Structures and Duality in Combinatorial Programming

HDR — Denis Cornaz

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Introduction

- $(P) = \{Ax \leq b\}$ is *TDI:*

$$\begin{aligned} & \min\{y^\top b : y^\top A \geq w^\top\} \\ = & \min\{y^\top b : y^\top A \geq w^\top, y \text{ integer}\} \qquad (\forall w \text{ integer}) \end{aligned}$$

- $P = \{x : Ax \leq b\}$ is *integer:*

$$\begin{aligned} & \max\{w^\top x : x \in P\} \\ = & \max\{w^\top x : x \in P, x \text{ integer}\} \qquad (\forall w) \end{aligned}$$

- A is 0-1: a clutter $H = (E, \mathcal{C})$

$$\begin{aligned} \alpha_w(H) := & \max\{w^\top x : Ax \leq \mathbf{1}, x \geq \mathbf{0}, x \text{ integer}\} \quad \text{(packing)} \\ \tau_w(H) := & \min\{w^\top x : Ax \geq \mathbf{1}, x \geq \mathbf{0}, x \text{ integer}\} \quad \text{(covering)} \end{aligned}$$

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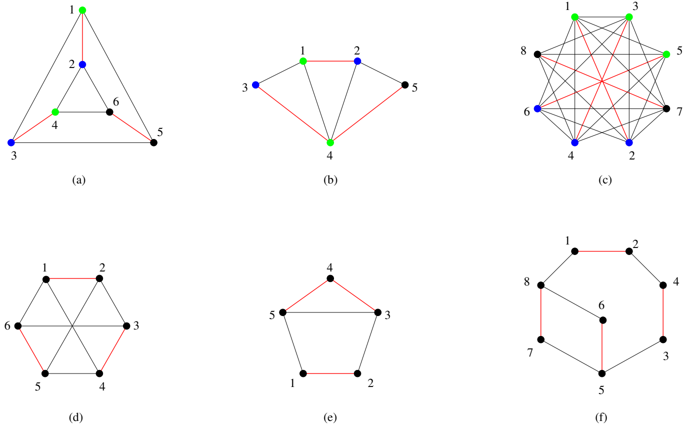
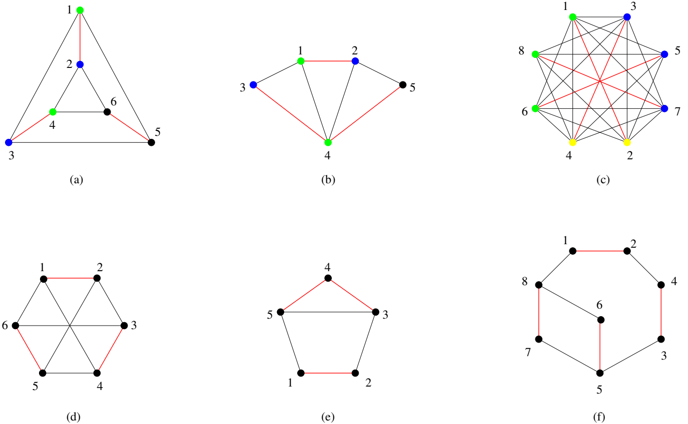
The minimum biclique cover problem (with Fonlupt)



$H = (E, \mathcal{C})$ is the minimal non-biclique clutter of $G = (V, E)$

$$\omega(G) - 1 \leq \max_{C \in \mathcal{C}} |C| \leq \omega(G) \quad \text{(with equality at the right if } \omega(G) \text{ is odd)}$$

The minimum biclique cover problem (with Fonlupt)



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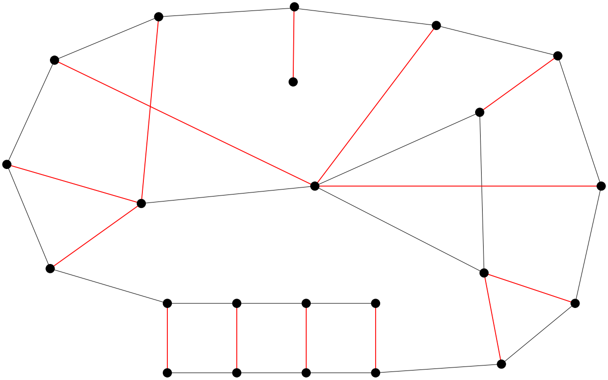
A general method (with Fonlupt, Kerivin, and Mahjoub)

- $\mathcal{B}$ is a set of edge subsets of G,
- $\mathcal{C}(\mathcal{B}) := \{C \subseteq E : C \not\subseteq B \text{ but } C' \subseteq B, \forall C' \subset C\}$

Finding a minimum cost $C \in \mathcal{C}(\mathcal{B})$ is P when $\mathcal{B}$ is:

(a) edge sets of complete bipartite subgraphs of G,
(b) edge sets of complete multipartite subgraphs of G,
(c) edge sets of vertex-induced bipartite subgraphs of G,
(d) arc sets of vertex-induced acyclic subdigraphs of G.

