An introduction to Nontransitive Decomposable Conjoint Measurement

with application to Noncompensatory Preferences

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Outline

- Introduction and Motivation
- Nontransitive Conjoint Measurement
  - Overview and summary of results
- Noncompensatory Preferences
  - Definitions
  - Related works and motivation
  - A general conjoint measurement model
  - Results
- Discussion
  - Extensions
  - Open problems

Introduction: Conjoint Measurement

- Set of *attributes* $N = \{1, 2, \dots, n\}$
- Set of *objects* evaluated on N: $Y \subseteq X_1 \times X_2 \times \cdots \times X_n$
- *Binary relation* on the set of objects: $\succsim$

Objective: Study/Build/Axiomatise numerical representations of $\succsim$

Interest of Numerical Representations

- Manipulation of $\succsim$
- Construction of numerical representations

Interest of Axiomatic Analysis

- Tests of models
- Understanding models

Introduction: Cartesian Product Structures

- **MCDM**
  - x is an "alternative" evaluated on "attributes"
- **DM under uncertainty**
  - x is an "act" evaluated on "states of nature"
- **Economics**
  - x is a "bundle" of "commodities"
- **Dynamic DM**
  - x is an "alternative" evaluated at "several moments in time"
- **Social Choice**
  - x is a "distribution" between several "individuals"

$x \succsim y$ means "x is at least as good as y"

Introduction: Additive Transitive Representation

Basic model: **Additive utility**

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$

Examples:

- MCDM: Weighted sum, Additive utility, Goal programming, Compromise Programming
- DM under uncertainty: SEU
- Dynamic DM: Discounting
- Social Choice: Inequality measures *à la* Atkinson/Sen

Well-developed Theory (Debreu 1960, Luce & Tukey 1964)

Introduction: Problems

- **Empirical problems**
  - Transitivity of $\sim$ (Luce 1956)
  - Transitivity of $\succ$ (May 1954, Tversky 1969)
  - Additional conditions: Independence (EU vs. Choquet EU)
- **Technical Problems**
  - Asymmetry: “finite” vs. “Rich” cases
  - Asymmetry: $n = 2$ vs. $n \geq 3$ cases

Study more general models

- X finite (Scott-Suppes 1958, Scott 1964)
  - Necessary and sufficient Conditions
  - Denumerable Set of "Cancellation Conditions"
  - No nice uniqueness results
  - Axioms hardly interpretable and testable
- X has a "rich structure" and $\succsim$ behaves consistently in this "continuum" (Debreu 1960, Luce-Tukey 1964)
  - (Topological assumptions + continuity) or (solvability assumption + Archimedean condition)
  - A finite (and limited) set of "Cancellation Conditions" entails the representation (independence, TC)
  - u_i define "interval scales" with common unit ($v_i = \alpha u_i + \beta_i$)
  - Asymmetry $n = 2$ vs. $n \geq 3$
  - Respective roles of necessary vs. structural conditions

Introduction: Possible extensions

- Additive utility = $\underbrace{\textit{Additive}}_{1}\ \underbrace{\textit{Transitive}}_{2}$ Conjoint Measurement
- Extensions
  1. Drop additivity
  2. Drop transitivity and/or completeness
- Other extensions: more complex additive forms (Choquet EU, Gini-like inequality measures)

Introduction: Extensions

Decomposable Transitive model (Krantz et al (1971))

$$x \succsim y \Leftrightarrow F(u_i(x_i)) \geq F(u_i(y_i)) \qquad F \text{ increasing}$$

Advantages Simple axiomatic analysis, Simple proofs

Drawbacks Transitivity and completeness

Introduction: Extensions

Additive Non Transitive Models

(Bouyssou 1986, Fishburn 1990, 1991, Vind 1991)

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0 \qquad p_i(x_i, x_i) = 0 \text{ or } p_i \text{ skew symmetric}$$

Advantages Flexible towards transitivity and completeness, Classical results are particular cases

Drawbacks Asymmetries, Complex proofs

Particular case: Additive Difference Model (Tversky 1969)

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0 \qquad \Phi_i \text{ increasing and odd}$$

Introduction: Models

Additive Transitive	$\longleftrightarrow$	Transitive Decomposable
$\updownarrow$		$\updownarrow$
Additive Nontransitive	$\longleftrightarrow$	**?**

Nontransitive decomposable models (Bouyssou and Pirlot)

$$x \succsim y \Leftrightarrow F(p_i(x_i, y_i)_{i=1,2,\ldots,n}) \geq 0$$

with additional properties:

- F increasing/nondecreasing and/or odd, p_i skew symmetric
- $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ (with $\varphi_i(\nearrow, \searrow)$))

Introduction: Analysis

Non Transitive Decomposable models:

- imply substantive requirements on $\succsim$
- may be axiomatized in a simple way avoiding the use of a denumerable number of conditions in the finite case and of unnecessary structural assumptions in the infinite case
- allow to study the “pure consequences” of cancellation conditions in the absence of transitivity, completeness and structural requirements on X
- are sufficiently general to include as particular cases most aggregation rules that have been proposed in the literature
- provide insights on the links and differences between methods

Noncompensatory preferences

Idea: show the usefulness of the general framework of Nontransitive decomposable Conjoint measurement to study a particular problem

Noncompensatory preferences: Preferences governed by an importance relation on the set of subsets of attributes.

