Simple but powerful models of stereotype formation *

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Abstract

We propose simple models aiming at explaining the formation of stereotypes. A stereotype is an overall judgement brought by an observer over a group of individuals or objects. For each member of the group, we suppose that we observe a characteristic that belongs to a denumerable set. The formation of a stereotype about the group is governed by a perception function. Our basic model consists in decomposing a perception function into three steps: (*i*) characteristics are recoded numerically so that higher numbers mean a higher support for the stereotype, (*ii*) this vector of numbers is consistently aggregated into a single number, and (*iii*) this number is compared to a threshold and the stereotype is accepted if the threshold is exceeded. We characterize perception functions that can be explained using such a model. We then study various extensions of our basic model.

Keywords: Stereotype, Perception function, Mental process.

1 Introduction

In this paper, we view a stereotype as an overall judgement brought by an observer over a given group of individuals or objects, such as "Frenchmen are arrogant", "Restaurants in the 16th arrondissement of Paris are expensive", or "Turks are smokers". Perceiving the world through stereotypes is a typical human attitude.

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Lee et al. (1995) see stereotypes as tools used by the mind to navigate its complex environment. In that respect, the formation of stereotypes is inevitable, at least to some extent.

The question we address here is not why we form stereotypes but how we do it. To be more precise, we wish to identify mathematical models that can help us understand the formation of stereotypes.

As Krueger et al. (2003) discuss, in the literature on stereotype formation, the *attribution hypothesis* is central (although other hypotheses can be envisaged). It asserts that an observer judges a group according to the observations she makes about the members of the group. There is a *stereotypical attribute* which the observer may think that the group as a whole exhibits. The members of the group carry certain *characteristics* which are somehow related to this stereotypical attribute. If the observer sees that these characteristics are "sufficiently prevalent" in a "sufficiently representative" subgroup then she concludes that the group as a whole exhibits the stereotypical attribute. For example, suppose that the stereotypical attribute is "being a smoker" as a group, say the Turks. Among the Turks the observer knows, she observes that some are smokers while some are not and if smoking is prevalent in a "sufficiently representative" subgroup of Turks, then she concludes that "Turks are smokers".[1]

We will propose several simple models for the formation of stereotypes. These models are inspired by related models in the area of categorization in mathematical psychology. Categorization models aim at explaining the conditions under which an individual decides that an object carrying several properties may or not belong to a certain category. This link with categorization problems should be no surprise since many papers have already discussed the links between categorization and social stereotyping (see, e.g., Bodenhausen and Peery, 2009; Haslam and Turner, 1992; Hugenberg and Sacco, 2008; Schaller, 1991).

On a technical level, our model draws on Goldstein (1991) and several extensions presented in Bouyssou and Marchant (2007a,b), and Słowiński et al. (2002). In these works, the objects that are categorized belong to a product set that is not homogeneous. In our setting, this would mean that the set of characteristics could be specific to each observed individual or object. We do not allow here for this possibility. We work with the same set of characteristics for all observed individuals or objects and, hence, we deal with a homogeneous Cartesian product.

It will turn out that these models also have strong links with models coming from social choice theory and cooperative game theory. Indeed, we wish to illustrate how these two fields can contribute to other ones, here social psychology. In

[1] To quote Zawadzki (1948, p. 135, 2nd col.), "The popular conception of a group characteristic seems to be a characteristic which is present in the majority of the members of the group. According to this concept, it is a necessary and sufficient condition for a group characteristic to be represented in at least 51% of the members of the group".

doing so, we feel close to many works authored by Philippe Mongin, in which the power and versatility of such models is brilliantly exposed.

An analysis of stereotype formation with a similar spirit is by Can and Sanver (2009). Comparing their analysis with the one presented here, several points should be noticed.

The model in Can and Sanver (2009) is Arrovian in nature and is inspired by the analysis in May (1952) of the method of majority decisions. Hence, the decision that is taken is threefold. The observer can accept the stereotype (Turks are smokers). She can accept its "opposite" (Turks are non-smokers). Finally, she can accept none of the above two stereotypes. Indeed, in May's model, when comparing two alternatives x and y, one may conclude that "x is preferred to y", that "y is preferred to x" or that "x and y are indifferent". Our model is more in line on this point with the literature on simple games (Grabisch, 2016; Owen, 2013) and their extensions (Bilbao, 2000; Hsiao and Raghavan, 1993). The decision taken in our basic model is either to accept a stereotype (because the coalition supporting the proposal is "winning") or not to accept it (because the coalition supporting the proposal is not "winning"). In order to illustrate the flexibility of our approach, we propose an extension of our basic model. The model of Can and Sanver is a particular case of this extension.

