The Biparticity of a Graph

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ABSTRACT

The biparticity $\beta(G)$ of a graph G is the minimum number of bipartite graphs required to cover G. It is proved that for any graph G, $\beta(G) = \{\log_2 \chi(G)\}$. In view of the recent announcement of the Four Color Theorem, it follows that the biparticity of every planar graph is 2.

Dedicated to the memory of Paul Turán

1. BIPARTICITY

This graphical invariant was defined in [3] along with several other concepts relating to coverings and packings of graphs. The *biparticity* $\beta(G)$ is the minimum number of spanning bipartite subgraphs which cover the lines of G. We will determine the biparticity of any graph G in terms of its chromatic number.

Our notation and terminology will follow the book [2]. As usual, write $\{x\}$ for the least integer not less than the real number x. Let $E(G)$ be the set of lines of G, $V(G)$ the set of points, and $p(G)$ the number of points.

THEOREM. For any graph G, $\beta(G) = \{\log_2 \chi(G)\}$.

Proof. For the upper bound we will proceed by induction to show that if $\chi(G) = 2^n$, then $\beta(G) \leq n$. This is obviously true if $n = 1$. Assume it is true for $n = k$, and that $\chi(G) = 2^{k+1}$. Let G_1 be the subgraph generated by 2^k of the colors, and let G_2 be the subgraph generated by the other 2^k colors. Then $x(G_i) = 2^k$. By induction $\beta(G) \leq 1 + \max_{i=1,2} \beta(G_i) \leq 1 + k$ as required.

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For the lower bound, let $E(G)=B_1 \cup B_2 \cup \cdots \cup B_t$ be a minimal decomposition of the lines of G into bipartite graphs. Let S be the set of all t-tuples with entries 1 or 0. Define a function $T: V(G) \to S$ as follows: The points of each bigraph B_i can be partitioned into two fixed classes, say blue and green. Let $T_i(v)$ denote the ith entry in the t-tuple $T(v)$, defined by

$$T_i(v)=\begin{cases} 1 & \text{if } v \text{ is blue in } B_i \\ 0 & \text{if } v \text{ is green in } B_i. \end{cases}$$

For each sequence $\sigma \in T(V(G))$, let

$$V(\sigma)=T^{-1}(\sigma)=\{v \in V(G)/T(v)=\sigma\}.$$

Notice that for any σ, whenever $u, w \in V(\sigma)$, then u and w are not adjacent. Hence the collection of sets $R=\{V(\sigma)/\sigma \in T(V(G))\}$ gives a coloring of G. By definition of chromatic number, $|\chi(G)| \le |R| \le |S| = 2^t$. ∎

As an interesting special case of the theorem, the biparticity of the complete graph K_p is $\{\log_2 p\}$.

2. OBSERVATIONS

A. The minimum number $\beta_n(G)$ of spanning subgraphs in a partition of the lines of a graph G into n-partite graphs is called its *n-particity*. For the n-particity of an arbitrary graph G, the theorem then generalizes as expected to the equation

$$\beta_n(G)=\{\log_n \chi(G)\}.$$

B. Writing $\beta=\beta(G)$ and $\chi=\chi(G)$ for a given graph G, the theorem implies at once the following bounds on the chromatic number:

$$2^{\beta-1} < \chi \le 2^{\beta}. \tag{1}$$

C. It is entertaining to observe that (1) gives the following new equivalence to the "Four Color Conjecture":

For all planar graphs G which are not bipartite, $\beta(G)=2$. (2)

In view of the recent announcement of the Four Color Theorem [1], statement (2) holds. Hedetniemi [4] had shown that $\beta(G) \le 3$.

D. We were anticipated in part by Nieminen [6] who observed that every graph G satisfies the inequality $\beta(G) \leq \beta(K_{\chi(G)})$.

E. A result equivalent to the theorem was obtained by Matula [5]. He defined a slicing of a graph G to be an ordered partition $(C_1, C_2, \ldots, C_m)$ of $E(G)$ such that the set of edges C_i is defined recursively by the following: C_i is a cutset of G if $i=1$ and of $G - \bigcup_{j<i} C_j$ if $i \leq 2$. He then showed that the minimum length of such an ordered partition is exactly $\{\log_2 \chi(G)\}$.

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