Motivation

- (Weighted) majorities
- MCDM: "outranking relations"
- Experimental Psychology

Strict Noncompensatory Preferences

Context: Conjoint measurement (MCDM)

Ingredients

- an *asymmetric* binary relation on each attribute $i \in N$: P_i
  - $P(x,y) = \{i \in N : x_i P_i y_i\}$
  - $P(y,x) = \{i \in N : y_i P_i x_i\}$
  - Asymmetry of P_i implies $P(x,y) \cap P(y,x) = \emptyset$
- an *asymmetric* importance relation $\rhd$ between disjoint subsets of attributes *monotonic* (wrt inclusion):

$$[A \rhd B, C \supseteq A, B \supseteq D, C \cap D = \emptyset] \Rightarrow [C \rhd D]$$

Strict Noncompensatory Preferences

Definition. A binary relation $\mathcal{P}$ on a set $Y \subseteq X_1 \times X_2 \times \cdots \times X_n$ of alternatives is said to be a *strict noncompensatory preference* if there are:

- an *asymmetric* binary relation $\rhd$ between disjoint subsets of N that is *monotonic* and
- an *asymmetric* binary relation P_i on each X_i $(i = 1, 2, \ldots, n)$

such that, for all $x, y \in Y$:

$$x\mathcal{P}y \Leftrightarrow P(x, y) \rhd P(y, x)$$

where $P(x, y) = \{i \in N : x_i P_i y_i\}$

- $\mathcal{P}$ is asymmetric
- $\mathcal{P}$ may not be transitive
- $\mathcal{P}$ may have circuits

Counterexample: (nontrivial) additive utility model

Questions

Suppose that you observe a binary relation $\succ$ on a set $X = X_1 \times X_2 \times \cdots \times X_n$

- What distinguishes $\succ$ if it is noncompensatory?
  - Characterization of strict noncompensatory relations
- In what sense a strict noncompensatory relation is different from a relation obtained using other aggregation approaches?
  - Characterization of strict noncompensatory relations using conditions that are not entirely specific to these relations

Notation

- $N = \{1, 2, \ldots, n\}$: set of attributes
- $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$: countable set of alternatives
- Abusing notations: (x_J, y_{-J}) and $(x_i, y_{-i}) \in X$, $X_{-J} = \prod_{i \notin J} X_i$, $X_{-i} = \prod_{j \neq i} X_j$

- $\succ$ asymmetric binary relation on X interpreted as "strict preference"
- for all $J \subseteq N$, define a*marginal preference relation*:

$$x_J \succ_J y_J \text{ iff } (x_J, z_{-J}) \succ (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J}$$

- attribute $i \in N$ is *essential* if for some $x_i, y_i \in X_i$ and some $z_{-i} \in X_{-i}$

$$(x_i, z_{-i}) \succ (y_i, z_{-i})$$

- attribute $i \in N$ is *influent* if for some $x_i, y_i, z_i, w_i \in X_i$ and some $x_{-i}, y_{-i} \in X_{-i}$

$$\begin{cases} (x_i, x_{-i}) \succ (y_i, y_{-i}) \\ \text{and} \\ Not[(z_i, x_{-i}) \succ (w_i, y_{-i})] \end{cases}$$

- essential $\Rightarrow$ influent; influent $\not\Rightarrow$ essential
- influence is innocuous, essentiality is *not*
- all attributes will be supposed influent (w(much)log)

Fishburn's Noncompensation (1976)

- Notation: $\succ(x,y) = \{i : x_i \succ_i y_i\}$

  $x_i \succ_i y_i$ *iff* $(x_i, z_{-i}) \succ (y_i, z_{-i})$, for all $z_{-i} \in X_{-i}$

- Asymmetry of $\succ$ $\Rightarrow$ asymmetry of $\succ_i$ $\Rightarrow \succ(x,y) \cap \succ(y,x) = \emptyset$

Definition: $\succ$ is *Fishburn noncompensatory* if

$$\left.\begin{array}{rcl} \succ(x,y) & = & \succ(z,w) \\ \succ(y,x) & = & \succ(w,z) \end{array}\right\} \Rightarrow [x \succ y \Leftrightarrow z \succ w]$$