A second point that deserves to be noticed is the following. The model of Can and Sanver requires the observed characteristic to be binary. In our example of smoking, each group member is either a smoker or not. Whether some group members are "heavy smokers" has no impact on the conclusion. Our model removes this restriction by considering a denumerable set of characteristics.

We conceive a *perception function* as a mapping from the set of characteristic profiles into the binary decision of whether the group as a whole admits the attribute or not. In our basic model the perception function takes the form of what we call a *mental process*. It consists in the combination of three steps: (i) a "numerical representation" assigns a real number to every characteristic in such a way that higher numbers mean higher support for the attribute, (ii) an "aggregation function" consistently transforms vectors of real numbers into a single real number, and (iii) the resulting number is compared to a threshold and the attribute is accepted if this threshold is exceeded.

We characterize the class of perception functions that can be explained through mental processes. As a particular case of our characterization, we consider mental processes where the numerical representation is dichotomous. We also extend our analysis so as to cover observers whose decisions are more general than being binary, as well as the ones who are unable to observe all members of the group.

The structure of the paper is as follows. Section 2 formalizes the basic concepts. Section 3 presents our main results. Section 4 discusses a possible extension of our basic model. Section 5 concludes.

2 Perception functions and mental processes

Let $N = \{1, \ldots, n\}$ with $n \geq 2$ be a finite set. We interpret the set N as being a group of *individuals* or objects (henceforth, we say "individuals" for the sake of brevity) that is scrutinized by an observer. Each individual carries one of the *characteristics* coming from a (nonempty) denumerable (i.e., finite or countably infinite, the necessity of this restriction will be discussed later) set X.

In order to avoid trivial cases, we suppose that X has at least two elements. Hence, an observer who looks at the set of individuals sees a *profile* $p = (p_i)_{i \in N} \in X^n$ of characteristics. Given a profile $p \in X^n$ and some $i \in N$, we write (x, p_{-i}) as a shorthand for the profile in which the characteristic of $i \in N$ is $x \in X$, whereas all other individuals carry the same characteristic as in the profile p.

There is a (stereotypical) *attribute* which the group of individuals as a whole may exhibit. Whether this group exhibits this attribute is somehow connected to the characteristics of the group members. Thus, the observer decides whether the group exhibits the attribute as a function of the profile of characteristics she observes. In other words, the observer has a *perception function* $\varphi : X^n \to \{0, 1\}$ where, for each $p \in X^n$, we interpret $\varphi(p) = 1$ as the group as a whole exhibiting the attribute and $\varphi(p) = 0$ as not making such a conclusion about the group.[2,3]

A *numerical representation (of characteristics)* is a mapping $v : X \to \mathbb{R}$ that assigns a real number to each characteristic, in such a way that the higher the number, the stronger the support for the attribute. As is customary, we denote by $v(X)$ the image of the set X induced by the mapping v. An *aggregation function* is a mapping $\mathcal{F} : [v(X)]^n \to \mathbb{R}$ that *consistently* transforms an n-tuple of real numbers into a single real number. The consistency requirement is interpreted here as meaning that $\mathcal{F}$ is non-decreasing in each of its arguments.

We call a pair $(v, \mathcal{F})$ a *mental process.* A mental process $(v, \mathcal{F})$ *explains* a perception function φ if:

$$\varphi(p) = 1 \Leftrightarrow \mathcal{F}(v(p_1), \ldots, v(p_n)) > 0, \tag{MP}$$

for all $p \in X^n$. In this case, we say that a perception function is explained by the Mental Process model (henceforth the MP model).

[2]Our analysis does not rule out constant perception functions where $\varphi(p)$ gets the same value (either 0 or 1) at every $p \in X^n$, although it is clear that such trivial perception functions are not particularly interesting.

[3]The use of the numbers "0" and "1" in the codomain is purely conventional. Any other numbers could have been used. The only constraint is that the largest number is interpreted as the acceptance of the attribute.

Remark 1
The definition of the MP model calls for several observations.

