A NOTE ON THE "MIN IN FAVOR" RANKING METHOD FOR VALUED PREFERENCE RELATIONS [1]

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Abstract.

This note deals with the general problem of ranking several alternatives on the basis of a valued preference relation. We present a system of five axioms (neutrality, ordinality, continuity, row monotonicity and row egalitarianism) which is shown to characterize a ranking method based on the Min operator.

I. Introduction

In order to compare a number of alternatives taking into account several criteria or the opinion of several voters, many methods associate with each ordered pair (a,b) of alternatives a number synthesizing the result of the comparison of a and b along the different points of view. In the field of MCDM, the number associated to the ordered pair (a, b) generally reflects the importance of the criteria for which "a is preferred to b" or "a is at least as good as b" (see, e.g., PROMETHEE, Brans et al. (1984) or ELECTRE III, Roy (1978)). Though such a synthesis is often useful, we know (at least since the work of Condorcet) that, when the different points of view taken into account are conflictual, it may not be easy to compare the alternatives on the basis of these members.

Many methods can be envisaged to rank alternatives on the basis of such information. In order to compare these methods we may study their behavior with regards to a number of "desirable" properties (see, e.g., Vincke (1991)). Alternatively (but not exclusively), we may try to find a set of axioms that would characterize a particular method. Following Bouyssou (1991), this is the route followed in this note in which we present a system

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of axioms characterizing a ranking method based on the Min operator.

After having introduced our definitions and notations in section 2, we present our axioms in section 3. Our results appear in section 4.

II. Definitions and Notations

Let A be a finite set of objects called "alternatives" such that $|A| = n \geq 2$. We define a valued (binary) relation on A as a function R associating with each ordered pair of alternatives $(a,b) \in A^2$ with [2]) $a \neq b$ an element of $[0,1]$. Let $R(A)$ be the set of all valued relations on A.

A crisp (binary) relation S on A is a subset of A^2. We will classically write a S b instead of $(a,b) \in S$. A crisp relation S on A is complete if for all a, b $\in$ A either a S b or b S a. It is transitive if for all a, b, c $\in$ A, a S b and b S c imply a S c. A complete and transitive crisp relation will be called a ranking.

We define a ranking method $\geq$ as a function associating a ranking $\geq(R)$ on A with any valued relation R on A.

Given a valued relation $R \in R(A)$, an obvious way to obtain a ranking method is to associate a score $S(a, R)$ with each alternative a and to rank the alternatives according to their scores, i.e.

$$a \geq (R)\ b \text{ iff } S(a,R) \geq S(b,R) \tag{1}$$

In this note, we present a characterization of the ranking method based on the following score:

$$SF_{Min}(a,R) = \operatorname*{Min}_{c \in A \setminus \{a\}} R(a,c) \tag{2}$$

We will refer to the ranking method defined by (1) and (2) as the Min In Favor method.

Besides its simplicity and intuitive appeal, a reason to study

2 The restriction could be omitted at the cost of minor modifications of our axioms.

the Min In Favor method is that it does not make use of the "cardinal" properties of the valuations.
Viewing the set of alternatives A as a subset of a universal set X, the Min In Favor score defined by (2) can easily be used to define a chice function on X (i.e. a function associating with each valued relation $R \in R(X)$ and each nonempty subset A of X a nonempty choice set included in A). Several authors (see, e.g. Barrett et al. (1990) or Moulin (1988)) have introduced such a choice function. Barrett at al. (1990) have shown that this choice function behaves nicely with regards to number of desirable properties. Though we will concentrate here on the Min In Favor ranking method, it is not difficult to extend our results so as to obtain a characterization of the Min In Favor choice function.

III. Axioms

In the sequel, we note $=(R)$ and $>(R)$ the symetric and asymetric parts of $\geq(R)$, i.e. [$a =(R)\ b$ iff ($a \geq(R)\ b$ and $b \geq(R)\ a$)] and [$a \geq(R)\ b$ iff ($a \geq(R)\ b$ and Not $b \geq(R)\ a$)]. A ranking method $\geq$ on A is said to be **neutral** if, for all permutations σ on A, all $R \in R(A)$ and all a, b $\in$ A:

$$a \geq(R)\, b \rightarrow \sigma(a) \geq (R^{\sigma})\, \sigma(b)$$

where R^{σ} is defined by $R^{\sigma}(\sigma(c), \sigma(d)) = R(c,d)$ for all c,d $\in$ A with $c \neq d$.
Neutrality expresses the fact that a ranking method should not discriminate between alternatives just because of their labels. It is a classical property in this context (see Rubinstein (1980)). The Min In Favor method is obviously neutral.
It is easily checked that neutrality implies that, for all $R \in R(A)$ and all a, b $\in$ A with $a \neq b$:
[$R(a,b) = R(b,a)$, $R(a,c) = R(b,c)$ and $R(c,a) = R(c,b)$ for all c $\in A\setminus\{a,b\}$] $\rightarrow a =(R) b$.
A ranking method $\geq$ is said to be **ordinal** if, for all $R \in R(A)$ and strictly increasing transformation ϕ from [0,1] to [0,1],

