + linear

A relation $\succsim$ is a *weighted majority relation with multiplicative threshold* $\rho \geq 1$ if

$$x \succsim y \text{ iff } \sum_{i \in N : x_i S_i y_i} w_i \geq \frac{1}{\rho} \sum_{j \in N : y_j S_j x_j} w_j, \tag{3.47}$$

with $[w_i]$ and S_i as in the case of an additive threshold.

Constructing preference relations using these rules resembles what is known as the TACTIC method; it was proposed and studied in Vansnick (1986) with the possible adjunction of vetoes. In the original version of TACTIC, the preference is defined as an asymmetric relation $\succ$; the symmetric version that we consider here obtains from the original one just by saying that $x \succsim y$ if and only if we have not $y \succ x$.

It is easy to provide a representation of a weighted majority relation with additive threshold in model $(D10)$ or $(D7)$, but a representation in model $(D11)$ is in general not possible (Bouyssou et al., 2006). Turning to weighted majority relations with multiplicative threshold, one observes that $\succsim$ is complete and can be represented in model $(D6)$ or $(D7)$ but $\succsim$ does not fit in model $(D11)$ since, in general, indifference is not "narrow" (Bouyssou et al., 2006).

Table 3.5 provides a summary of the main models applicable to preferences that distinguish no more than three classes of differences of preference on each dimension.

3.4.6 *Examples of Models Using Vetoes*

Vetoes could be introduced in all the examples dealt with in the previous section (Sect. 3.4.5). We shall only consider the cases of qualified weighted majority relations (see Sect. 3.4.5.3) with vetoes (the relations that are the basic ingredients in the ELECTRE I and II methods).

The intuition one can have about a *veto* is the following. Consider an alternative x and a criterion i on which the level of the performance x_i of x is much worse than the level y_i of another alternative y. A veto of y on x on criterion i consists in rejecting the possibility that x be globally preferred to y irrespective of the performances of x and y on the criteria other than i. In other words, a veto on criterion i forbids to declare that $x \succsim y$ if (x_i, y_i) is a "negative" difference that is "large enough in absolute value", with respect to relation $\succsim_i^*$ or $\succsim_i^{**}$ (in the latter case, this is equivalent to saying that (y_i, x_i) is a large enough "positive" difference). Of course, in case the difference (x_i, y_i) leads to a veto on declaring x preferred to y, it is certainly because we do not have $x_i S_i y_i$, but, instead, $y_i P_i x_i$, and "even more". We thus define the veto relation V_i as a subset of relation P_i consisting of all pairs $(y_i; x_i)$ such that the presence of the reverse pair (x_i, y_i) in two alternatives x and y prohibits $x \succsim y$; V_i is an asymmetric relation.

Suppose that, for all i, X_i is a subset of the real numbers (X can be seen, in a sense, as a performance table) and that S_i is a semiorder determined by the following condition:

$$x_i S_i y_i \Leftrightarrow x_i \geq y_i - \tau_{i,1} \tag{3.48}$$

where $\tau_{i,1}$ is a non-negative threshold. This is similar to the situation described in Sect. 3.4.5.3 with the example of the cost (except that the cost is to be minimised; here we prefer the larger values): the values x_i and y_i are indifferent ($x_i I_i y_i$) if they differ by less than the threshold $\tau_{i,1}$; x_i is strictly preferred to y_i ($x_i P_i y_i$) if it passes y_i by at least the value of the threshold. In such a case, a convenient way of defining the veto relation V_i, a subset of P_i, is by means of another threshold $\tau_{i,2}$ that is larger than $\tau_{i,1}$. We say that the pair (y_i, x_i) belongs to the veto relation V_i if the following condition is satisfied:

$$y_i V_i x_i \Leftrightarrow y_i > x_i + \tau_{i,2}. \tag{3.49}$$

Clearly, the veto relation defined above is included in P_i. Assume indeed that $y_i V_i x_i$; since $\tau_{i,2}$ is larger than $\tau_{i,1}$, we have $y_i > x_i + \tau_{i,2} > x_i + \tau_{i,1}$, yielding $y_i P_i x_i$. We call $\tau_{i,2}$, a *veto threshold*; the relation V_i defined by (3.49) is a strict semiorder, i.e. the asymmetric part of a semiorder; it is contained in P_i that is also a strict semiorder, namely, the asymmetric part of the semiorder S_i. In such a situation, when comparing an arbitrary level x_i to a fixed level y_i, we can distinguish four relative positions of x_i with respect to y_i that are of interest. These four zones are shown on Fig. 3.9; they correspond to relations described above, namely:

If x_i belongs to:	Then:
Zone I	$x_i P_i y_i$
Zone II	$x_i I_i y_i$
Zone III	$y_i P_i x_i$ and *Not*$[\, y_i V_i x_i \,]$
Zone IV	$y_i P_i x_i$ and $y_i V_i x_i$

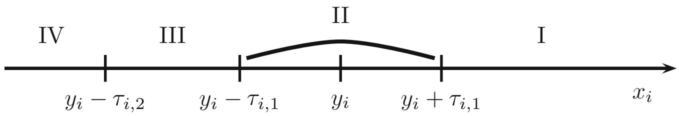


Fig. 3.9 Relative positions of an arbitrary level x_i with respect to a fixed level y_i

3.4.6.1 Weighted Qualified Majority with Veto

Starting with both an a priori preference relation S_i (a semiorder) and an a priori veto relation V_i (a strict semiorder included in P_i) on each set X_i, we can define a global preference relation of the ELECTRE I type as follows:

$$x \succsim y \text{ iff } \begin{cases} \sum_{i \in N : x_i S_i y_i} w_i \geq \delta \\ \text{and} \\ \text{there is no dimension } i \text{ on which } \; y_i V_i x_i, \end{cases} \tag{3.50}$$

where $(w_1, \ldots, w_n)$ denotes a vector of normalised weights and δ, a threshold between $1/2$ and 1. The global preference of the ELECTRE I type is thus a weighted qualified majority relation (in which the a priori preferences may be semiorders instead of weak orders) that is "broken" as soon as there is a veto on any single criterion, i.e. as soon as the performance of an alternative on some dimension is sufficiently low as compared to the other. It is not difficult to provide a representation of such a preference relation $\succsim$ in model ($D8$) letting:

$$p_i(x_i, y_i) = \begin{cases} w_i \text{ if } x_i S_i y_i \\ 0 \text{ if } y_i P_i x_i \text{ but } Not[\, y_i V_i x_i \,] \\ -M \text{ if } y_i V_i x_i, \end{cases} \tag{3.51}$$

where M is a large positive constant and

$$G([p_i]) = \sum_{i \in N} p_i - \delta. \tag{3.52}$$

If no veto occurs in comparing x and y, then $G([p_i(x_i, y_i)]) = \sum_{i : x_i S_i y_i} w_i - \delta$, which is the same representation as for the weighted qualified majority without veto (Sect. 3.4.5.3). Otherwise, if, on at least one criterion j, one has $y_j V_j x_j$, then $G([p_i(x_i, y_i)]) < 0$, regardless of x_{-j} and y_{-j}. The effect of the constant M in the definition of p_i is to make it impossible for G to pass or reach 0 whenever any of the terms p_i is equal to $-M$.

The above numerical representation of an ELECTRE I type of preference relation in model $(D8)$ is regular since p_i, as defined by (3.51), is a numerical representation of the weak order $\succsim_i^*$ on the differences of preference. This order distinguishes three equivalence classes of differences of preference, namely those corresponding respectively to the cases where $x_i\, S_i\, y_i$, $y_i\, P_i\, x_i$ but $Not[\, y_i\, V_i\, x_i\,]$ and $y_i\, V_i\, x_i$.