- Setting the threshold at 0 is arbitrary but is without loss of generality. Any other threshold could have been used instead.
- We will later observe that the strict inequality used to compare $\mathcal{F}(v(p_1), \ldots, v(p_n))$ to the threshold 0 can equivalently be replaced by a nonstrict inequality.
- Finally, we will also show that the apparently more restrictive model in which $\mathcal{F}$ is *increasing* (instead of simply *non-decreasing*) in each of its arguments is, in fact, equivalent the MP model. •

3 Main results

3.1 A characterization of the MP model

We introduce a condition over perception functions that allows us to explain them by mental processes. We say that a perception function φ satisfies the *ordering of the characteristics* condition (henceforth, condition OC) if

$$\left.\begin{array}{c}\varphi\big((x, p_{-i})\big) = 1 \\ \text{and} \\ \varphi\big((y, q_{-j})\big) = 1\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\varphi\big((y, p_{-i})\big) = 1, \\ \text{or} \\ \varphi\big((x, q_{-j})\big) = 1,\end{array}\right. \tag{OC}$$

for all $i, j \in N$, all $x, y \in X$ and all $p, q \in X^n$.

Remark 2
In the expression of the above condition, the "or" is not exclusive (the same will be true in conditions DOC and GOC, respectively defined in Sections 3.2 and 4). A related, albeit different, condition is called "linearity" in Bouyssou and Marchant (2007a). Linearity can be traced back to Goldstein (1991). It is also used in Słowiński et al. (2002). Linearity applies to Cartesian products that are not necessarily homogeneous. Condition OC strengthens this condition taking advantage of the fact that we work on homogeneous Cartesian products. •

Condition OC is tantamount to requiring that elements of X are (weakly) ordered: either x is above y or vice versa. In the first case, if $\varphi\big((y, p_{-i})\big) = 1$, then we must have $\varphi\big((x, p_{-i})\big) = 1$. In the second case, if $\varphi\big((x, p_{-i})\big) = 1$, then we must have $\varphi\big((y, p_{-i})\big) = 1$. For another interpretation of this condition, let us give an example of its failure. Let X be the set of natural numbers that reflect the income levels of the n observed individuals of a group, while the attribute of the group is "being egalitarian". For each $p \in X^n$, let $\varphi(p) = 1 \Leftrightarrow p_i = p_j$, $\forall i, j \in N$. In other words, the observer thinks that the group is egalitarian if and only if each of its members has the same income level. We have in this example, $\varphi(1, 1, \ldots, 1) = 1$ and $\varphi(2, 2, \ldots, 2) = 1$, while $\varphi(2, 1, 1, \ldots, 1) = 0$ and $\varphi(1, 2, 2, \ldots, 2) = 0$, which violates condition OC. Note that in this example, the perception function does not admit an ordering of the income levels such that getting individual observations higher in that order can only contribute to concluding that the group exhibits the attribute. In fact, this is precisely what condition OC ensures: the perception function induces an ordering (in fact a weak ordering) of the characteristics such that getting individual observations higher in that order cannot be detrimental to concluding that the group exhibits the attribute.

We define the binary relation $\succsim_\varphi$ on X as follows:

$$x \succsim_\varphi y \text{ if } \big[\varphi(x, p_{-i}) \geq \varphi(y, p_{-i}), \text{ for all } i \in N \text{ and all } p \in X^n\big].$$

Let $\succ_\varphi$ (resp. $\sim_\varphi$) be the asymmetric (resp. symmetric) part of $\succsim_\varphi$. Notice that, by construction, the relation $\succsim_\varphi$ is transitive. It may not be complete however. The following lemma formalizes the consequences of condition OC.

Lemma 3
The perception function φ satisfies condition OC if and only if (iff) $\succsim_\varphi$ is complete.

Proof
Necessity is easily shown. Suppose that $\varphi(y, p_{-j}) = 1$ and $\varphi(x, p_{-i}) = 1$. We have $x \succsim_\varphi y$ or $y \succsim_\varphi x$. In the first case, $\varphi(y, p_{-j}) = 1$ implies $\varphi(x, p_{-j}) = 1$. In the second case, $\varphi(x, p_{-i}) = 1$ implies $\varphi(y, p_{-i}) = 1$. Hence, φ orders the characteristics.

Let us show sufficiency. Suppose that the relation $\succsim_\varphi$ is not complete, so that, for some $x, y \in X$, we have neither $x \succsim_\varphi y$, nor $y \succsim_\varphi x$. The former implies that $\varphi(y, p_{-i}) = 1$ and $\varphi(x, p_{-i}) = 0$, for some $p \in X^n$ and some $i \in N$. The latter implies that $\varphi(x, q_{-j}) = 1$ and $\varphi(y, q_{-j}) = 0$, for some $q \in X^n$ and some $j \in N$. This contradicts the fact that φ satisfies condition OC. □

When $\succsim_\varphi$ is a weak order, i.e., when φ satisfies condition OC, let $v_\varphi : X \to \mathbb{R}$ be any numerical representation of $\succsim_\varphi$. This means that

$$v_\varphi(x) \geq v_\varphi(y) \Leftrightarrow x \succsim_\varphi y,$$

for all $x, y \in X$. The existence of such a function v_φ is ensured, since we have supposed that X is denumerable.

