A fundamental result of linear algebra can be stated as follows :

$$\{a : Ax = a, \exists x\} \quad = \quad \{a : Ca = \mathbf{0}\} \tag{1}$$

meaning that every linear subspace generated by linear combination of a finite number of points of $\mathbb{R}^n$, namely the columns of A, is the set of solutions of an homogeneous linear system ($\forall A \, \exists C$), and any set of solutions of an homogeneous linear system, namely $Ca = \mathbf{0}$, is a finitely generated linear subspace ($\forall C \, \exists A$). The proof is based on an algorithm, namely Gaussian Elimination, which provides a sequence of invertible matrices $B_1, \ldots, B_k$ (here B_i is either, a *pivoting matrix*, *i.e.* obtained from the identity by replacing one column of the identity by any column with a nonzero element at the position where the 1 was, or a *permutation matrix*, *i.e.* obtained by permuting rows, equivalently columns, of the identity), the product $B = B_k B_{k-1} \ldots B_1$ of these invertibles matrices is an invertible matrix and hence $\{x : Ax = a\} = \{x : BAx = Ba\}$ since there must have equalities every where in:

$$\{x : Ax = a\} \subseteq \{x : BAx = Ba\} \subseteq \{x : B^{-1}BAx = B^{-1}Ba\} = \{x : Ax = a\}$$

The algorithm stops with an equivalent system $BAx = Ba$ which has the following form (up to column permutation): $[I\, M]\, x = B'a$ and $\mathbf{0} = B''a$ where $B = \left[\begin{smallmatrix} B' \\ B'' \end{smallmatrix} \right]$. The size of I is the *dimension* of the linear space spanned by A, equivalently the *rank* of A. Now the the proof is easy since:

Proof of (1).

($\forall A \, \exists C$) Take $C = B''$, thus any a for which there is a solution $x = \left(\begin{smallmatrix} x' \\ x'' \end{smallmatrix} \right)$ of $BAx = Ba$ must satisfy $Ca = \mathbf{0}$; furthermore $x' = B'a$, $x'' = \mathbf{0}$ is then a solution.

($\forall C \, \exists A$) Since $Ca = \mathbf{0}$ is equivalent to $[I\, M]\, a = \mathbf{0}$ (and $\mathbf{0} = \mathbf{0}$), then $a = \left(\begin{smallmatrix} a' \\ a'' \end{smallmatrix} \right)$ is a solution if and only if $a' = -Ma''$, which is equivalent to $a = Ax$ for some x, with $A = \left[\begin{smallmatrix} -M \\ I \end{smallmatrix} \right]$.

□

It has the following consequence:

$$(\exists x : Ax = a) \quad \Longleftrightarrow \quad (c^\top A = \mathbf{0}^\top \Rightarrow c^\top a = 0) \tag{2}$$

Proof of (2).

$\Rightarrow$: $(Ax = a \text{ and } c^\top A = \mathbf{0}^\top) \Rightarrow (0 = c^\top Ax = c^\top a)$.

$\Leftarrow$: Since $CA = O$ is a zero matrix, then $(c^\top A = \mathbf{0}^\top \Rightarrow c^\top a = 0)$ implies $Ca = \mathbf{0}$, which implies $Ax = a$ for some x.

□

Let n be the dimension of the linear space spanned by the columns of a matrix A. The so-called fundamental theorem of linear inequalities is:

$$(\exists x \geq \mathbf{0} : Ax = a) \quad \text{XOR} \quad \left(\begin{array}{ll} \exists c : c^\top A \geq \mathbf{0}^\top & \text{with equality for at least } n-1 \\ & \text{linearly independent columns, and} \\ \quad c^\top a < 0 & \end{array} \right) \tag{3}$$

meaning that either a belongs to the cone generated by the columns of A, or, exclusively, there is a hyperplan (of the spanned space) through the origin and $n-1$ columns which separates all other columns from a. Since both the cone and the hyperplan belong to the spanned space, we can assume that A is full row-rank (as a counterexample with A not full row-rank woud give one with A full row-rank by projection unto the spanned space).

