Exercise 1 *Prove that, given (a set defined by) a linear system* $\{x : Ax = a\}$ *and an invertible matrix* B, *then* $\{x : Ax = a\} = \{x : BAx = Ba\}$ *holds (with size of* B *so that* BA *is a matrix).*

Exercise 2 *Prove that a pivoting matrix (i.e. obtained from the identity by replacing one column of the identity by any column with a nonzero element at the position where the 1 was) is invertible.*

Exercise 3 *Prove that a permutation matrix (i.e. obtained by permuting rows, equivalently columns, of the identity) is invertible.*

Exercise 4 *Prove that* $B = B_k B_{k-1} \dots B_1$ *is invertible, where the* B_i*'s are pivoting or permutation matrices.*

Exercise 5 *Give an algorithm which, given a linear system* $\{x : Ax = a\}$ *as input, outputs an equivalent linear system* $\{x : Ax = a\} = \{x : BAx = Ba\}$ *which has the following form:*

$$\begin{array}{rcl} [I\, M]\, x & = & B'a \\ \mathbf{0} & = & B''a \end{array}$$

where $B = \left[\begin{smallmatrix} B' \\ B'' \end{smallmatrix} \right]$.

Exercise 6 *Prove that, given a matrix* A, *there is a matrix* C *so that*

$$\{a : Ax = a, \exists x\} \quad = \quad \{a : Ca = \mathbf{0}\} \tag{1}$$

Exercise 7 *Prove that, given a matrix* C, *there is a matrix* A *so that* (1) *holds.*

Exercise 8 *Prove that* $(\exists x : Ax = a)$ *only if* $(c^\top A = \mathbf{0}^\top \Rightarrow c^\top a = 0)$.

Exercise 9 *Prove that if* $(\exists x : Ax = a)$, *then* $(c^\top A = \mathbf{0}^\top \Rightarrow c^\top a = 0)$.

Exercise 10 *Let* n *be the dimension of the linear space spanned by the columns of a full row-rank matrix* A. *Prove the exclusivity in*

$$(\exists x \geq \mathbf{0} : Ax = a) \quad XOR \quad \left(\begin{array}{ll} \exists c : c^\top A \geq \mathbf{0}^\top & \textit{with equality for at least } n-1 \\ & \textit{linearly independent columns, and} \\ \quad c^\top a < 0 & \end{array} \right) \tag{2}$$

Let A_j for $j \in J$ denotes the columns of A, let A_B be the submatrix of A obtained by removing the columns A_j with $j \in J \setminus B$, and assume that A_B is invertible.

Exercise 11 *Prove that, in the the following algorithm, the vector* c *exists indeed at Step 2.*

Step 1. *Let* $x_B = A_B^{-1} a$. *If* $x_B \geq \mathbf{0}$ *stop.*

Step 2. *Let* σ *be the minimum* $\sigma \in B$ *with* $x_\sigma < 0$. *There exists* c *with* $c^\top A_{B \setminus \{\sigma\}} = \mathbf{0}^\top$ *and* $c^\top A_\sigma = 1$. *Thus* $c^\top a = c^\top A_B x = x_\sigma < 0$.

Step 3. *If* $c^\top A \geq \mathbf{0}^\top$ *stop.*

Step 4. *Let* ρ *be the minimum* $\rho \in J$ *with* $c^\top A_\rho < 0$, *reset* $B := B \setminus \{\sigma\} \cup \{\rho\}$ *and go to Step 1.*

Exercise 12 *Prove that the columns indexed in $B \setminus \{\sigma\} \cup \{\rho\}$ form indeed an invertible matrix at Step 4.*

If the algorithms described above loops, the same subset $B \subseteq J$ is used at some iteration and at a later iteration. Let μ be the maximum $\mu \in J$ which is removed and added, between these two iterations, and say μ leaves B at iteration k and enters at iteration ℓ. Denote $B^i, x^i, \sigma^i, c^i, \rho^i$ the objects B, x, ρ, c, σ of the algorithm at iteration i.