Remark 4
Our wish to obtain a numerical representation of the weak order $\succsim_\varphi$ on X is the only motivation for having restricted X to be denumerable. Adding an appropriate order-denseness condition would allow us to alleviate this restriction. Since this is not crucial for our purposes, we do not develop this point. •

In fact, perception functions satisfying condition OC are exactly the ones that can be explained by mental processes. In order to show this, the following lemma will be useful.

Lemma 5
Let $p, q \in X^n$. If $p_i \sim_\varphi q_i$, $\forall i \in N$, then $\varphi(p) = \varphi(q)$.

Proof
Take any $p, q \in X^n$ and suppose that $p_i \sim_\varphi q_i$, $\forall i \in N$. Using, $p_1 \sim_\varphi q_1$ we obtain $\varphi(p) = \varphi(q_1, p_{-1})$. Now, using $p_2 \sim_\varphi q_2$, we obtain $\varphi(q_1, p_{-1}) = \varphi(q_1, q_2, p_{-\{1,2\}})$, abusing notation in an obvious way. Hence, we have $\varphi(p) = \varphi(q_1, q_2, p_{-\{1,2\}})$. Iterating the above reasoning leads to the desired conclusion. □

We are now in position to state the main result in this section.

Theorem 6
A perception function φ can be explained by some mental process $(v, \mathcal{F})$ iff φ satisfies condition OC.

Proof
Necessity. Let φ be explained by some mental process $(v, \mathcal{F})$. Take any (x, p_{-i}), $(y, q_{-j}) \in X^n$ with $\varphi(x, p_{-i}) = 1$ and $\varphi(y, q_{-j}) = 1$. This implies:

$$\mathcal{F}(v(x), (v(p_k))_{k \neq i}) > 0 \text{ and } \mathcal{F}(v(y), v(q_\ell)_{\ell \neq j}) > 0,$$

abusing notation in an obvious way. Now, if $v(x) \geq v(y)$, the non-decreasingness of $\mathcal{F}$ in all its arguments implies $\mathcal{F}(v(x), v(q_\ell)_{\ell \neq j}) > 0$, so that $\varphi(x, q_{-j}) = 1$. In the case $v(y) \geq v(x)$, we similarly conclude that $\mathcal{F}(v(y), (v(p_k))_{k \neq i}) > 0$, so that $\varphi(x, q_{-i}) = 1$.

Sufficiency. As φ satisfies condition OC, we know from by Lemma 3 that $\succsim_\varphi$ is complete, so that it is a weak order. Let v_φ be any numerical representation of $\succsim_\varphi$. Using Lemma 5 and the definition of v_φ ensures that the aggregation function such that:

$$\mathcal{F}(v_\varphi(p_1), \ldots, v_\varphi(p_n)) = \begin{cases} \exp(\sum_{i \in N} v_\varphi(p_i)) & \text{if } \varphi(p) = 1, \\ -\exp(-\sum_{i \in N} v_\varphi(p_i)) & \text{if } \varphi(p) = 0, \end{cases}$$

is well-defined. Using the definition of v_φ and the definition of $\mathcal{F}$, it is easy to show that $\mathcal{F}$ is increasing in all its arguments. This completes the proof since it is clear that $(v_\varphi, \mathcal{F})$ explains φ. □

Remark 7
The above result prompts a number of observations.

- The aggregation function built in the above proof is increasing in all its arguments. This shows that the model in which $\mathcal{F}$ is increasing is equivalent to the apparently more general MP model.
- It is also clear that our aggregation function is never equal to the threshold 0. Hence, in the expression of the MP model, we can always replace the strict inequality by a non-strict one, without any change.
- Finally considering a mapping of $\mathbb{R}^n$ into $(-\infty, \kappa)$ and a mapping of $\mathbb{R}^n$ into $(\kappa, +\infty)$, both being non-decreasing in all their arguments show that we may replace the threshold 0 by any threshold $\kappa \in \mathbb{R}$.
- Constant perception functions trivially order the characteristics. It follows that they are covered by Theorem 6 and, hence, can be explained by mental processes. •

Remark 8
A perception function φ that can be explained by some mental process induces on the set of characteristics X a partition that has a simple and intuitive interpretation.