$\geq(R) = \geq(\phi[R])$

where $\phi[R]$ is a valued relation such that $\phi[R](c,d) = \phi(R(c,d))$ for all $c,d \in A$ with $c \neq d$.

Ordinality implies that a ranking method should not make use of the "cardinal" properties of the valuations. It is a crucial property for our purposes. It is obvious that the Min In Favor method is ordinal.

Consider a sequence of valued relations $(R^i \in R(A),\ i=1,2,\ldots)$. We say that this sequence converges to $R \in R(A)$ if, for all $\varepsilon>0$, there is an integer k such that, for all $j>k$ and all $a,b \in A$ with $a \neq b$, $|R^j(a,b) - R(a,b)| < \varepsilon$.

A ranking method $\geq$ is said to be **continuous** if, for all $R \in R(A)$, all sequences $(R^i \in R(A),\ i=1,2,\ldots)$ converging to R and all $a,b \in A$,

$[a \geq(R^i)\ b$ for all R^i in the sequence$] \rightarrow [a \geq(R)\ b]$.

Continuity says that "small" changes in a valued relation should not lead to radical changes in the associated ranking. In the context of decision under uncertainty, a similar axiom has been used by Milnor (1954). It is easy to see that if a sequence of valued relations $(R^i, i=1,2,\ldots)$ converges to a valued relation R, then, for all $a \in A$, the sequence $(SF_{Min}(a,R^i),\ i=1,2,\ldots)$ converges (in the usual sense) to $SF_{Min}(a,R)$. This shows that the Min In Favor method is continuous.

A ranking method $\geq$ is said to be **row monotonic** if for all $R \in R(A)$ and all $a,b \in A$, $a \geq(R)\ b \rightarrow a >(R')\ b$

where R' is identical to R except that $R'(a,c) > R(a,c)$ for all $c \in A\backslash\{a\}$.

Row monotonicity says that the position of an alternative should improve in the ranking if its position is improved *vis-a-vis* all other alternatives in the valued relation.

Though row monotonicity is a fairly strong property, it is obvious that the Min In Favor method is row monotonic.

Suppose that R' is identical to R except that $R(b,d) > R'(b,d)$ for all $d \in A\backslash\{b\}$. It is not difficult to see that row monotonicity implies that $[a \geq(R)\ b \rightarrow a >(R'')\ b]$.

The Min In Favor method is not the only ranking method that is neutral, ordinal, continuous and row monotonic. For instance, this is also the case for the Max In Favor method defined by (1) with the following score:

$$SF_{Max}(a,R) = \underset{c\in A\setminus\{a\}}{Max} R(a,c)$$

This calls for an axiom that would be more specific to the Min In Favor method. Though other axioms could be envisaged, we will use an axiom saying that, viewing a valued relation as a matrix, averaging the row associated to an alternative cannot decrease its position in the renking. More formally, we say that a ranking method $\succeq$ is **row egalitarian** if for all $R \in R(A)$ and all $a,b \in A$,
$a \succeq(R)\ b \rightarrow a \succ(R_a)\, b$
where R_a is identical to R except that
$R_a(a,c) = \sum_{d\in A\setminus\{a\}} R(a,d)/(n-1)$ for all $c\in A\setminus\{a\}$.

The Min In Favor method is obviously row egalitarian, whereas the Max In Favor method is not.

IV. RESULTS

The main purpose of this note is to prove the following:

Proposition. The Min In Favor method is the only ranking method that is neutral, ordinal, continuous row monotonic and row egalitarian.

The following lemmas will be useful in the proof of the proposition.

Lemma 1. If a ranking method $\succeq$ is neutral, continuous, ordinal and row monotonic then, for all $R \in R(A)$ and all $a, b \in A$ with $a \neq b$,
$[R(a,c) = 1$ for all $c \in A\setminus\{a\} \rightarrow a \succeq(R)\ b]$.

Proof of lemma 1.