The maximum complete multipartite subgraph problem

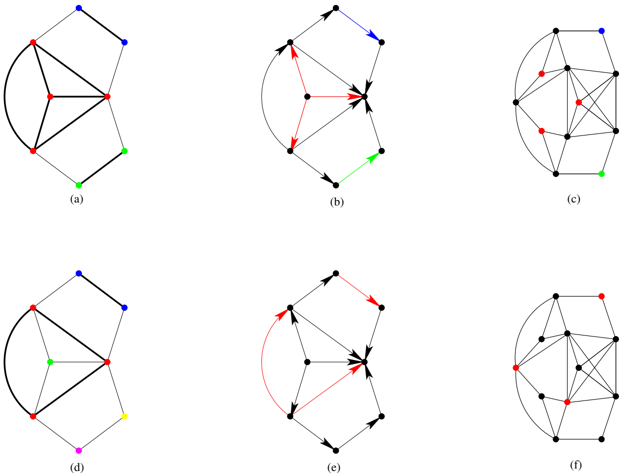


$H = (E, \mathcal{C})$ is the minimal non-multiclique clutter of G

$$|C| = 2 \quad (\forall C \in \mathcal{C}) \qquad \Longleftrightarrow \qquad G \text{ is fan- and prism-free}$$

1.
2. Graph coloring
3.
4.

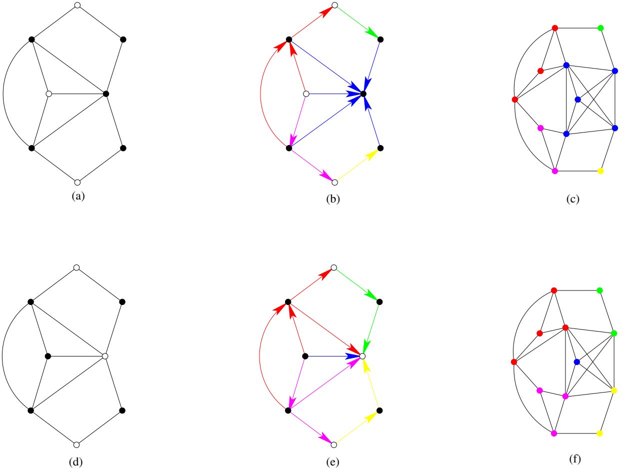
Chromatic Gallai identities (with Jost and Meurdesoif)



$\vec{G}$ is an orientation of G, without 3-dicycle

$$\overline{\chi}(G) + \alpha(S(\vec{G})) = |V(G)|$$

Chromatic Gallai identities (with Jost and Meurdesoif)



$\vec{G}$ is an orientation of G, without 3-dicycle

$$\overline{\chi}(G) + \alpha(S(\vec{G})) = |V(G)| = \alpha(G) + \overline{\chi}(S(\vec{G}))$$

Appl. in mathematical programming (with Meurdesoif)

$$\overline{\chi}_f(G) = \left\{ \begin{array}{lll} \max & \mathbf{1}^\top x & \\ \text{s.t.} & x(K) \leq 1 & (\forall \text{clique } K) \\ & x_v \geq 0 & (\forall v \in V) \end{array} \right.$$

$$= \left\{ \begin{array}{lll} \min & \mathbf{1}^\top y & \\ \text{s.t.} & \sum_{K \ni v} y_K = 1 & (\forall v) \\ & y_K \geq 0 & (\forall \text{clique } K) \end{array} \right.$$

y optimal for $G \Rightarrow x$ feasible for $S(\vec{G})$

$$x_{uv} = \sum_{\text{clique-stars } S_K \ni uv} y_K$$

Appl. in mathematical programming (with Meurdesoif)

$$\vartheta(G) = \begin{cases} \max & \sum_v \|x_v\|^2 \\ \text{s.t.} & \|x_o\|^2 = 1 \\ & x_o^\top x_v = \|x_v\|^2 \qquad (\forall v) \\ & x_u^\top x_v = 0 \qquad (\forall uv \in E) \end{cases}$$

$$= \begin{cases} \min & \|y_o\|^2 \\ \text{s.t.} & \|y_v\|^2 = 1 \qquad (\forall v) \\ & y_o^\top y_v = 1 \qquad (\forall v) \\ & y_u^\top y_v = 0 \qquad (\forall uv \notin E) \end{cases}$$