Suppose that we have observed the same characteristic $z \in X$ for all $i \in N$. We have either $\varphi(\mathbf{z}) = 1$ or $\varphi(\mathbf{z}) = 0$, where $\mathbf{z}$ stands for the n-tuple $(z, z, \ldots, z)$. Let $X^+_\varphi = \{z \in X : \varphi(\mathbf{z}) = 1\}$ and $X^+_\varphi = \{z \in X : \varphi(\mathbf{z}) = 0\}$. It is clear that X^+_φ and X^-_φ partition the set of characteristics X (as soon as φ is not constant).

The elements in X^+_φ are, in some sense, "pro-stereotype" whereas the opposite interpretation holds for the elements in X^-_φ. Notice however that the elements in X^+_φ do not necessarily form an equivalence class of the relation $\succsim_\varphi$. Some elements in X^+_φ might be "very strongly" pro-stereotype, while other will be only "mildly" so. The same applies to the elements of X^-_φ.

Hence, the model has a weak order together with a partition between "positive" and "negative" levels[4]. A similar construction can be performed for all the models defined in the paper. •

3.2 Dichotomous mental processes

Dichotomous mental processes occur when the relation $\succsim_\varphi$ is a weak order having at most two distinct equivalence classes. This means that, although there may be

[4]This resembles the preference approval framework in social choice theory (see Sanver, 2010) in which voters not only rank alternatives but also qualify them as "good" or as "bad".

many characteristics in the set X, only two types, at most, can be distinguished with respect to their impact on the attribute. When this is the case, it is always possible to use the set $\{0,1\}$ as a codomain for v_φ.

A mental process of this kind has a remarkably simple interpretation. Each individual in $i \in N$ votes "yes" or "no", according to whether the value of $v_{\succsim_\varphi(x)}$ is 1 or 0. Now the aggregation function maps this n-tuple of 0 and 1 into the set[5] $\{0,1\}$ and is non-decreasing in all its arguments. Hence, we may view the aggregation function of a simple game that consists in dividing the power set 2^N into winning and loosing coalitions in a way that is compatible with set inclusion. This is easily formalized.

A *simple game* is a mapping $\gamma : 2^N \to \{0,1\}$ such that $\gamma(A) \geq \gamma(B)$, whenever $A \supseteq B$ (we omit here the condition saying that the grand set N is a winning coalition, whereas the empty set $\varnothing$ is not. This is because, we have not excluded trivial perception functions from our analysis). For more information about simple games, we refer to Grabisch (2016) or Owen (2013).

A simple game γ induces an aggregation function $\mathcal{F}_\gamma : \{0,1\}^n \to \{0,1\}$ defined, for all $r = (r_1, \ldots, r_n) \in \{0,1\}^n$, letting $\mathcal{F}_\gamma(r) = \gamma(\{i \in N : r_i = 1\})$. In other words, when a perception function φ dichotomously orders the characteristics, there is a simple game γ that induces an aggregation function $\mathcal{F}_\gamma$ such that the mental process $(v_\varphi, \mathcal{F}_\gamma)$ explains φ. This corollary to Theorem 6 is stated below.

Theorem 9
A perception function φ can be explained by some dichotomous mental process $(v, \mathcal{F}_\gamma)$ induced by a simple game γ iff φ dichotomously orders the characteristics.

Remark 10
A *capacity* is a simple game γ that is normalized, i.e., such that $\gamma(\varnothing) = 0$, $\gamma(N) = 1$. When φ is not constant, i.e., $\varphi(p) = 1$ and $\varphi(q) = 0$ for some $p, q \in X^n$, and dichotomously orders the characteristics, then it is clear that the simple game allowing to represent φ, as in the above theorem is a capacity. •

It is not difficult to find a condition strengthening condition OC and ensuring that $\succsim_\varphi$ is a weak order having at most two equivalence classes. Building on Bouyssou and Marchant (2007a) and Bouyssou and Pirlot (2009), it is simple to show that the following condition is exactly what is needed.

We say that φ satisfies the *dichotomous ordering of characteristics* condition

[5] Although the codomain of $\mathcal{F}$ is here the same as the codomain of v_φ, this is purely anecdotal. Instead of taking $\{0,1\}$ as the codomain of φ, any codomain $\{\lambda, \tau\}$ with $\lambda < \tau$ could have been chosen.