Proof of (3)**.**

(X) $(x \geq \mathbf{0},\ Ax = a \text{ and } c^\top A \geq \mathbf{0}^\top) \Rightarrow (0 \leq c^\top Ax = c^\top a)$.

(OR) Let A_j for $j \in J$ denotes the columns of A, let A_B be the submatrix of A obtained by removing the columns A_j with $j \in J \setminus B$, and assume that A_B is invertible. The proof follows from the finiteness of the following algorithm, which we prove just after:

Step 1. Let $x_B = A_B^{-1}a$. If $x_B \geq \mathbf{0}$ stop.

Step 2. Let σ be the minimum $\sigma \in B$ with $x_\sigma < 0$. There exists c with $c^\top A_{B\setminus\{\sigma\}} = \mathbf{0}^\top$ and $c^\top A_\sigma = 1$. Thus $c^\top a = c^\top A_B x = x_\sigma < 0$.

Step 3. If $c^\top A \geq \mathbf{0}^\top$ stop.

Step 4. Let ρ be the minimum $\rho \in J$ with $c^\top A_\rho < 0$, reset $B := B \setminus \{\sigma\} \cup \{\rho\}$ and go to Step 1.

If the algorithms loops, the same subset $B \subseteq J$ is used at some iteration and at a later iteration. Let μ be the maximum $\mu \in B$ which is removed and added, between these two iterations, and say μ leaves B at iteration k and enters at iteration ℓ. Denote $B^i, x^i, \sigma^i, c^i, \rho^i$ the objects B, x, ρ, c, σ of the algorithm at iteration i.

So $B^{i+1} = B^i \setminus \{\sigma^i\} \cup \{\rho^i\}$, $B = B^k = B^{\ell+1}$, and $\mu = \sigma^k = \rho^\ell$. Let $j \in B$; we have:

if $j > \mu$, then, by maximality of μ, we have $j \in B^\ell$ and $j \neq \sigma^\ell$, hence ${c^\ell}^\top A_j = 0$

if $j = \mu$, then $j = \sigma^k$, hence $x_j^k < 0$, and $j = \rho^\ell$ hence ${c^\ell}^\top A_j < 0$

if $j < \mu$, by minimality of σ^k, we have $x_j^k \geq 0$, and by minimality of ρ^ℓ, we have ${c^\ell}^\top A_j \geq 0$

It follows that $0 < {c^\ell}^\top A_{B^k} x_{B^k} = {c^\ell}^\top a$; a contradiction since ${c^\ell}^\top a < 0$ by Step 2.

□

To see that c exists at Step 2, it suffices to notice that, for every b, the system $y^\top A_B = b^\top$ has always a unique solution y (in particular when $b_j = 0$ except for one coordinate). Suppose that the columns indexed in $B \setminus \{\sigma\} \cup \{\rho\}$ do not form an invertible matrix at Step 4. Then the system $y^\top A_{B\setminus\{\sigma\}\cup\{\rho\}} = (0, \dots, 0, 1)$ has several solutions y ; impossible, since it would imply that $y^\top A_B = (0, \dots, 0, 1)$ has several solutions as well.

(3) has the following consequences (4)-(9).

$$\{a : Ax = a, \exists x \geq \mathbf{0}\} \quad = \quad \{a : Ca \geq \mathbf{0}\} \tag{4}$$

meaning that every finitely generated cone is a polyhedral cone and vice-versa.

Proof of (4)**.**

($\forall A\, \exists C$) By enumerating all subsets of $n-1$ columns, we can construct a matrix C the rows $c^\top$ of which correspond to all the hyperplans so that $c^\top A \geq \mathbf{0}^\top$ with at least $n-1$ equalities. By (3), $Ca \geq \mathbf{0}$ if and only if $Ax = a$ for some x.