Suppose, that, for some neutral, ordinal, continuos and row monotonic ranking method $\succeq$, some $R \in R(A)$ and some $a,b \in$ with $a \neq b$, we have $R(a,c) = 1$ for all $c \in A\setminus\{a\}$ and $b \succ(R)\ a$.
Consider a sequence $(\varepsilon_i \in (0,1],\ i=1,2,\ldots)$ converging to 0. To this sequence we associate a sequence of valued relations

($R^i \in R(A)$, i=1,2,...) such that, for all c,d $\in$ A with c $\neq$ d, $R^i(c,d)$ = R(c,d) if and only if c = a and $R^i(c,d)$ = Max{0,R(c,d) - ε_i} otherwise.

The sequence ($R^i \in R(A)$, i=1,2,...) converges to R so that, by continuity, we must have b >(R^j) a for some R^j in the sequence.

Consider now a sequence of strictly increasing transformations (ϕ, i=1,2,...) from [0,1] to [0,1] such that $\phi_i(x) = x^i$ for all x $\in$ [0,1]. The sequence ($\phi_i[R^j] \in R(A)$, i=1,2,...) converges to a valued relation R* $\in$ R(A) such that, for all c,d $\in$ A with c $\neq$ d, $R^*(c,d)$ = 1 if and only if c = a and $R^*(c,d)$ = 0 otherwise.

Ordinality implies that b >($\varepsilon_i[R^j]$ a for all $\varepsilon_i[R^j]$ in the sequence so that continuity leads to b $\geq$(R^*) a.

Consider now a valued relation $\underline{R}$ such that $\underline{R}$(c,d) = 0 for all c,d $\in$ A with c $\neq$ d. Neutrality implies a =($\underline{R}$) b and row monotonicity leads to a >(R^*) b, a contradiction.

Lemma 2. If a ranking method $\geq$ is neutral, ordinal, continuous, row monotonic and row egalitarian then, for all R $\in$ R(A) and all a,b $\in$ A with a $\neq$ b,

[R(a,c) = 1 for all c $\in$ A\{a} and R(b,d) < 1 for some d $\in$ A\{a} $\rightarrow$ a >(R) b].

Proof of Lemma 2.

Suppose that, for some neutral, ordinal, continuous, row monotonic and row egalitarian ranking method $\geq$, some R $\in$ R(A) with a $\neq$ b, we have:

R(a,c) = 1 for all c $\in$ A\{a} and R(b,d) < 1 for some d $\in$ A\{b} and b $\geq$(R) a.

By row egalitarianism, we have b $\geq$(R_b) a. Since, by hypothesis, R_b(b,d) < 1. we can find a valued relation R′ identical to R_b except that R′(b,d) > R_b(b,d) for all d $\in$ A\{b}. Thus, row monotonicity leads to b >(R′)a, which contradicts lemma 1.

Proof of the Proposition.

We already observed that the Min In Favor method is neutral, row monotonic, ordinal, continuous and row egalitarian. Thus all we have to prove is that if a ranking method $\geq$ is neutral, ordinal, continuous, row monotonic and row egalitarian then:

$a \geq(R)\ b \leftrightarrow SF_{Min}(a,R) \geq SF_{Min}(b,R)$, i.e.,

$$SF_{Min}(a,R) = SF_{Min}(b,R) \rightarrow a =(R)\ b \text{ and} \tag{3}$$

$$SF_{Min}(a,R) > SF_{Min}(b,R) \rightarrow a >(R)\ b. \tag{4}$$

First suppose that $SF_{Min}(a,R) > SF_{Min}(b,R)$ for some $R \in R(A)$ and some $a,b \in A$.

Let $\underline{A}_b = \{c \in A\backslash\{b\} : R(b,c) = S_{Min}(b,R)\}$. Consider a sequence of strictly increasing transformations (ϕ_i, i=1,2,...) such that:

$\phi_i(x) = x$ if $x \leq SF_{Min}(b,R)$ and

$\quad = x^{1/i}$ otherwise.

The sequence ($\phi_i[R]$, i=1,2,...) converges to a relation R^* for which:

$R^*(a,c) = 1$ for all $c \in A\backslash\{a\}$,

$R^*(b,c) = SF_{Min}(b,R)$ for all $c \in \underline{A}_b$ and

$\quad = 1$ otherwise.

Since $\underline{A}_b$ is nonempty and $SF_{Min}(b,R) < 1$, we know from lemma 2 that $a >(R^*)\ b$.

If $b \geq(R)\ a$, then ordinality implies that $b \geq(\phi_i[R])\ a$ for all $\phi_i[R]$ in the sequence. Thus, continuity leads to $b \geq(R^*)\ a$, a contradiction. This establishes (4).