(henceforth condition DOC) if

$$\left.\begin{array}{c}\varphi\big((x,p_{-i})\big)=1\\ \text{and}\\ \varphi\big((y,q_{-j})\big)=1\end{array}\right\}\Rightarrow\left\{\begin{array}{c}\varphi\big((y,p_{-i})\big)=1,\\ \text{or}\\ \varphi\big((z,q_{-j})\big)=1,\end{array}\right. \tag{DOC}$$

for all $i, j \in N$, all $x, y, z \in X$ and all $p, q \in X^n$.

It is clear that this condition strengthens condition OC (take z equal to x in condition DOC). It is easy to check that condition DOC implies that the weak order $\succsim_\varphi$ induced on X by the perception function φ can have at most two distinct equivalence classes. But it is also clear that condition DOC is implied by a dichotomous mental process. Indeed, suppose that $\varphi(y, p_{-i}) = 0$. Because we know that $\varphi(x, p_{-i}) = 1$, this means that x and y cannot belong to the same equivalence class of $\succsim_\varphi$. This proves the following.

Theorem 11
A perception function φ can be explained by some mental process $(v, \mathcal{F})$ in which v takes at most two distinct values iff φ satisfies condition DOC.

3.3 Prejudice in the dichotomous model

Can and Sanver (2009) have studied the concept of prejudice in their model of stereotype formation. The idea is simple. In their approach the set of characteristics is binary (they take it to be $\{-1, 1\}$) and it is understood that these two characteristics are ordered with respect to the attribute. Hence, given a profile p, it makes sense to speak of the "opposite" of p, denoted by $-p$, in which all observations 1 are turned into -1 and vice versa. In this setting, a model of formation of stereotype is *impartial* (i.e., has no positive or negative prejudice with respect to the stereotype) if, when going from p to $-p$, the results of φ is switched from 1 to 0 and vice versa [6].

In the MP model, the set X may contain many elements. Hence, it does not make clear sense here to speak of the opposite of a profile, which complicates the analysis of prejudice in the MP model. However, in the particular case of the dichotomous MP model, this becomes possible again. We can now define the inverse of a profile p as the profile $-p$ in which all characteristics in p belonging to upper equivalence class of $\succsim_\varphi$ are replaced by a characteristic in the lower equivalence class of $\succsim_\varphi$ and vice versa.

With this definition, we can say, e.g., that a perception function φ has *no negative prejudice* if $\varphi(p) + \varphi(-p) \geq 1$, which forbids to have at the same time

[6] In the model of Can and Sanver, the codomain is $\{-1, 0, 1\}$. We have simply adapted their idea to our model in which the codomain is $\{0, 1\}$.

$\varphi(p) = 0$ and $\varphi(-p) = 0$ (we can define analogously the notion of having *no positive prejudice* if $\varphi(p) + \varphi(-p) \leq 1$, which forbids to have at the same time $\varphi(p) = 1$ and $\varphi(-p) = 1$).

Results similar to the ones in Can and Sanver (2009) can then be established. Indeed, suppose that a perception function φ is nontrivial and satisfies condition DOC. In view of Theorem 11, we know that φ can be explained by some mental process $(v, \mathcal{F})$ in which v takes at most two distinct values. Clearly it is not restrictive to suppose that these two values are 0 and 1. In view of Theorem 9 and Remark 10, it follows that there is a normalized capacity γ such that:

$$\varphi(p) = 1 \iff \gamma(\{i \in N : v(p_i) = 1\}) = 1,$$

for all profiles $p \in X^n$. In such a case, we say that the perception function φ is explained by the normalized capacity γ.

In this framework, it is clear that the normalized capacity γ that explains a nonconstant dichotomous MP model with no negative prejudice (resp. no positive prejudice), is such that, for all proper and nonempty subsets A of N, $\gamma(A) + \gamma(N \setminus A) \geq 1$ (resp. $\gamma(A) + \gamma(N \setminus A) \leq 1$). This proves the following.

Theorem 12
A nontrivial perception function φ that is explained by some dichotomous mental process can also be explained by a normalized capacity. If the perception function has no negative prejudice (resp. no positive prejudice), the normalized capacity can be chosen so as to satisfy $\gamma(A) + \gamma(N \setminus A) \geq 1$ (resp. $\gamma(A) + \gamma(N \setminus A) \leq 1$), for all proper and nonempty subsets A of N.