($\forall C\,\exists A$) Given a matrix C, by above, there is a matrix B, and then a matrix D, so that

$$\begin{array}{rcl} \{c:\, y^\top C = c^\top,\, \exists y \geq \mathbf{0}\} & = & \{c:\, Bc \geq \mathbf{0}\} \\ \{b:\, y^\top B = b^\top,\, \exists y \geq \mathbf{0}\} & = & \{b:\, Db \geq \mathbf{0}\} \end{array}$$

It suffices to prove that $\{a:\, Ca \geq \mathbf{0}\} = \{b:\, y^\top B = b^\top,\, \exists y \geq \mathbf{0}\}$.

$\supseteq$: For each row $b^\top$ of B and each row $c^\top$ of C, we have $0 \leq b^\top c = c^\top b$. Thus $Cb \geq \mathbf{0}$, and it follows that $C(B^\top y) \geq \mathbf{0}$ for all $y \geq \mathbf{0}$.

$\subseteq$: For each row $d^\top$ of D and each row $b^\top$ of B, we have $0 \leq d^\top b = b^\top d$. Thus $Bd \geq \mathbf{0}$ and so $d^\top = y^\top C$ for some $y \geq \mathbf{0}$. If $Ca \geq \mathbf{0}$ then $0 \leq y^\top Ca = d^\top a$; hence $Da \geq \mathbf{0}$ and it follows that $a^\top = z^\top B$ for some $z \geq \mathbf{0}$.

□

$$\{x:\, Ax \leq a\} \quad = \quad \{x:\, Bb = x,\, \exists b \geq \mathbf{0}:\, \mathbf{1}^\top b = 1\} + \{x:\, Cc = x,\, \exists c \geq \mathbf{0}\} \tag{5}$$

meaning that every polyhedron is the sum of a polytope and a polyhedral cone. Indeed, the first term in the sum is the convex hull of the columns of B, and the second term is a finitely generated cone, so a polyhedral cone.

Proof of (5)**.**

($\forall A, a\,\exists B, C$): Given a matrix $[A\ -a]$, by (4), there is a matrix D so that

$$\left\{\begin{pmatrix} x \\ \mu \end{pmatrix}:\, Ax - \mu.a \leq \mathbf{0},\, \mu \geq 0\right\} = \left\{\begin{pmatrix} x \\ \mu \end{pmatrix}:\, Dy = \begin{pmatrix} x \\ \mu \end{pmatrix}, \exists y \geq \mathbf{0}\right\}$$

Moreover, we can whose a matrix D of the form

$$D = \begin{pmatrix} B & C \\ \mathbf{1}^\top & \mathbf{0}^\top \end{pmatrix}$$

So $Ax \leq a$ if and only if $\binom{x}{1} = Dy$ for some $y \geq \mathbf{0}$. Which is equivalent to

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Bb + Cc \\ \mathbf{1}^\top b \end{pmatrix} \quad \text{for } b \geq \mathbf{0} \text{ and } c \geq \mathbf{0}$$

($\forall B, C\,\exists A, a$): Given B, C, by (4), there is a matrix $[A\ -a]$ so that

$$\left\{\begin{pmatrix} x \\ \mu \end{pmatrix}:\, \begin{pmatrix} B & C \\ \mathbf{1}^\top & \mathbf{0}^\top \end{pmatrix} y = \begin{pmatrix} x \\ \mu \end{pmatrix}, y \geq \mathbf{0}\right\} = \left\{\begin{pmatrix} x \\ \mu \end{pmatrix}:\, [A\ -a]\begin{pmatrix} x \\ \mu \end{pmatrix} \leq \mathbf{0}\right\}$$

which, restricted to $\mu = 1$ is equivalent to (5).

□

Another consequence of (3) is the so-called Farka's lemma:

$$(\exists x \geq \mathbf{0}:\, Ax = a) \quad \Longleftrightarrow \quad (c^\top A \geq \mathbf{0}^\top \Rightarrow c^\top a \geq 0) \tag{6}$$

meaning that a point a does not belong to a cone if and only if there is a hyperplane separating point a from the cone.