The representation above is probably the most natural and intuitive one. Since the set of relations that can be described by (3.50) contains the weighted qualified majority relations, it is clear from Sect. 3.4.5.3 that one cannot expect that weighted qualified majority relations with veto admit a representation in model $(D7)$ or $(D11)$. They however admit a representation in model $(D6)$ and in its strictly increasing yet equivalent version $(D10)$. For a representation in model $(D6)$, we may choose for p_i a numerical representation of the weak order $\succsim_i^{**}$ that determines five equivalence classes of differences of preference, namely:

$$p_i(x_i, y_i) = \begin{cases} M & \text{if } x_i\, V_i\, y_i \\ w_i & \text{if } x_i\, P_i\, y_i \text{ and } Not[\, x_i\, V_i\, y_i\,] \\ 0 & \text{if } x_i\, I_i\, y_i \\ -w_i & \text{if } y_i\, P_i\, x_i \text{ and } Not[\, y_i\, V_i\, x_i\,] \\ -M & \text{if } y_i\, V_i\, x_i, \end{cases} \tag{3.53}$$

where M is a positive constant larger than w_i. The function G can be defined by

$$G([p_i(x_i, y_i)]) = \begin{cases} \sum_{i: x_i S_i y_i} \min(p_i(x_i, y_i), w_i) - \delta & \text{if, for all } j \in N,\ Not[\, y_j\, V_j\, x_j\,] \\ -1 & \text{if, for some } j \in N,\ y_j\, V_j\, x_j. \end{cases} \tag{3.54}$$

A strictly increasing representation [in model $(D10)$] obtains by the usual construction (with an exponential function).

Remark 3.1 The relations defined by means of vetoes that are described in this section constitute a very particular subclass of relations for which five classes of differences of preference can be distinguished. There are of course many other ways of defining models of preference that distinguish five classes of differences.

3.4.7 Examples of Preferences that Distinguish a Large Variety of Differences

Contrary to the examples discussed so far in which the relations $\succsim_i^*$ or $\succsim_i^{**}$ distinguish a small number of classes of preference differences (typically three or five classes for $\succsim_i^{**}$ in the above examples), there are very common cases where there is a large number of distinct classes, possibly an infinite number of them.

The most common model, the additive value model, usually belongs to the class of models in which $\succsim_i^{**}$ makes subtle distinctions between differences of preferences; indeed its definition, Eq. (3.1), p. 37, can be rewritten in the following manner:

$$x \succsim y \text{ iff } \sum_{i=1}^{n} (u_i(x_i) - u_i(y_i)) \geq 0. \tag{3.55}$$

The difference $u_i(x_i) - u_i(y_i)$ can often be interpreted as a representation $p_i(x_i, y_i)$ of $\succsim_i^{**}$; the preference then satisfies model $(D11)$. Let us take a simple example; assume that $X_i = \mathbb{R}$, that the number of dimensions n is equal to 2 and that $u_i(x_i) = x_i$ for $i = 1, 2$. The preference is defined by:

$$\begin{aligned} x \succsim y \text{ iff } & x_1 + x_2 \geq y_1 + y_2 \\ \text{iff } & (x_1 - y_1) + (x_2 - y_2) \geq 0. \end{aligned} \tag{3.56}$$

In such a case, $p_1(x_1, y_1) = x_1 - y_1$ is a numerical representation of the relation $\succsim_1^{**}$ on the differences of preference on the first dimension X_1 (and similarly for $x_2 - y_2$ on X_2). The pair (x_1, y_1) corresponds to an at least as large difference of preference as (z_1, w_1) iff $x_1 - y_1 \geq z_1 - w_1$; indeed, if $(z_1, a_2) \succsim (w_1, b_2)$ for some "levels" a_2, b_2 in X_2, then substituting (z_1, w_1) by (x_1, y_1) results in $(x_1, a_2) \succsim (y_1, b_2)$ and, conversely, if $(y_1, c_2) \succsim (x_1, d_2)$ for some c_2, d_2 in X_2, then $(w_1, c_2) \succsim (z_1, d_2)$ [by definition of $\succsim_1^{**}$, see (3.28) and (3.27)]. We have furthermore that both preferences obtained after these substitutions are strict as soon as $(x_1, y_1) \succ_1^{**} (z_1, w_1)$, i.e. as soon as $x_1 - y_1 > z_1 - w_1$. This strict responsiveness property of $\succsim$ is characteristic of model $(D11)$, in which indifference is "narrow" as was already mentioned at the end of Sect. 3.4.5.3. Indeed if $(z_1, a_2) \succsim (w_1, b_2)$, we must have:

$$(z_1 - w_1) + (a_2 - b_2) = 0$$

and substituting (z_1, w_1) by (x_1, y_1) results in $(x_1 - y_1) + (a_2 - b_2) > 0$ as soon as $x_1 - y_1 > z_1 - w_1$.