4 A simple extension of the basic model

In the model of Can and Sanver, each member of the group is observed to exhibit one of two possible characteristics which are implicitly assumed to be ordered beforehand. In this respect, the MP model that allows one to observe on each member of the group a characteristic belonging to a denumerable set is a direct generalization of the model of Can and Sanver. Moreover, the MP model does not rely on an a priori ordering of the characteristics.

The model of Can and Sanver, inspired by the formal framework of May (1952), allows the observer to conclude one of $\{-1, 0, +1\}$, meaning that we accept the attribute $(+1)$, we reject it (-1), or we do not conclude (0). In this respect, it is more general than the MP model that only allows for a binary conclusion.

We show in this section how it is possible to extend our basic model in order to allow for richer codomains. This will make the model of Can and Sanver a particular case of this extension. This will also illustrate the flexibility and richness of the framework presented here.

Instead of taking the set $\{0,1\}$ to be the codomain of the perception function, let us suppose now that the elements of the codomain belong to a finite set and are ordered with respect to the acceptance of the attribute, e.g., the stereotype may be "strongly accepted", "weakly accepted", "neither accepted nor rejected", "weakly rejected", or "strongly rejected", which will lead to the codomain[7] $\{1,2,3,4,5\}$. After such an extension, the model of Can and Sanver will clearly become a particular case of ours.

This extension is easy to analyze but a full formal analysis would require the introduction of a cumbersome notation. Hence, we only outline how this is done, leaving the details to the interested reader.

A *generalized perception function* Φ maps the set X^n into the ordered set $R = \{1, 2, \ldots, r\}$. The elements of R are interpreted as levels of acceptance of the stereotype, r being the highest level and 1 being the lowest level.

We say that a *generalized mental process* $(V, \mathcal{G})$ explains the generalized perception function Φ if V is a mapping from X to $\mathbb{R}$, $\mathcal{G}$ is a mapping from $[V(X)]^n$ to $\mathbb{R}$ that is non-decreasing in each of its arguments, and there are real numbers σ_1, σ_2, ..., $\sigma_{r+1} \in \mathbb{R}$ satisfying $\sigma_1 < \sigma_2 < \cdots < \sigma_{r+1}$, such that:

$$\Phi(p) = k \Leftrightarrow \sigma_k < \mathcal{G}(V(p_1), \ldots, V(p_n)) \leq \sigma_{k+1}, \qquad \text{(GMP)}$$

for all $p \in X^n$ and $k \in R$. This is what we call the generalized mental process model (henceforth, GMP model).

Remark 13
As above with the MP model, the choice of strict and nonstrict inequalities is conventional. Similarly, the apparently more restrictive model in which Φ is required to be increasing in all its arguments is equivalent to the GMP model. •

Starting from a generalized perception function Φ, we build the following binary relation on X, letting, for all $x, y \in X$,

$$x \succsim_\Phi y \Leftrightarrow \big[\Phi(x, p_{-i}) \geq \Phi(y, p_{-i}), \text{ for all } i \in N \text{ and } p \in X^n\big].$$

Similarly to what was the case in the MP model, it is clear that the relation $\succsim_\Phi$ is transitive but may not be complete. In the case it is complete, it will thus be a weak order. Consider now any numerical representation V_Φ of this weak order $\succsim_\Phi$ mapping X into $\mathbb{R}$ such that

$$x \succsim_\Phi y \Leftrightarrow V_\Phi(x) \geq V_\Phi(y),$$

[7] As in footnote 3, it is not restrictive to suppose that the codomain is a subset of $\mathbb{N}$. The numbers that are chosen are clearly immaterial for our analysis.

for all $x, y \in X$. Given any such a function V_Φ, we proceed as follows. Take any σ_1, σ_2, ..., $\sigma_{r+1} \in \mathbb{R}$ such that $\sigma_1 < \sigma_2 < \cdots < \sigma_{r+1}$. For all $k \in R$, consider any increasing function Λ_k mapping $\mathbb{R}^n$ into (σ_k, σ_{k+1}). Finally define $\mathcal{G}$ in the following way:

$$\mathcal{G}(V_\Phi(p)) = \Lambda_k \left(\sum_{i=1}^{n} V_\Phi(p_i) \right) \text{ if } V_\Phi(x) = k.$$

The well-definedness of $\mathcal{G}$ is easy to establish, following the same arguments as in Lemma 3. Its non-decreasingness easily follows from its construction and the definition of V_Φ.

