Models for the representation of preferences and similarities on a product set

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Introduction

Fuzzy control / Fuzzy logic

INPUT $\longrightarrow$ RULES $\longrightarrow$ OUTPUT

Problem

- model the input
- elaborate the rules
- interpret the output in terms of decision

Scheme similar to *constructive* decision making or decision aiding

Functional (or inferential) scheme

	Input	Black box	Output
Control	control variables	control mechanism	command
Fuzzy control	control variables	rules	decisions
Reasoning	data	inference rules	consequences
Decision making	evaluations	aggregation mechanism	decision
Classification	data	rules or inference	assignment

Simplistic view : strong emphasis on the modelling process

- Input have to be modelled:
  - DM : evaluations $\longrightarrow$ preferences
  - Classification : data $\longrightarrow$ similarity
  - Control ; control variables $\longrightarrow$ degree of membership
- Output have to be interpreted:
  - Decision-Aiding : fuzzy binary relation $\longrightarrow$ prescription
  - DM : fuzzy binary relation $\longrightarrow$ decision
  - Classification: fuzzy membership or fuzzy relation $\longrightarrow$ assignment to a class

The Box

The nature of the mechanism implemented may vary a lot :

- rules of various natures : logical, statistical inference, ...
- functional relationship

Legitimation of the mechanism implemented: it may implement :

- rules or dependencies extracted from observation (*descriptive* approach);
- consistency and rationality requirements (*normative* approach);
- “reasonable” ways of transforming the available information (input) in order to reach a goal (*constructive* approach)

Legitimation principles :

- observation of real behaviour (phenomenological models)
- logic
- consensus (of shareholders and analyst)

What is a method ?

A method is a comprehensive ensemble ($\neq$ a set) of procedures for dealing with various aspects of a problem

Example of aspects:

- modelling input;
- selecting or building a transformation mechanism;
- interpreting output;
- conceiving the interaction with the shareholders;
- ...

A method usually borrows to all approaches (descriptive, normative and constructive)

Our goal

Mainly interested in providing a **framework** for building black box contents

- in the *decision* and *classification* contexts
- transformation mechanisms *representable* as a *functional* model

Not a model for the way of thinking

Not a specific approach

Model of a class of models

Summary

1. Models of multiple criteria crisp preferences
2. Models of multiple criteria fuzzy preferences
3. Models for ordinal dissimilarity indices
4. Models for decision under uncertainty

Models for crisp preference and their characterisation

General model

$$x \succsim y \text{ iff } F(p_1(x_1,y_1),\ldots,p_n(x_n,y_n)) \geq 0$$

Model M_0**:**	$p_i(x_i,x_i)=0$ and $F(\mathbf{0})\geq 0$
Model M_1**:**	M_0 + F non-decreasing
Model M_2**:**	M_1 + p_i skew-symmetric
Model M_3**:**	M_2 + F odd

Model M_i': like M_i but F **increasing**

p_i skew-symmetric: $p_i(y_i,x_i) = -p_i(x_i,y_i)$
and F odd: $F(-\mathbf{p}) = -F(\mathbf{p})$

Examples of models

Multi-attribute value model

$$x \succsim y \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i)$$
$$\quad \text{iff} \quad \sum_{i=1}^{n} [u_i(x_i) - u_i(y_i)]$$

$p_i(x_i, y_i) = u_i(x_i) - u_i(y_i)$
$F = \sum$

Examples of models (Cont.)

Condorcet-like model (majority, concordance)

$$x \gtrsim y \quad \text{iff} \quad w\left(\{i : x_i \geq y_i\}\right) \geq 50\%$$

$$p_i(x_i, y_i) = \begin{cases} 1 & \text{if } x_i \geq y_i \\ -1 & \text{if } x_i < y_i \end{cases}$$

$F = w(\ldots) - 50\%$

The outranking methods (B. Roy) are based on this kind of principles

Key issues in these models

- modelling “differences of preference” : $p_i(x_i, y_i)$
- aggregating “differences of preference” : F
  If the balance between differences of preference is “in favour”, then: preference

Remark : the modelling of the preference differences may be precise or rough:

- many classes of differences of preference (value theory)

- few classes (majority rule, outranking)

Notations

$X = \prod_{i=1}^{n} X_i$

a is a vector of X: $a = (a_1, \ldots, a_n)$

$X_{-i} = \prod_{j:j \neq i} X_j$

a_{-i} is a vector of X_{-i} : $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$

$(x_i, a_{-i}) \in X$: substitute a_i of $a \in X$ by $x_i \in X_i$

$I \subseteq \{1, \ldots, n\} : (x_I, a - I) \in X$

Theorem 1

1. $\succsim$ satisfies model M_0 iff $\succsim$ is reflexive and independent;
2. $\succsim$ satisfies model M_1 (or equivalently M_1') iff $\succsim$ is reflexive, independent and satisfies RC_1;
3. $\succsim$ satisfies model M_2 (or equivalently M_2') iff $\succsim$ is reflexive, independent and satisfies RC_1 and $RC2$;
4. $\succsim$ satisfies model M_3 iff $\succsim$ is complete and satisfies RC_1 and RC_2;
5. $\succsim$ satisfies model M_3' iff $\succsim$ is complete and satisfies TC;

If $|X_i|$ is infinite, suitable technical conditions must be added.

