Given $x, y \in \mathbb{R}^n$, we denote the scalar product by $x^\top y = \sum_{j=1}^n x_j y_j$, and the norm by $||x|| = (x^\top x)^{1/2}$. The convex hull of two points $a, b \in \mathbb{R}^n$ is the segment $[a, b]$, that is, the set of all point $x \in \mathbb{R}^n$ so that $x = \alpha.a + (1 - \alpha).b$ for some real $0 \leq \alpha \leq 1$. More generally, the convex hull of k points, that is, the columns of some (finite) matrix $B \in \mathbb{R}^{n \times k}$, is

$$conv.hull(B) = \{x \in \mathbb{R}^n : x = Bb, \exists b \in \mathbb{R}^k, b \geq \mathbf{0} : \mathbf{1}^\top b = 1\}$$

A polytope P is the convex hull of a finite set of points, that is, a subset of $\mathbb{R}^n$ of the form $P = conv.hull(B)$ for some matrix B. *Given a graph G, its matching polytope is the convex hull of the 0-1 characteristic vectors of all its matchings.*

A classical result is that every polytope is also the set of all solutions of a linear system, that is, given B, there always exists a matrix A such that

$$\{x : Ax \leq a\} \quad = \quad \{x : Bb = x, \exists b \geq \mathbf{0} : \mathbf{1}^\top b = 1\} \tag{1}$$

The converse holds provided that the set at the left is bounded, that is, given a bounded set of the form $P = \{x : Ax \leq a\}$, it is the convex hull of points, called extreme points of P. For instance, the cube in the space $\mathbb{R}^3$, denoted by $[\mathbf{0}, \mathbf{1}]$, is both

1. the set of points x with $0 \leq x_j \leq 1$, and
2. the convex hull of all points with 0-1 coordinates.

It holds for all hypercubes as well, and generally, in $\mathbb{R}^n$, the extrem points of $P = \{x : Ax \leq a\}$ are both the set of points of P

1. which satisfy n linearly independent equations of the system $Ax = a$
2. which are the convex hull of no pair of distinct points in P

We will prove that the matching polytope of any bipartite graph G is described by a system $Ax \leq a$ where the rows of A are the characteristic vector of the stars of G, and $a = \mathbf{1}$.

1 Geometrical insight

Pythagorean theorem, which sates that $||x||^2 = x_1^2 + x_2^2$, where $x = (x_1, x_2) \in \mathbb{R}^2$ (for a proof: express $(a + b)^2$ in two ways), implies $||x||^2 = x_1^2 + x_2^2 + \ldots + x_n^2$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, assuming that 3 points always define a plane (proof by induction on n). Pythagorean theorem implies also that $x^\top y = ||x||.|||y||. \cos\theta$, where θ is the angle between the two lines $(\mathbf{0}x)$ and $(\mathbf{0}y)$. (Proof: express $||x - y||$ in two different ways.) This implies that $x^\top y = 0$ if and only if the two lines $(\mathbf{0}x)$ and $(\mathbf{0}y)$ are orthogonal (or one of x, y is the origin). This is why linear spaces have two possible descriptions, for instance, a line $(\mathbf{0}a)$ of $\mathbb{R}^2$ is both the set of x so that

1. $x = \alpha.a$ for some real α

2. $c^\top x = 0$ for some non-zero c with $c^\top a = 0$

or, plane $(\mathbf{0}ab)$ of $\mathbb{R}^3$ is both the set of x so that

1. $x = \alpha.a + \beta.b$ for some reals α, β
2. $c^\top x = 0$ for some non-zero c with $c^\top a = c^\top b = 0$

(and more generally a linear subspace of $\mathbb{R}^n$ is both

1. $\{a :\ Ax = a,\ \exists x\}$
2. $\{a :\ Ca = \mathbf{0}\}$

)

For lines (ab), planes (abc) (and, more generally, affine spaces not containing the origin), this gives: a line (ab) of $\mathbb{R}^2$ is both the set of x so that

1. $x = a + \alpha.(b - a)$ for some real α
2. $c^\top x = 1$ for some non-zero c with $c^\top (b - a) = 0$

or, a plane (abc) of $\mathbb{R}^3$ is both the set of x so that

1. $x = a + \alpha.(b - a) + \beta.(c - a)$ for some reals α, β
2. $c^\top x = 1$ for some non-zero c with $c^\top (b - a) = c^\top (c - a) = 0$

2 Linear systems and graphs

Given a set X, typically X is the edge-set or the vertex-set of some graph, the characteristic vector of a subet $Y \subseteq X$ is the vector $\chi^Y \in \mathbb{R}^X$ (that is, a vector of $\mathbb{R}^{|X|}$ with a one-to-one correspondence between its coordinates and the elements of X) so that

$$\chi^Y_x = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

Throughout, we let A be the 0-1 vertex-edge incidence matrix of G, that is, equivalently

1. the rows are the characteristic vector of stars of G (in other words, each row of A is the vector $\chi^{\delta(v)}$ for some vertex $v \in V$)
2. the columns are the characteristic vector of the edges of G (in other words, each column of A is the vector $\chi^{\{u,v\}}$ for some edge $e = uv \in E$)

Then the 0-1 vectors x satisfying $Ax \leq \mathbf{1}$ are the characteristic vectors of matchings of G.

Which other familiar objects from graph theory can be described in a similar way with the matrix A ?

3 Matching polytope of bipartite graphs

A bipartite graph $G = (V, E)$ is both

1. a graph with $\chi(G) \leq 2$
2. a graph with no odd cycle

(Proof: partition the vertex-set V into subsets V_i at distance i from some $s \in V$.)

First, let us show that it is easy to describe the perfect matching polytope of a bipartite graph:

- Let x be a point of $P = \{x : Ax = \mathbf{1},\, x \geq \mathbf{0}\}$ and let $F = \{e \in E : 0 < x_e < 1\}$. Clearly, the subgraph (V, F) of G has no vertex of degree 1 (otherwise a constraint would be violated).
- Suppose that there is an even cycle $C \subseteq F$. So C has a bipartition into matchings M, N; and hence, there is a $\varepsilon > 0$, so that both $y = x + \varepsilon.(\chi^M - \chi^N)$ and $z = x - \varepsilon.(\chi^M - \chi^N)$ are in P. In this case, $x = (y + z)/2$ is not an extreme point.
- So extreme points of P have no fractional coordinate.

Finally, this leads to a description of the matching polytope:

- Let x be a point of $P = \{x : Ax \leq \mathbf{1},\, x \geq \mathbf{0}\}$.
- Let $G' = (V', E')$ be a copy of G and let x' be the vector with $x_{e'} = x_e$.
- Let $\tilde{G}$ be the graph obtained from the disjoint union of G and G', by adding $|V| = |V'|$ edges $e = vv'$ linking a vertex v and its copy v'. Let $\tilde{x}$ be the vector with $\tilde{x}_e = \tilde{x}_{e'} = x_e$ and $\tilde{x}_{vv'} = 1 - x(\delta(v))$.
- So $\tilde{x}$ is a point of the perfect matching polytope of $\tilde{G}$, and hence its the convex hull of characteristic vectors $\chi^{\tilde{M}}$ of perfect matchings $\tilde{M}$ of $\tilde{G}$.
- Since $\tilde{M} \cap E = M$ is a matching of G, then x is the convex hull of characteristic vectors χ^M of matchings M of G.