Remark: Neutrality-like condition

Properties of Fishburn noncompensatory preferences

Proposition. If $\succ$ is Fishburn noncompensatory then

1. $\succ$ is independent:
   $(x_J, z_{-J}) \succ (y_J, z_{-J})$ for some $z_{-J} \in X_{-J} \Rightarrow x_J \succ_J y_J$
2. $x_i \sim_i y_i$ for all $i \in N \Rightarrow x \sim y$
3. $x_j \succ_j y_j$ for some $j \in N$ and $x_i \sim_i y_i$ for all $i \in N \setminus \{j\} \Rightarrow x \succ y$
4. all influent attributes are essential

Important remark

A strict noncompensatory relation may well violate *all* these conditions except independence

Example: semi-ordered weighted majorities

$$x\mathcal{P}y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \sum_{j \in P(y,x)} w_j + \varepsilon$$

where $\varepsilon > 0$ and $w_i \geq 0$ for all $i \in N$

If $w_j < \varepsilon$ attribute j is NOT essential (but may well be influent)

Is Fishburn's original idea useful?

Fishburn Monotonic Noncompensation

Definition: $\succ$ in *Fishburn monotonically noncompensatory* if

$$\left.\begin{array}{rcl} \succ(x,y) & \subseteq & \succ(z,w) \\ \succ(y,x) & \supseteq & \succ(w,z) \end{array}\right\} \Rightarrow [x \succ y \Rightarrow z \succ w]$$

Theorem (adapted from Fishburn 1976). The following are equivalent:

1. $\succ$ is a strict noncompensatory relation in which all attributes are *essential*
2. $\succ$ is an asymmetric relation being Fishburn monotonically noncompensatory

Problems

Basing the analysis of noncompensation on Fishburn's definition:

- leads to a *narrow view* of noncompensation excluding all relations in which attributes may be influent without being essential
- does NOT allow to point out the *specific features* of strict noncompensatory relations within a general framework of conjoint measurement
- amounts to using *very strong* conditions

An alternative approach

Nontransitive Decomposable Conjoint Measurement

Model (M)

$$x \succ y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) > 0$$

with:

- p_i *skew symmetric*: $p_i(x_i, y_i) = -p_i(y_i, x_i)$
- F *odd*: $F(\mathbf{x}) = -F(-\mathbf{x})$
- F *nondecreasing* in all its arguments

Interpretation: $p_i(x_i, y_i)$ are “preference differences” adequately combined by F

Axioms

$ARC1_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \succ (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, c_{-i}) \succ (y_i, d_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \succ (w_i, b_{-i}),\end{array}\right.$$

- $ARC1_i$ (Asymmetric inteR-attribute Cancellation) suggests that $\succ$ induces on X_i^2 a relation that compares "preference differences" in a well-behaved way
- $ARC1$ if $ARC1_i$ for all $i \in N$

Axioms

$ARC2_i$ if

$$\left.\begin{array}{r}(x_i, a_{-i}) \succ (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \succ (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succ (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \succ (z_i, d_{-i}),\end{array}\right.$$

- $ARC2_i$ suggests that the preference difference (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i)
- $ARC2$ if $ARC2_i$ for all $i \in N$

Induced Comparison of Preference Differences

Two quaternary relations

$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow$

$$[\text{for all } a_{-i}, b_{-i} \in X_{-i}, (z_i, a_{-i}) \succ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succ (y_i, b_{-i})]$$

$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow$

$$[(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)]$$

- $\succsim_i^*$ and $\succsim_i^{**}$ are transitive by construction
- $\succsim_i^*$ and $\succsim_i^{**}$ may not be complete
- $\succsim_i^{**}$ is reversible $(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow (w_i, z_i) \succsim_i^{**} (y_i, x_i)$

Results

Theorem. Let $\succ$ be a binary relation on a finite or countably infinite set $X = \prod_{i=1}^{n} X_i$. Then $\succ$ satisfies model (M) *iff* it is asymmetric and satisfies $ARC1$ and $ARC2$.

(can be extended to the general case using NS conditions)

Remark. Model (M) contains as particular cases:

1. Additive utilities: $x \succ y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) > \sum_{i=1}^{n} u_i(y_i)$
2. Additive differences: $x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) > 0$
3. Additive Nontransitive preferences: $x \succ y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) > 0$

Characterization of strict noncompensatory relations

Theorem. The following are equivalent

1. $\succ$ is a strict noncompensatory relation
2. $\succ$ has a representation in model (M) with all relations $\succsim_i^{**}$ having three distinct equivalence classes
3. $\succ$ is asymmetric, satisfies $ARC1$ and $ARC2$ and all relations $\succsim_i^{**}$ have three distinct equivalence classes

Remarks

- the condition that all $\succsim_i^{**}$ have three distinct equivalence classes can be expressed in terms of $\succ$ (technical, not very informative)
- full characterization of strict noncompensatory relations
- conditions $ARC1$ and $ARC2$ are NOT specific to strict noncompensatory relations
- asymmetry, $ARC1$ and $ARC2$ are independent conditions
- *specific feature* of strict noncompensatory relations: very rough differentiation of preference differences on each attribute (3 classes: positive, neutral, negative differences)

Discussion

Question:

Why not suppose in the definition of strict noncompensatory relations that P_i have nice properties (weak orders, strict semi-orders)?