Mutual Independence

$\succsim$ *independent in the sense of preferences*

$\forall I \subseteq \{1, \ldots, n\}1$, $\forall a, b \in X$ and $\forall x_I, y_I \in X_I$,

$$(x_I, a_{-I}) \succsim (y_I, a_{-I}) \Longrightarrow (x_I, b_{-I}) \succsim (y_I, b_{-I}) \qquad (1)$$

Weak independence

$\forall i = 1, \ldots, n$, $\forall a, b \in X$ and $\forall x_i, y_i \in X_i$,

$$(x_i, a_{-i}) \succsim (y_i, a_{-i}) \Longrightarrow (x_i, b_{-i}) \succsim (y_i, b_{-i}) \qquad (2)$$

Note that

$$\text{Weak independence} \not\Longrightarrow \text{Mutual independence}$$

interRCriteria decomposability

RC1

$\forall i = 1, \ldots, n$, $\forall a, b, c, d \in X$ and $\forall x_i, y_i, z_i, w_i \in X_i$,

$\mathbf{RC1_i}$:

$$\left. \begin{array}{ccc} (x_i, a_{-i}) & \succsim & (y_i, b_{-i}) \\ \text{and} & & \\ (z_i, c_{-i}) & \succsim & (w_i, d_{-i}) \end{array} \right\} \Longrightarrow \left\{ \begin{array}{ccc} (z_i, a_{-i}) & \succsim & (w_i, b_{-i}) \\ \text{or} & & \\ (x_i, c_{-i}) & \succsim & (y_i, d_{-i}) \end{array} \right.$$

$RC1$ allows to define quaternary relations: $\{\succsim_i^*\}$ on X_i^2

$(x_i, y_i) \succsim_i^* (z_i, w_i)$ iff $\forall a_{-i}, b_{-i}$,

$$\left[(z_i, a_{-i}) \succsim_(w_i, b_{-i})\right] \Longrightarrow \left[(x_i, a_{-i}) \succsim (y_i, b_{-i})\right]$$

Results

$RC1_i$ is equivalent to $\succsim_i^*$ being complete;

$\succsim_i^*$ is transitive by definition;

$\succsim_i^*$ is thus a complete preorder on the “differences” of preference

interCriteria decomposability with skew-symmetry

RC2

$\forall i = 1, \ldots, n$, $\forall a, b, c, d \in X$ and $\forall x_i, y_i, z_i, w_i \in X_i$,

$\mathbf{RC2_i}$:

$$\left.\begin{array}{ccc} (x_i, a_{-i}) & \succsim & (y_i, b_{-i}) \\ \text{and} & & \\ (y_i, c_{-i}) & \succsim & (x_i, d_{-i}) \end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc} (z_i, a_{-i}) & \succsim & (w_i, b_{-i}) \\ \text{or} & & \\ (w_i, c_{-i}) & \succsim & (z_i, d_{-i}) \end{array}\right.$$

$RC1$ and $RC2$ allow to define quaternary relations: $\{\succsim_i^{**}\}$ on X_i^2

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) iff (x_i, y_i) \succsim_i^{*} (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^{*} (y_i, x_i)$$

Results

$RC1_i$ and $RC2_i$ is equivalent to $\succsim_i^{**}$ being complete;
$\succsim_i^{**}$ is transitive by definition;
$\succsim_i^{**}$ is thus a complete preorder on the "differences" of preference; it is also "skew-symmetric".