Thus, any increase or decrease of $p_i(x_i, y_i)$ breaks indifference. This is also the case with the additive difference model (3.25) (with $p_i(x_i, y_i) = \Phi_i(u_i(x_i) - u_i(y_i))$) and the nontransitive additive model (3.26).

Remark 3.2 (From Ordinal to Cardinal) The framework based on marginal traces on differences that we studied in the present Sect. 3.4 is general enough to encompass both "non-compensatory" and "compensatory" preferences, for instance, preferences based on a majority or a lexicographic rule (three classes of differences of preference) and those represented in an additive manner (that can potentially distinguish an unbounded number of differences). A weighted qualified majority rule, for instance, can be said to be *ordinal* or *purely non-compensatory*; from the representation of the procedure [Eqs. (3.41), (3.42)], one can see that the full weight w_i associated to a dimension is credited to an alternative x, as compared to an alternative y, as soon as the preference difference $p_i(x_i, y_i)$ is in favour of x on that dimension. In this model, the preference difference $p_i(x_i, y_i)$ is positive as soon as x_i is preferred to y_i, w.r.t. some a priori preference relation S_i on X_i, hence the denomination of "ordinal".

On the opposite, in the additive value model [Eq. (3.55)], a large difference of preference on one dimension can be compensated by small differences of opposite sign on other dimensions: the procedure is compensatory and it uses the full power of the numbers p_i in arithmetic operations like sums and differences; we call it "cardinal".

Between those two extremes, the other procedures can be sorted in increasing order of the number of classes of differences of preference they permit to distinguish. This can be seen as a picture of a transition from "ordinal" to "cardinal" or, alternatively, from non-compensatory to compensatory procedures. Of course, the type of model is determined by the richness of the preferential information available.

3.5 Models with Weakly Differentiated Preference Differences

In Sect. 3.4.5, we have investigated a variety of models in which the number of classes of differences of preference is reduced to at most three. Can one provide a unified framework for discussing and understanding all those variants of a majority rule? It is our aim in this section to briefly describe such a framework. All the preferences described in the above-mentioned sections have some right to be called *concordance* relations. The term "concordance" was introduced by Roy (1968, 1971) in the framework of the ELECTRE methods [see also Roy (1996), Roy and Bouyssou (1993, sections 5.2 and 5.3) and Roy (1991); Roy and Vanderpooten (1996)]. It specifies an index (the so-called *concordance index*) that measures the strength of the coalition of criteria saying that an alternative x is at least as good as an alternative y. Here we use this term in the same spirit for qualifying a preference relation that results from the comparison of the strengths of coalitions of criteria: we have in mind all preference relations studied in Sect. 3.4.5.[5]

An earlier investigation of preference relations of this type in a conjoint measurement framework is due to Fishburn (1976) through its definition of *non-compensatory* preferences [see also Bouyssou and Vansnick (1986)]. Recently, Bouyssou and Pirlot (2002a), Bouyssou and Pirlot (2005) proposed a precise definition of concordance relations and showed that they can be described within the family of models that rely on traces on differences (Sect. 3.4.3). It is the goal of this section to outline those results. Similar ideas have been developed by Greco et al. (2001).

[5] The lexicographic preference described in Sect. 3.4.5.4 enters into this framework but can be seen as a limit case.

3.5.1 Concordance Relations

In a conjoint measurement context, a concordance relation is characterised by the following features.

Definition 3.12 A reflexive relation $\succsim$ on X is a concordance relation if there are:

- a complete binary relation S_i on each X_i,
- a binary relation $\trianglerighteq$ between subsets of N having N for union that is monotonic with respect to inclusion, i.e. such that for all $A, B, C, D \subseteq N$,

$$[A \trianglerighteq B, C \supseteq A, B \supseteq D, C \cup D = N] \Rightarrow C \trianglerighteq D, \tag{3.57}$$

such that, for all $x, y \in X$,

$$x \succsim y \;\Leftrightarrow\; S(x, y) \trianglerighteq S(y, x), \tag{3.58}$$

where $S(x, y) = \{i \in N : x_i S_i y_i\}$.