$x \in \mathbb{R}^{d \times m+1}$ optimal for $S(\vec{G}) \Rightarrow y \in \mathbb{R}^{nd \times n+1}$ feasible for G

$$y_{o\,v} = x_o - \sum_{u:uv \in E(\vec{G})} x_{uv} \text{ and } y_{v\,u} = \begin{cases} y_{o\,v} & \text{if } u = v \\ x_{uv} & \text{if } uv \in E(\vec{G}) \\ 0 & \text{otherwise} \end{cases}$$

Appl. in mathematical programming (with Meurdesoif)

If $\alpha(G) \leq \beta(G) \leq \overline{\chi}(G)$, $\forall G$, then

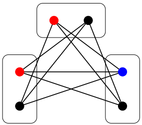
$$\alpha(G) \leq \quad \Phi_\beta(G) := |V(G)| - \beta(S(\vec{G})) \quad \leq \overline{\chi}(G) \qquad (\forall G)$$

Improving Lovász's ϑ bound for coloring

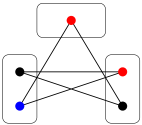
$$\alpha(G) \leq \Phi_{\overline{\chi}_f}(G) \leq \overline{\chi}_f(G) \quad \text{and} \quad \vartheta(G) \leq \Phi_\vartheta(G) \leq \overline{\chi}(G) \qquad (\forall G)$$

	$\lvert V\rvert$	$\lvert E\rvert$	min $\rho := \frac{\Phi_\vartheta - \vartheta}{\vartheta}$	mean ρ	max ρ
$\overline{M_3}$	5	5	–	23.6%	–
$\overline{M_4}$	11	35	26.8%	27.3%	28.7%
$\overline{M_5}$	23	182	26.5%	27.6%	29.5%
$\overline{M_6}$	47	845	26.0%	27.8%	29.5%

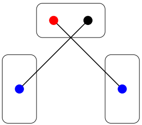
Coloring clustered graphs (with Bonomo, Ekim and Ries)



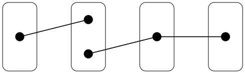
(a)



(b)



(c)



(d)

(e)

$$M(G,\mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } M(G/\mathcal{V}) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\omega(G/\mathcal{V}) \leq \chi(G/\mathcal{V}) \leq \chi_{sel}(G,\mathcal{V})$$

Coloring clustered graphs (with Bonomo, Ekim and Ries)

G is *selective-perfect* if $M(G,\mathcal{V})$ is perfect for all $\mathcal{V}$

G is *i-threshold* if

$$uv \in E \Updownarrow \Re(t(u)+t(v)) - \Im(t(u))\Im(t(v)) > 0$$

Minimal clustered graphs such that $M(G,\mathcal{V}) \neq M(G/\mathcal{V})$ are (a)-(c)

G is selective-perfect $\iff$ G is *i*-threshold

The clique-connecting forest polytope

$$
\text{(CCFO)}\begin{cases} 0 \le x_e \le 1 & (\forall e \in E) \\ x(E(U)) \le |U| - \begin{cases} 1 & \text{if } U \text{ is a clique of } G \\ 2 & \text{otherwise} \end{cases} & (\forall U \subseteq V) \end{cases}
$$

(a) (b)

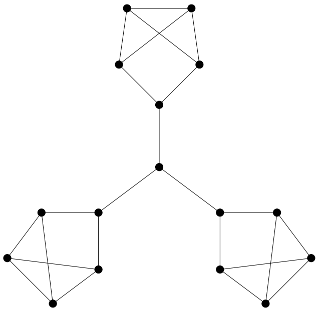
Facets of the clique-connecting forest polytope

Inequalities induced by complete sets or by the clique polytope are facets

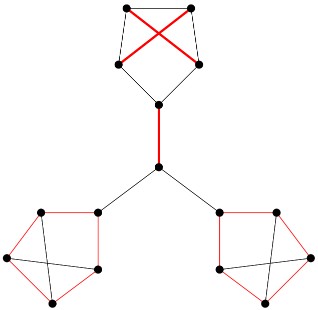
1.
2.
3. Min-max relations
4.