Triple Cancellation

TC:

$$\left.\begin{array}{rcl}(x_i, a_{-i}) & \succsim & (y_i, b_{-i}) \\ \text{and} & & \\ (z_i, b_{-i}) & \succsim & (w_i, a_{-i}) \\ \text{and} & & \\ (w_i, c_{-i}) & \succsim & (z_i, d_{-i})\end{array}\right\} \Longrightarrow (x_i, c_{-i}) \succsim (y_i, d_{-i})$$

Results

If $\succsim$ is complete, TC implies $RC1$ and $RC2$

Uniqueness of the representation

For Model M_3

$$x \succsim y \quad \text{iff} \quad F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) \geq 0$$

If p_i are forced to be numerical representations of $\succsim_i^{**}$
and F is forced to take its values among $\{-1, 0, 1\}$
Then p_i are unique up to a strictly increasing transformation and F is unique

Rmk: If F is supposed to be strictly increasing, i.e. in the models M'_i
there is a "waste of information" by just cutting F at the 0 level

Models for fuzzy preference and their characterisation

Fuzzy **ordinal** preferences

$$x \succsim_\alpha y \quad \text{iff} \quad F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) \geq \varphi(\alpha),$$

where

$x, y \in X$,

$\alpha \in A$, A an ordered index set,

$p_i : X_i^2 \longrightarrow \mathbb{R}$, with $p_i(x_i, x_i) = 0$,

$F : \prod_{i=1}^n p_i(X_i^2) \longrightarrow \mathbb{R}$, a function of n arguments

and $\varphi : A \longrightarrow \mathbb{R}$, an increasing function.

Generalises e.g. :

$$x \succsim_\alpha y \quad \text{iff} \quad \sum_{i=1}^{n} [u_i(x_i) - u_i(y_i)] \geq \varphi(\alpha)$$

Hypotheses:

Independence

inteRCriteria decomposability (RC) :

$\forall i = 1, \ldots, n$, $\forall \alpha, \alpha'$, $\forall a, b, c, d \in X$ and $\forall x_i, y_i, z_i, w_i \in X_i$,

$\mathbf{RC1_i}(\alpha, \alpha')$:

$$\left.\begin{array}{ccc} (x_i, a_{-i}) & \succsim_{\alpha} & (y_i, b_{-i}) \\ \text{and} & & \\ (z_i, c_{-i}) & \succsim_{\alpha'} & (w_i, d_{-i}) \end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc} (z_i, a_{-i}) & \succsim_{\alpha} & (w_i, b_{-i}) \\ \text{or} & & \\ (x_i, c_{-i}) & \succsim_{\alpha'} & (y_i, d_{-i}) \end{array}\right.$$

$RC1$ allows to define quaternary relations $\{\succsim^*_{i,\alpha}\}$ that compare "differences of preference" above the level α

Example of result

Theorem 2
The family of relations $\succsim_\alpha$ can be represented by

$$x \succsim_\alpha y \quad \text{iff} \quad F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) \geq \varphi(\alpha)$$

with
$p_i(x_i, x_i) = 0$,
F non-decreasing in each argument and ... a little more
ϕ increasing

iff the family $\succsim_\alpha$

- is non-increasing
- is independent in the sense of preferences
- satisfies $RC1_i(\alpha, \alpha'), \forall i, \alpha, \alpha'$
- is lower semi-continuous;

Hypotheses

Property of F

F weakly strictly increasing if
$\forall p_i, p'_i \in \mathbb{R}$, if $p_i > p'_i$, there is p_{-i} such that

$$F(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n) >$$
$$F(p_1, \ldots, p_{i-1}, p'_i, p_{i+1}, \ldots, p_n)$$

Properties of $\succsim_\alpha$

Non-Increasing: $\alpha > \alpha' \Longrightarrow \succsim_\alpha \subseteq \succsim_{\alpha'}$

Lower Semi-Continuity : (only for $|A| = \infty$)
If there is an upper bound to the set of indices α for which $a \succsim_\alpha$ then $a \succsim_{\sup \alpha}$

Remarks

Uniqueness : the representation is unique up to appropriate transformations

Alternative formulation
The family of lower semi-continuous relations $\succsim_\alpha$ is equivalent to an (ordinally) valued relation:

$$v : X^2 \longrightarrow \mathbb{R}$$

Definition : Suppose that $\psi : A \longrightarrow \mathbb{R}$ is an order isomorphism

$$v(x, y) = \psi(\alpha) \text{ iff } x \succsim_\alpha y \text{ and } \neg[x \succsim_{\alpha'} y] \ \forall \alpha' > \alpha$$

Reformulation of $RC1_i(\alpha, \alpha')$:

$$\left.\begin{array}{rcl} v((x_i, a_{-i}), (y_i, b_{-i})) & \geq & \alpha \\ & \text{and} & \\ v((z_i, c_{-i}), (w_i, d_{-i})) & \geq & \alpha' \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} v((z_i, a_{-i}), (w_i, b_{-i})) & \geq & \alpha \\ & \text{or} & \\ v((x_i, c_{-i}), (y_i, d_{-i})) & \geq & \alpha' \end{array}\right.$$

Alternative formulation (cont.)