Answer:

We could have done so. However this would not have allowed to improve the characterization.

New conditions: $AAC1$, $AAC2$ and $AAC3$ (traces of $\succ_i^{**}$)

Discussion

Question:

It is easy to generalize Arrow-like theorems to the case of MCDM using Fishburn's noncompensation or monotonic noncompensation.

Is it so with strict noncompensatory relations?

Answer:

YES because in a strict noncompensatory relation it is always true that

$$\left.\begin{array}{rcl} P(x,y) & \subseteq & P(z,w) \\ P(y,x) & \supseteq & P(w,z) \end{array}\right\} \Rightarrow [x \succ y \Rightarrow z \succ w]$$

Sample result

Theorem. Let $\succ$ be a *nonempty* strict noncompensatory relation on a finite set $X = \prod_{i=1}^{n} X_i$. Suppose that $\succ$ has been obtained using, on each $i \in N$, a relation P_i for which there are $a_i, b_i, c_i \in X_i$ such that $a_i P_i b_i$, $b_i P_i c_i$ and $a_i P_i c_i$.

Then, if $\succ$ is *transitive*, it has an *oligarchy*, i.e. there is a unique nonempty $O \subseteq N$ such that, for all $x, y \in X$:

- $x_i P_i y_i$ for all $i \in O \Rightarrow x \succ y$,
- $x_i P_i y_i$ for some $i \in O \Rightarrow Not[y \succ x]$.

Discussion

Question:

Does the analysis generalize to "large" preference relations?

Answer:

YES with an alternative general model:

$$x \succsim y \Leftrightarrow F(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0$$

- p_i *skew symmetric*: $p_i(x_i, y_i) = -p_i(y_i, x_i)$
- $F(\mathbf{0}) \geq 0$
- F *nondecreasing* in all its arguments

More difficult however because $\succsim$ may not be complete

Discussion

Question:

How to define "degrees" of noncompensation?

Answer: the analysis provides a mean to define the "degree of compensatoriness" of a binary relation using the number c_i^{**} of equivalence classes of $\succsim_i^{**}$

Degree of compensatoriness of $\succ$

$$c^{**} = \max_{i=1,2,\ldots,n} c_i^{**}$$

$c^{**} = 3$ *iff* $\succ$ is a strict noncompensatory relation

Discussion

Question:

What about Decision under Uncertainty?

Finite number of states $\Rightarrow$ Homogeneous Cartesian product: $X = C^n$

Counterpart of strict noncompensatory relations = Strict Lifting Rules (Dubois et al. 1997)

$$x\mathcal{P}y \Leftrightarrow P(x,y) \rhd P(y,x) \qquad \text{with } P(x,y) = \{i \in N : x_i P y_i\}$$

($\rhd$ model likelihood)

Examples: Probabilistic lifting, Possibilist Lifting

A full characterization of strict lifting rules is at hand using a variant of model (M) taking into account the homogeneity of the Cartesian product

Open Problems

- There are "intuitively" noncompensatory preference relations that do not enter our framework
  - Min, Max (particular cases of Choquet or Sugeno), Conjunctive, Disjunctive
- All these relations violate independence (and even weak independence).
- The present framework should be enlarged in order to encompass non-independent relations

Conjunctive rule

$X_i = A_i \cup Ui$ with $A_i \cap U_i = \emptyset$

$x \in A \Leftrightarrow x_i \in A_i$ for all $i \in N$

$x \succ y \Leftrightarrow x \in A$ and $y \in U$

Example

$x_i \in A_i$, $y_i \in U_i$, $a_{-i} \in A_{-i}$, $b_{-i} \in U_{-i}$

$(x_i, a_{-i}) \in A$, $(y_i, a_{-i}) \in U \Rightarrow (x_i, a_{-i}) \succ (y_i, a_{-i})$

$(x_i, b_{-i}) \in U$, $(y_i, b_{-i}) \in U \Rightarrow (x_i, b_{-i}) \sim (y_i, a_{-i})$

Weak separability

$(x_i, a_{-i}) \succ (y_i, a_{-i})$ and $(y_i, b_{-i}) \succ (y_i, b_{-i})$ is impossible