In this definition, we interpret S_i as the *a priori* preferences on the scale co-domain X_i of each dimension; in cases of practical interest, S_i will usually be a weak order or a semiorder (but we do not assume this for the start) and the global preference of x over y results from the comparison of the coalitions of criteria $S(x, y)$ and $S(y, x)$. The former can be seen as the list of reasons for saying that x is at least as good as y, while the latter is a list of reasons supporting that y is at least as good as x. In order to compare coalitions of criteria, we assume that there is a relation $\trianglerighteq$ on the power set of the set N that allows us to decide whether a subset of criteria constitutes a stronger argument than another subset of criteria; the interpretation of such a relation is straightforward when the compared subsets are the lists of dimensions $S(x, y)$ and $S(y, x)$ involved in the comparison of two alternatives x and y. Note that $\trianglerighteq$ enables us only to compare "complete" coalitions of criteria, i.e. those having N for their union.

The weighted majority relation (Sect. 3.4.5.2), typically, fulfills the requirements for a concordance relation as defined above. In this example, the strength of a subset of criteria can be represented by the sum of their weights and comparing $S(x, y)$ to $S(y, x)$ amounts to comparing two numbers, namely the sums of the weights of the dimensions that belong respectively to $S(x, y)$ and $S(y, x)$. In such a case, $\trianglerighteq$ can be extended to a weak order on the power set of N and this weak order admits a numerical representation that is additive with respect to individual dimensions:

$$S(x, y) \trianglerighteq S(y, x) \text{ iff } \sum_{i \in S(x,y)} w_i \geq \sum_{i \in S(y,x)} w_i. \tag{3.59}$$

In our general definition however, we neither postulate that $\trianglerighteq$ is a weak order nor that it can be additively represented on the basis of "weights" of individual criteria.

On the relation $\trianglerighteq$, we only impose a quite natural property (3.57), namely that it is monotonic with respect to the inclusion of subsets of criteria.

The interesting feature of concordance relations is that they can easily be characterised within the family of models (Dk) that rely on preference differences. The main result, obtained in Bouyssou and Pirlot (2005, Theorem 1), establishes that concordance relations are exactly those preferences for which the traces on differences $\succsim_i^{**}$ are weak orders with no more than three equivalence classes. This result will be part 1 of Theorem 3.4 stated below on p. 85. Concordance relations consequently form a subclass of the relations belonging to model $(D6)$.

3.5.1.1 The Relation $\trianglerighteq$

As a consequence of this result, *all* preferences described in Sect. 3.4.5 admit a representation as a concordance relation and can be described by (3.58), i.e.:

$$x \succsim y \Leftrightarrow S(x, y) \trianglerighteq S(y, x),$$

for some $\trianglerighteq$ and some S_i satisfying the requirements of Definition 3.12. We emphasise that this is true not only for simple weighted majorities (Sect. 3.4.5.2) but also for *qualified* majorities (Sect. 3.4.5.3) or lexicographic preferences (Sect. 3.4.5.4) that are not primarily defined through comparing coalitions (qualified majority is defined through comparing the "pros" in favour of x against y to a threshold; lexicographic relations arise from considering the most important criterion and only looking at the others when alternatives are tied on the most important one). Part 1 of Theorem 3.4 says that all these relation can *also* be represented according with Eq. (3.58) using an appropriate definition of $\trianglerighteq$ and S_i. Of course, we cannot ensure that $\trianglerighteq$ can be represented in general according with Eq. (3.59), i.e. in an additive manner.

3.5.1.2 The Relations S_i

The link between S_i and $\succsim$ is given by:

$$x_i \, S_i \, y_i \Leftrightarrow (x_i, y_i) \succsim_i^* (x_i, x_i). \tag{3.60}$$

The interpretation of this definition is clear (at least for reflexive and independent preferences $\succsim$ with which all "null differences" (x_i, x_i), for $x_i \in X_i$, are indifferent with respect to relation $\succsim_i^*$): $x_i \, S_i \, y_i$ means that the difference of preference (x_i, y_i) is "non negative", in the sense that it is at least as large as the "null difference" (x_i, x_i) or any other null difference (z_i, z_i).

