On the Decomposition of K_n into Complete Bipartite Graphs

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Abstract

A short proof is given of the impossibility of decomposing the complete graph on n vertices into n-2 or fewer complete bipartite graphs.

Let G be a finite graph, and let $V_1, \ldots, V_r$ be sets of vertices of G. Assume that for each i, G_i is a subgraph of $G(V_i)$, the subgraph induced by G on V_i. Assume furthermore that the G_i are edge-disjoint and between them contain all edges of G. Then $G_1, \ldots, G_r$ form a *decomposition* of G. It was proved in [1] that if $G = K_n$, the complete graph on n vertices, and each G_i is a complete bipartite graph, then $r \geq n - 1$. This inequality is a consequence of a theorem in [1], and in [2] it is remarked that the application of that theorem still seems to be the only known way of proving it (a similar remark was made in [1]). Below I give a direct proof. I thank K.P. Villanger for bringing the problem to my attention.

Let the vertices of K_n be $1, \ldots, n$ and let $V_i = A_i \cup B_i$, $i = 1, \ldots, r$, such that the edges of G_i are all the edges between A_i and B_i. Let L_i (M_i) be the polynomial $\Sigma X_j; j \in A_i$ ($\Sigma X_j; j \in B_i$); Then, as the G_i form a decomposition of K_n, we have the equation

$$X_1X_2 + X_1X_3 + \cdots + X_{n-1}X_n = L_1M_1 + \cdots + L_rM_r. \quad (1)$$

If $r \leq n - 2$ the set of homogeneous linear equations

$$L_1 = L_2 = \cdots = L_r = X_1 + \cdots + X_n = 0$$

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has a nontrivial real solution $(a_1, \ldots, a_n)$. But this gives, by (1), the contradiction

$$0 < a_1^2 + \cdots + a_n^2 = (a_1 + \cdots + a_n)^2 - 2(a_1a_2 + a_1a_3 + \cdots + a_{n-1}a_n)$$
$$= 0^2 - 2[0 \cdot M_1(a_1, \ldots, a_n) \cdots + 0 \cdot M_r(a_1, \ldots, a_n)] = 0.$$

It would still be nice to have a nonalgebraic proof, and also a treatment of infinite complete graphs.

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