Proof of (6)**.**

$\Rightarrow$: It follows from the exclusivity (X) in (3).

$\Leftarrow$: It follows from the (OR) in (3).

$\square$

Farkas's lemma (6) has the two following different variants (7)-(8).

$$(\exists x \geq \mathbf{0} : \; Ax \leq a) \quad \iff \quad (c^\top A \geq \mathbf{0}^\top \text{ and } c \geq \mathbf{0} \Rightarrow c^\top a \geq 0) \tag{7}$$

Proof of (7)**.**

$\Rightarrow$: $(Ax \leq a,\, x \geq \mathbf{0},\, c \geq \mathbf{0} \text{ and } c^\top A \geq \mathbf{0}^\top) \Rightarrow (0 \leq c^\top Ax \leq c^\top a)$.

$\Leftarrow$: There is a $x \geq \mathbf{0}$ so that $Ax \leq a$ if and only if there is a $x \geq \mathbf{0}$ so that $[A\,I]\,x = a$. By (6), the later is equivalent to the fact that $c^\top [A\,I] \geq \mathbf{0}$ implies $c^\top a \geq 0$.

$\square$

$$(\exists x : \; Ax \leq a) \quad \iff \quad (c^\top A = \mathbf{0}^\top \text{ and } c \geq \mathbf{0} \Rightarrow c^\top a \geq 0) \tag{8}$$

Proof of (8)**.**

$\Rightarrow$: $(Ax \leq a,\, c \geq \mathbf{0} \text{ and } c^\top A = \mathbf{0}^\top) \Rightarrow (0 = c^\top Ax \leq c^\top a)$.

$\Leftarrow$: There is a x so that $Ax \leq a$ if and only if there is a $x \geq \mathbf{0}$ so that $[A \; -A \;\; I]\,x = a$. By (6), the later is equivalent to the fact that $c^\top [A \; -A \;\; I] \geq \mathbf{0}$ implies $c^\top a \geq 0$.

$\square$

The duality theorem of linear programming is a consequence of (7), it states that, if both polyhedra are nonempty, then the following equality holds:

$$\min\{c^\top x : \; Ax \leq b,\, x \geq \mathbf{0}\} = \max\{y^\top b : \; y^\top A \geq c^\top,\, y \geq \mathbf{0}\} \tag{9}$$

Proof of (9)**.**

$\leq$: If $y^\top A \geq c^\top$ and $x \geq \mathbf{0}$, then $c^\top x \leq y^\top Ax$. If $Ax \leq b$ and $y \geq \mathbf{0}$, then $y^\top Ax \leq y^\top b$.

$\geq$: It suffices to prove that there are $x \geq \mathbf{0}$ and $y \geq \mathbf{0}$ so that

$$\begin{bmatrix} A & O^\top \\ O & -A^\top \\ -c^\top & b^\top \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$$

where O is a zero matrix. By (7), this is equivalent to the fact that if there are $u \geq \mathbf{0}$, $v \geq \mathbf{0}$, and $\mu \geq 0$ so that $u^\top A \geq \mu.c^\top$ and $Av \leq \mu.b$, then $u^\top b \geq v^\top c$. If $\mu > 0$, then

$$u^\top b = u^\top(\mu^{-1}\mu\, b) \geq u^\top(\mu^{-1}Av) = \mu^{-1}(u^\top Av) \geq \mu^{-1}\mu\, c^\top v = c^\top v$$

If $\mu = 0$, let $\bar{x} \geq \mathbf{0}$ and $\bar{y} \geq \mathbf{0}$ so that $A\bar{x} \leq b$ and $\bar{y}^\top A \geq c^\top$. Thus $u^\top b \geq u^\top A\bar{x} \geq 0$ and $c^\top v \leq \bar{y}^\top Av \leq 0$.

$\square$