In order to prove (3) suppose that $SF_{Min}(a,R) = SF_{Min}(b,R) = \lambda$ for some $R \in R(A)$ and some $a,b \in A$.

If $\lambda = 1$ then $\underline{A}_a = \underline{A}_b = A$ so that $a =(R)\ b$ by lemma 1.

Suppose now that $\lambda \neq 1$ and that $b >(R)\ a$, the proof for the other case being similar. It is easy to build a sequence ($R^i \in R(A)$, i=1,2,...) converging to R and such that $SF_{Min}(a,R^i) > SF_{Min}(b,R^i)$ for all R^i in the sequence, e.g. letting R^i identical to R except that $R^i(a,c) = Min\{1, R(a,c) + 1/i\}$ for all $c \in \underline{A}_a$. Thus (4) implies $a >(R^i)\ b$ for all R^i in the sequence. Using continuity, we have $a \geq(R)\ b$, a contradiction. This establishes (3) completing the proof of the proposition.

It is not difficult to see that a similar method of proof can be used to characterize the Max In Favor method modifiying in an obvious way row egalitarianism. Furthermore replacing row monotonicity and row egalitarianism by similar axioms applying to columns leads to a straightforward characterization of the Min Against and the Max Against methods respectively defined by (1) with the following scores:

$$SA_{Min}(a,R) = -\operatorname*{Min}_{c\in A\setminus\{a\}} R(c,a) \text{ and } SA_{Max}(a,R) = -\operatorname*{Max}_{c\in A\setminus\{a\}} R(c,a)$$

Let us finally observe that it is impossible to deduce any of the five axioms characterizing the Min In Favor method from the other four as shown by the following examples:

i- Let $\phi : A \to \{1,2,\ldots,|A|\}$ be a one-to-one function.

Define $\geq$ as:

$a \geq(R)\ b$ iff $SF_{Min}(a,R)/\phi(a) \geq SF_{Min}(b,R)/\phi(b)$.

This ranking method is ordinal, continuous, row monotonic and row egalitarian but not neutral.

ii- Define $\geq$ as:

$a \geq(R)\ b$ iff $L(a,R) \geq L(b,R)$, where, for all $c \in A$,

$$L(c,R) = \sum_{d\in A\setminus\{a\}} R(c,d)$$

This ranking method is neutral, continuous, row monotonic and row egalitarian but not ordinal.

iii- To each $c \in A$, a valued relation $R \in R(A)$ associates an element $(R(c,d); d \in A\setminus\{a\})$ of R^{n-1}. Let $R(c)$ be the element of R^{n-1} obtained by reordering the components of $(R(c,d); d \in A\setminus\{c\})$ in increasing order. Define $\geq$ as:

$a\ _{\geq}(R)\ b$ iff $R(a) \geq^L R(b)$

where $\geq^L$ is the usual lexicographic relation between vectors, i.e., for all $x,y \in R^{n-1}$,

$x \geq^L y$ iff $[x =^L y$ or $x >^L y]$ with

$x =^L y$ iff $x_i = y_i$ for $i = 1,2,\ldots,n-1$ and

$x >^L y$ iff $x \neq y$ and $x_i > y_i$ for the smallest i such that $x_i \neq y_i$, where x_i (resp. y_i) isthe ith component of x (resp. y).

This ranking method is neutral, ordinal, row monotonic and row egalitarian but not continuous. To show that this method is not continuous, consider two alternatives $a,b \in A$ and a valued relation $R \in R(A)$ such that $R_1(a) = SF_{Min}(a,R) = SF_{Min}(b,R) = R_1(b)$

and $R_2(b) > R_2(a)$ so that $b >(R)$ a, where Ri(c) is the ith component of R(c). It is easy to build a sequence of valued relations ($R^i \in R(A)$, i=1,2,...) converging to R and such that $SF_{Min}(a,R^i) > SF_{Min}(b,R^i)$ so that $a >(R^i)$ b for all R^i in the sequence and thus violating continuity.

iv- Let $\geq$ be a ranking method such that a =(R) b for all R(A) and all a,b $\in$ A. This ranking method is neutral, ordinal, continuous and row eligatarian but not row monotonic.

v- The Max In Favor method is neutral, ordinal, continuous and row monotonic but not row egalitarian.

An alternative characterization of the Min In Favor Method has been obtained by Pirlot (1991). The comparison of our results with that Pirlot (1991) and several extensions of the method of proof used here will be considered in a subsequent paper.

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