The star polytope (with Nguyen)

- $G_1 \cup \ldots \cup G_k = G$ with $G_i \subseteq G$
- Cost $= \sum_i \Delta(G_i)$



(a)



(b)

Figure : If $\max \Delta(G_i) = 2$, then Cost $> \Delta(G)$

The star polytope (with Nguyen)

ocm set (C, M):

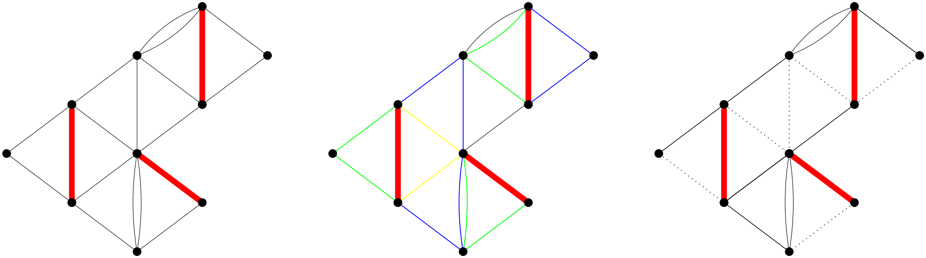
- M is a matching
- $C = C_1 \cup \ldots \cup C_k$ (C_i odd-circuit)
- $C_1, \ldots, C_k, M$ are pairwise vertex-disjoint

Generalizing Kőnig's min-max relation to a "best-posible" one

$$(P_{star}) \begin{cases} x_e \geq 0 & \text{for all } e \in E \\ \frac{1}{2}x(C) + x(M) \leq 1 & \text{for all maximal ocm set } (C, M) \text{ of } G, \end{cases}$$

is minimally TDI

Max-multiflow vs. min-multicut



(a) (b) (c)

- A *multicut* disconnects the demands

Max-multiflow vs. min-multicut

The multicut polytope: $conv.hull\{\chi^{\delta(V_1,\dots,V_p)}\}$ $(\forall p)$

$$(\mathrm{MCUT})\left\{\begin{array}{rl} 0 \le x_e \le 1 & (\forall e) \\ x(C \setminus \{e\}) \le x_e & (\forall \text{ circuit } C \ni e) \end{array}\right.$$

Weights: $(+)$ demands and $(-)$ links

(MCUT) is TDI $\iff$ the graph is series-parallel

$\Downarrow$

$G + H$ is series-parallel $\implies$ max-multiflow=min-multicut

1.
2.
3.
4. Election problems

Structure and algorithm in elections (with Galand and Spanjaard)

$$\mathcal{P}' = \begin{pmatrix} a & c & b \\ b & b & a \\ c & a & c \end{pmatrix} \quad \text{is a subprofile of} \quad \mathcal{P} = \begin{pmatrix} a & c & b & b & a \\ b & d & a & a & c \\ c & b & c & d & b \\ d & a & d & c & d \end{pmatrix}$$

- $\chi_v^c :=$ position of candidate $c \in C$ in voter $v \in V$
- *representative k-set* $C' \subseteq C$ minimizes $\sum_{v \in V} \min_{c \in C'} \chi_v^c$
- $\mathcal{P}$ *single-peaked* if $\exists$ path P with vertex-set C:

$$\{c \in C : \chi_v^c \leq i\} \text{ induces a connected subgraph of } P$$

$$(\forall v \in V, \forall i \in \{1, \ldots, |C|\})$$

Structure and algorithm in elections (with Galand and Spanjaard)

- *Kemeny voter* u minimizes $\sum_{v \in V} d(u,v)$, where

$$d(u,v) = \sum_{ab \text{ pair of } C} (\mathbf{1}^u_{ab} - \mathbf{1}^v_{ab})^2 \quad , \quad \mathbf{1}^u_{ab} = \begin{cases} 1 & \text{if } \chi^a_u < \chi^b_u \\ 0 & \text{otherwise} \end{cases}$$

(u may be outside V)

- $\mathcal{P}$ *single-crossing* if $\exists$ path P with vertex-set V:

$$\{v \in V : \chi^a_v < \chi^b_v\} \text{ induces a connected subgraph of } P$$

($\forall ab$ pair of C)

Structure and algorithm in elections (with Galand and Spanjaard)

- $\mathcal{C}$ = partition of C into intervals

$$\mathcal{P} = \begin{pmatrix} d & d \\ x & a \\ y & v \\ c & b \\ b & c \\ a & x \\ v & y \end{pmatrix} = \begin{pmatrix} d & d \\ x & a \\ y & v \\ c & b \\ b & c \\ a & x \\ v & y \end{pmatrix}$$