Therefore the GMP model holds as soon as $\succsim_\Phi$ is complete. It is easy to devise a condition that ensures this. We say that the generalized perception function Φ satisfies the *generalized ordering of characteristics* condition (henceforth, condition GOC) if

$$\left.\begin{array}{c} \Phi((x, p_{-i})) = k \\ \text{and} \\ \Phi((y, q_{-j})) = \ell \end{array}\right\} \Rightarrow \left\{\begin{array}{c} \Phi((y, p_{-i})) \geq k, \\ \text{or} \\ \Phi((x, q_{-j})) \geq \ell, \end{array}\right. \tag{GOC}$$

for all $i, j \in N$, all $x, y \in X$, all $p, q \in X^n$ and all $k, \ell \in R$. Observe that when R contains two elements, condition GOC reduces to condition OC.

This condition is new. Related conditions for non-homogeneous Cartesian products can be traced back to in Bouyssou and Marchant (2007b), Goldstein (1991), and Słowiński et al. (2002).

In view of the above observations, it is routine to prove the following.

Theorem 14
A generalized perception function Φ can be explained by a GMP $(V, \mathcal{G})$ iff Φ satisfies the condition GOC.

The model of Can and Sanver is clearly a particular case of the GMP model. It obtains when X consists of only two elements (-1 and 1, as used in Can and Sanver (2009)) and R consists of three elements (-1, 0 and 1, as used in Can and Sanver (2009)).

5 Conclusion

We have proposed several models for the formation of stereotypes. Our basic model (the MP model) is quite simple and decomposes a perception function using the notion of a mental process. This model has a transparent interpretation and is easy to analyze from an axiomatic point of view. Moreover, we have shown that it can easily be extended in various directions.

This model allows us to deal with observed characteristics that are not restricted to be binary (contrarily to what is the case in the model of Can and Sanver). A drawback of having a richer set of characteristics is that the notion of prejudice, as studied in Can and Sanver (2009), becomes more difficult to analyze. This is because there is no obvious algebraic structure on the set of characteristics that would allow us to speak of the opposite of a profile. This is however quite possible with dichotomous mental processes, i.e., the special case of the MP model in which there are only at most two types of characteristics.

Our basic model only allows for a binary decision: either the stereotype is accepted or not. This is at variance with the model of Can and Sanver in which three decisions can be made: the stereotype can be accepted, it can be rejected or no decision is taken. We have shown that our framework can easily be extended to allow for a rich variety of conclusions (the GMP model), making the model of Can and Sanver a particular case of our framework.

It is not difficult to generalize our approach to cover the case of "imperfect observation". A similar extension is presented in Can and Sanver (2009). The model of Can and Sanver handles imperfect observation by redefining its perception function, its axioms and its explanatory tools, restricting attention to a set $T \subset N$ of "visible" members of the group, i.e., members of the group we have been to observe a characteristic.

Suppose that T has $m < n$ elements. We can proceed as in Can and Sanver (2009). We replace the domain X^n of the perception function in the MP model with X^m. The MP model uses a single axiom, condition OC. It is easy to modify this condition to cope with a set of visible members $T \subset N$: in the expression of condition OC, just replace "for all $i, j \in N$" with "for all $i, j \in T$" and "X^n" with "X^m". The rest of the analysis is almost without change, once in the definition of the relation $\succsim_\varphi$ we also replace "for all $i \in N$ and $p, q \in X^n$" with "for all $i \in T$ and $p, q \in X^m$". Indeed, once the relation $\succsim_\varphi$ is defined and is constrained to be complete, the definition of v_φ and $\mathcal{F}$ readily follows, replacing whenever needed n with m and N with T. These modifications allows us to state Theorems 6 and 11 without change, in the case of partial observation. Similar modifications would allow us to state Theorem 14 for this new context.

A more difficult question would be to relate several models corresponding to several levels of imperfect observation. This more difficult but more interesting question would allow us to deal with the attitude of the individual towards the lack of information, a notion that is absent from the present paper. This clearly calls for further studies.

Let us conclude with a remark on a related strand of literature. When studying social stereotyping, we investigate how an individual observing a group draws a stereotypical conclusion about the group that is observed. In some sense, this is a kind of dual problem to the now classic literature on the "Who is a J.?" question (see, e.g., Çengelci and Sanver, 2010; Dimitrov and Puppe, 2011; Houy, 2007; Kasher and Rubinstein, 1997; Samet and Schmeidler, 2003) in which a group has to take a decision on whether of not an individual belongs to a group (the J.). Clearly, the techniques used in both fields are rather different. In particular, the notion of "liberalism" that is fundamental in this literature does not seem to have a direct counterpart in our framework.

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