For a general concordance relation $\succsim$, it can be shown that S_i is complete but not necessarily transitive; the marginal traces $\succsim_i^+$ and $\succsim_i^-$ are included in S_i, which in turn is contained in the marginal preference $\succsim_i$.

We summarise the above results in the following theorem that is based on Bouyssou and Pirlot (2005, theorems 2 and 4). Note that the latter paper provides conditions, expressed in terms of the relation $\succsim$, that are equivalent to requiring that the traces on differences $\succsim_i^{**}$ have at most three equivalence classes.

Theorem 3.4 (Concordance Relation)

1. *A relation $\succsim$ on X is a concordance relation iff it is reflexive, satisfies $RC12$ and its traces on differences $\succsim_i^{**}$ have at most three equivalence classes.*
2. *The relations S_i that intervene in the definition of concordance relations are semiorders iff $\succsim$ satisfies, in addition, $AC123$.*
3. *These relations are weak orders as soon as $\succsim$ satisfies $AC4$.*

3.5.1.3 Concordance-Discordance Relations

Concordance-discordance relations are similar to concordance relations but, in addition, their representation also involves a veto. They can be studied and characterized in the same spirit as concordance relations (Bouyssou and Pirlot, 2009).

References

Arrow, K. J. (1951). *Social choice and individual values*. New York: Cowles Foundations and Wiley.

Bouyssou, D. (1986). Some remarks on the notion of compensation in MCDM. *European Journal of Operational Research, 26*, 150–160.

Bouyssou, D., & Pirlot, M. (2002a). A characterization of strict concordance relations. In D. Bouyssou, É. Jacquet-Lagrèze, P. Perny, R. Słowinski, D. Vanderpooten, & Ph. Vincke (Eds.), *Aiding decisions with multiple criteria: Essays in honour of Bernard Roy* (pp. 121–145). Dordrecht: Kluwer.

Bouyssou, D., & Pirlot, M. (2002b) Nontransitive decomposable conjoint measurement. *Journal of Mathematical Psychology, 46*, 677–703.

Bouyssou, D., & Pirlot, M. (2004). Preferences for multi-attributed alternatives: Traces, dominance and numerical representations. *Journal of Mathematical Psychology, 48*(3), 167–185.

Bouyssou, D., & Pirlot, M. (2005). A characterization of concordance relations. *European Journal of Operational Research, 167*, 427–443.

Bouyssou, D., & Pirlot, M. (2009). An axiomatic analysis of concordance-discordance relations. *European Journal of Operational Research, 199*, 468–477.

Bouyssou, D., & Vansnick, J.-Cl. (1986). Noncompensatory and generalized noncompensatory preference structures. *Theory and Decision, 21*, 251–266.

Bouyssou, D., Marchant, Th., Pirlot, M., Perny, P., Tsoukiàs, A., & Vincke, Ph. (2000). *Evaluation and decision models: A critical perspective*. Dordrecht: Kluwer.

Bouyssou, D., Marchant, Th., Pirlot, M., Tsoukiàs, A., & Vincke, Ph. (2006) *Evaluation and decision models with multiple criteria: Stepping stones for the analyst*. New York: Springer.

Brans, J.P., & Vincke, Ph. (1985) A preference ranking organisation method. (The PROMETHEE method for multiple criteria decision-making). *Management Science, 31*(6), 647–656.

Debreu, G. (1960) Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences* (pp. 16–26). Stanford: Stanford University Press.

Dubois, D., & Fortemps, Ph. (1999). Computing improved optimal solutions to max-min flexible constriant satisfaction problems. *European Journal of Operational Research, 118*, 95–126.

Dubois, D., Fargier, H., & Prade, H. (1995). Fuzzy constraints in job-shop scheduling. *Journal of Intelligent Manufacturing, 6*, 215–234.

Dubois, D., Fargier, H., Perny, I., & Prade, H. (2003). A characterization of generalized concordance rules in multicriteria decision-making. *International Journal of Intelligent Systems, 18*(7), 751–774.

Edwards, W. (1971). Social utilities. *Engineering Economist, 6*, 119–129.

Fishburn, P. C. (1967). Methods of estimating additive utilities. *Management Science, 13*(7), 435–453.

Fishburn, P. C. (1970). *Utility theory for decision making*. New York: Wiley.