Theorem 2'

The ordinally valued relation v can be represented by

$$v(x,y) = \psi(\alpha) \quad \text{iff} \quad F(p_1(x_1,y_1),\ldots,p_n(x_n,y_n)) \geq \varphi(\alpha)$$

with the properties stated in Theorem 2

iff

the function v

- is independent in the sense of preferences
- satisfies the valued version of $RC1_i(\alpha,\alpha'), \forall i, \alpha, \alpha'$

Analogous results in the context of classification

Definition The valued relation $\sigma : X^2 \longrightarrow \mathbb{R}$ is a (ordinal) dissimilarity index if

- $\sigma(x, y) = \sigma(y, x)$
- $\sigma(x, x) \leq \sigma(y, x)$
- σ independent
- σ satisfies $RC1(\alpha, \alpha')$

Theorem 2"

The ordinally valued relation σ can be represented by

$$\sigma(x, y) = F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n))$$

with
$p_i(x_i, x_i) = 0$,
$p_i(x_i, y_i) = p_i(y_i, x_i) \geq 0$,
F non-decreasing and weakly strictly increasing

iff

the function σ is an ordinal dissimilarity index

Decision under uncertainty

(work in progress with P. Perny)

Usual formalism:

S : state space $= \{s_1, \ldots, s_n\}$

X : set of consequences

X^S : set of acts

$f \in X^S : f : s \longrightarrow f(s) \in X$

GOAL: build a relation $\succsim$ on acts

Examples

SEU (Subjective Expected Utility) model:

$$f \succsim g \quad \text{iff} \quad \sum_{s\in S} p(s)u(f(s)) \geq \sum_{s\in S} p(s)u(g(s))$$

Examples (Cont.)

Pessimistic qualitative utility model (Dubois et al 1998)

$$f \succsim g \quad \text{iff} \quad \min_{s \in S} \max\{1 - \pi(s), u(f(s))\} \geq \min_{s \in S} \max\{1 - \pi(s), u(g(s))\}$$

Lifting rule (Dubois et al 1997)

$$f \succsim g \quad \text{iff} \quad [f \succsim_P g] \succsim_U [g \succsim_P f]$$

with

$[f \succsim_P g] = \{s \in S : f(s) \succsim_P g(s)\}$

$\succsim_p$: an order of preference on the consequences

$\succsim_U$: an order of uncertainty on the subsets of states.

A reformulation

$X_i = X, \forall i = 1, \ldots, n$

$Y = X \times X \times \ldots \times X$ (n factors)

$y \in Y$ is an act:

y_1 = consequence if state = s_1

$\vdots$

y_n = consequence if state = s_n

Theorem 3 (Model UM3')

$$x \succsim y \quad \text{iff} \quad F(p(x_1, y_1), \ldots, p(x_n, y_n)) \geq 0$$

iff $\succsim$ complete and UTC

UTC:

$$\left.\begin{array}{rcl}(x_i, a_{-i}) & \succsim & (y_i, b_{-i}) \\ \text{and} & & \\ (z_i, b_{-i}) & \succsim & (w_i, a_{-i}) \\ \text{and} & & \\ (y_j, c_{-j}) & \succsim & (x_j, d_{-j})\end{array}\right\} \Longrightarrow (z_j, c_{-j}) \succsim (w_j, d_{-j})$$

where $x_i = x_j = \alpha$, $y_i = y_j = \beta$, $z_i = z_j = \gamma$, $w_i = w_j = \delta$.

p is a representation of the complete preorder on the differences of preference; it is identical on all copies of the set of consequences X

Property : $\succsim$ is independent (sure-thing principle P_2)

Models encompassed

SEU model:

$$f \succsim g \quad \text{iff} \quad \textstyle\sum_{s \in S} p(s)[u(f(s)) - u(g(s))] \geq 0$$

$$x \succsim y \quad \text{iff} \quad \textstyle\sum_{s \in S} p(i)[u(x_i) - u(y_i)] \geq 0$$

Lifting rule

Ordering the consequences $X = \{\beta, \gamma\, \delta, \epsilon\}$

$$\beta \succsim_P \gamma \quad \text{if} \quad (\beta, \gamma) \succsim^* 0 = (\delta, \delta) = (\beta, \beta)$$

Ordering the sets of states (according with their likelihood):

$$A \succsim_U B \quad \text{if} \quad \beta \succ_P \gamma \quad \text{s.t.} \quad \beta_A \gamma_{-A} \succsim \beta_B, \gamma_{-B})$$

Result: In Model $UM'3$ if there are at most 3 equivalence classes of difference of preference, then $\succsim$ can be described by a lifting rule

Conclusion

- Our model(s) are based on the aggregation of differences of preference
- They encompass many particular models

Usefulness

- Better understanding of key features of models
- Offer a framework for characterising specific methods