Exercise 13 *Let $j \in B^k$, prove that if $j > \mu$, then $c^{\ell\top} A_j = 0$.*

Exercise 14 *Let $j \in B^k$, prove that if $j = \mu$, then $c^{\ell\top} A_j < 0$.*

Exercise 15 *Let $j \in B^k$, prove that if $j < \mu$, then $c^{\ell\top} A_j \geq 0$.*

Exercise 16 *Prove that the algorithm of exercise 11 stops.*

Exercise 17 *Prove that* (2) *holds.*

Exercise 18 *Prove that, given a matrix A, there is a matrix C so that*

$$\{a : Ax = a, \exists x \geq \mathbf{0}\} \quad = \quad \{a : Ca \geq \mathbf{0}\} \tag{3}$$

Given a matrix C, by above, there is a matrix B, and then a matrix D, so that

$$\begin{array}{rcl} \{c : y^\top C = c^\top, \exists y \geq \mathbf{0}\} & = & \{c : Bc \geq \mathbf{0}\} \\ \{b : y^\top B = b^\top, \exists y \geq \mathbf{0}\} & = & \{b : Db \geq \mathbf{0}\} \end{array}$$

Exercise 19 *Prove that $\{a : Ca \geq \mathbf{0}\} \supseteq \{b : y^\top B = b^\top, \exists y \geq \mathbf{0}\}$.*

Exercise 20 *Prove that $\{a : Ca \geq \mathbf{0}\} \subseteq \{b : y^\top B = b^\top, \exists y \geq \mathbf{0}\}$.*

Exercise 21 *Prove that, given a matrix C, there is a matrix A so that* (3) *holds.*

Exercise 22 *Prove that, given a matrix A and a vector a, then there are two matrices B, C so that*

$$\{x : Ax \leq a\} \quad = \quad \{x : Bb = x, \exists b \geq \mathbf{0} : \mathbf{1}^\top b = 1\} + \{x : Cc = x, \exists c \geq \mathbf{0}\} \tag{4}$$

Exercise 23 *Prove that, given two matrices B, C, there are a matrix A and a vector a, so that* (4) *holds.*

Exercise 24 *Prove that $(\exists x \geq \mathbf{0} : Ax = a)$ only if $(c^\top A \geq \mathbf{0}^\top \Rightarrow c^\top a \geq 0)$*

Exercise 25 *Prove that if $(\exists x \geq \mathbf{0} : Ax = a)$, then $(c^\top A \geq \mathbf{0}^\top \Rightarrow c^\top a \geq 0)$*

Exercise 26 *Prove that $(\exists x \geq \mathbf{0} : Ax \leq a)$ only if $(c^\top A \geq \mathbf{0}^\top$ and $c \geq \mathbf{0} \Rightarrow c^\top a \geq 0)$*

Exercise 27 *Prove that if $(\exists x \geq \mathbf{0} : Ax \leq a)$, then $(c^\top A \geq \mathbf{0}^\top$ and $c \geq \mathbf{0} \Rightarrow c^\top a \geq 0)$*

Exercise 28 *Prove that $(\exists x : Ax \leq a)$ only if $(c^\top A = \mathbf{0}^\top$ and $c \geq \mathbf{0} \Rightarrow c^\top a \geq 0)$*

Exercise 29 *Prove that if $(\exists x : Ax \leq a)$, then $(c^\top A = \mathbf{0}^\top$ and $c \geq \mathbf{0} \Rightarrow c^\top a \geq 0)$*

Exercise 30 *Prove that $\min\{c^\top x : Ax \leq b, x \geq \mathbf{0}\} \leq \max\{y^\top b : y^\top A \geq c^\top, y \geq \mathbf{0}\}$ (where both sets are assumed non-empty).*

Exercise 31 *Prove that if there are* $u \geq \mathbf{0}$, $v \geq \mathbf{0}$, *and* $\mu > 0$ *so that* $u^\top A \geq \mu.c^\top$ *and* $Av \leq \mu.b$, *then* $u^\top b \geq v^\top c$.

Exercise 32 *Prove that if there are* $u \geq \mathbf{0}$, $v \geq \mathbf{0}$, *and* $\mu = 0$ *so that* $u^\top A \geq \mu.c^\top$ *and* $Av \leq \mu.b$, *then* $u^\top b \geq v^\top c$.

Exercise 33 *Prove that there are* $x \geq \mathbf{0}$ *and* $y \geq \mathbf{0}$ *so that*

$$\begin{bmatrix} A & O^\top \\ O & -A^\top \\ -c^\top & b^\top \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$$

where O *is a zero matrix.*

Exercise 34 *Prove that* $\min\{c^\top x : \, Ax \leq b, \, x \geq \mathbf{0}\} \geq \max\{y^\top b : \, y^\top A \geq c^\top, \, y \geq \mathbf{0}\}$.