Fishburn, P. C. (1976). Noncompensatory preferences. *Synthese, 33*, 393–403.

Fishburn, P. C. (1990a). Continuous nontransitive additive conjoint measurement. *Mathematical Social Sciences, 20*, 165–193.

Fishburn, P. C. (1990b). Additive non-transitive preferences. *Economic Letters, 34*, 317–321.

Fishburn, P. C. (1991a). Nontransitive preferences in decision theory. *Journal of Risk and Uncertainty, 4*, 113–134.

Fishburn, P. C. (1991b). Nontransitive additive conjoint measurement. *Journal of Mathematical Psychology, 35*, 1–40.

Goldstein, W. M. (1991). Decomposable threshold models. *Journal of Mathematical Psychology, 35*, 64–79.

Greco, S., Matarazzo, B., & Słowiński, R. (1991). Rough approximation of a preference relation by dominance relations. *European Journal of Operational Research, 117*, 63–83.

Greco, S., Matarazzo, B., & Słowiński, R. (2001, 30 May–2 June). Axiomatic basis of noncompensatory preferences. In Communication to *FUR X*, Torino.

Keeney, R. L. (1992). *Value-focused thinking. A path to creative decision making*. Cambridge: Harvard University Press.

Keeney, R. L., & Raiffa, H. (1976). *Decisions with multiple objectives: Preferences and value tradeoffs*. New York: Wiley.

Krantz, D. H. (1964). Conjoint measurement: The Luce-Tukey axiomatization and some extensions. *Journal of Mathematical Psychology, 1*, 248–277.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement,* vol. 1: *Additive and polynomial representations*. New York: Academic Press.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica, 24*, 178–191.

Luce, R. D. (2000). *Utility of gains and losses: Measurement-theoretic and experimental approaches*. Mahwah, NJ: Lawrence Erlbaum Association.

Luce, R. D., Tukey, J. W. (1964). Simultaneous conjoint measurement: A new type of fundamental measurement. *Journal of Mathematical Psychology, 1*, 1–27.

Perny, P. (1992). *Modélisation, agrégation et exploitation de préférences floues dans une problématique de rangement*. Thèse de doctorat, Université Paris Dauphine.

Raiffa, H. (1969, December). Preference for multi-attributed alternatives. RAND Memorandum, RM-5868-DOT/RC.

Roy, B. (1968). Classement et choix en présence de points de vue multiples (la méthode ELECTRE). *RIRO, 2*, 57–75.

Roy, B. (1971). Problems and methods with multiple objective functions. *Mathematical Programming, 1*, 239–266.

Roy, B. (1991). The outranking approach and the foundations of ELECTRE methods. *Theory and Decision, 31*, 49–73.

Roy, B. (1996). *Multicriteria methodology for decision aiding*. Dordrecht: Kluwer. Original version in French (1985). *Méthodologie multicritère d'aide à la décision*. Paris: Economica.

Roy, B., & Bouyssou, D. (1993). *Aide multicritère à la décision : méthodes et cas*. Paris: Economica.

Roy, B., & Vanderpooten, D. (1996). The european school of mcda: Emergence, basic features and current works. *Journal of Multi-Criteria Decision Analysis, 5*(1), 22–37.

Tversky, A. (1969). Intransitivity of preferences. *Psychological Review, 76*, 31–48.

Vansnick, J. Cl. (1986). De Borda et Condorcet à l'agrégation multicritère. *Ricerca Operativa, 40*, 7–44.

Vincke, Ph. (1992). *Multicriteria decision-aid*. New York: Wiley.

Vind, K. (1991). Independent preferences. *Journal of Mathematical Economics, 20*, 119–135.

von Winterfeldt, D., & Edwards, W. (1986). *Decision analysis and behavorial research*. Cambridge: Cambridge University Press.

Wakker, P. P. (1989). *Additive representations of preferences: A new foundation of decision analysis*. Dordrecht: Kluwer.

Wakker, P. P. (1991). Additive representations of preferences, a new foundation of decision analysis; the algebraic approach. In J.-P. Doignon & J.-Cl. Falmagne (Eds.), *Mathematical psychology: Current developments* (pp. 71–87). Berlin: Springer.