- $\mathcal{P}/\mathcal{C}$ = subprofile with one candidate per interval of $\mathcal{C}$

$$\mathcal{P}/(abcv, d, x, y) = \begin{pmatrix} d & d \\ x & a \\ y & x \\ a & y \end{pmatrix} \text{ and } \mathcal{P}/(abv, d, cxy) = \begin{pmatrix} d & d \\ c & a \\ a & c \end{pmatrix}$$

Structure and algorithm in elections (with Galand and Spanjaard)

single-peaked width:=
maximum size of an interval in $\mathcal{C}$ such that $\mathcal{P}/\mathcal{C}$ is single-peaked

finding the single-peaked width is P

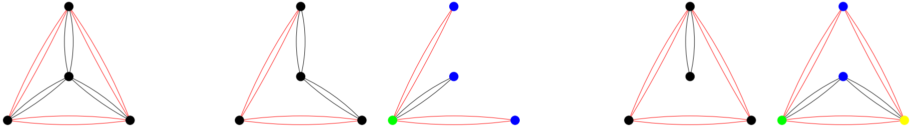
bounded single-peaked width $\Longrightarrow$ representative k-set becomes P

single-crossing width:=
maximum size of an interval in $\mathcal{C}$ such that $\mathcal{P}/\mathcal{C}$ is single-crossing

finding the single-crossing width is P

bounded single-crossing width $\Longrightarrow$ Kemeny voter becomes P

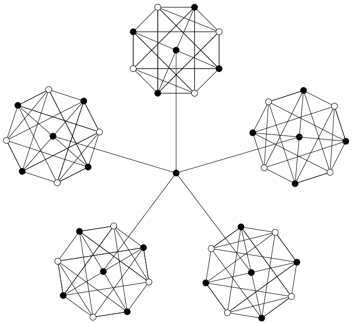
Question: Max-multiflow vs. min-multicut



(a) (b) (c)

- (G, R) signed graph: R is the set of demands (red)
- C odd-circuit: $|C \cap R|$ odd
- C flow: $|C \cap R| = 1$
- $T = R \triangle D$ (D cut or multicut)

Question: The star polytope


(a)

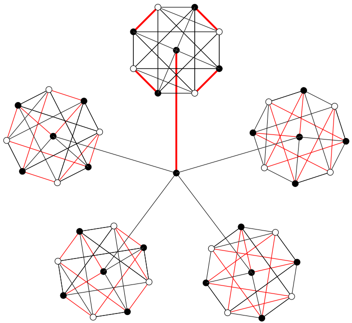

(b)

Figure : If $\max \Delta(G_i) = 4$, then Cost $> \Delta(G)$

Publications

- Denis Cornaz, Hervé Kerivin, Ali Ridha Mahjoub: Structure of minimal arc-sets vertex-inducing dicycles. (submitted)
- Denis Cornaz, Philippe Meurdesoif: Chromatic Gallai identities operating on Lovász number. *Mathematical Programming, Series A* (to appear) (online `springer.com`)
- Flavia Bonomo, Denis Cornaz, Tinaz Ekim, Bernard Ries: Perfectness of clustered graphs. *Discrete Optimization* 10(4) : 296-303 (2013)
- Denis Cornaz, Viet Hung Nguyen: Kőnig's edge-colouring theorem for all graphs. *Operations Research Letters* 41: 592-596 (2013)
- Denis Cornaz, Lucie Galand, Olivier Spanjaard: Kemeny elections with bounded single-peaked or single-crossing width. *In Proceedings of 23rd. IJCAI* : 76-82 (2013)
- Cédric Bentz, Denis Cornaz, Bernard Ries: Packing and covering with linear programming: A survey. *European Journal of Operational Research* 227(3): 409-422 (2013)
- Denis Cornaz, Lucie Galand, Olivier Spanjaard: Bounded single-peaked width and proportional representation. *In Proceedings of 20th. ECAI* : 270-275 (2012)
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- Denis Cornaz, Philippe Meurdesoif: The sandwich line graph. *Electronic Notes in Discrete Mathematics* 36: 955-960 (2010)
- Denis Cornaz: Clique-connecting forest and stable set polytopes. *RAIRO - Operations Research* 44(1): 73-83 (2010)
- Denis Cornaz, Vincent Jost: A one-to-one correspondence between colorings and stable sets. *Operations Research Letters* 36(6): 673-676 (2008)
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- Denis Cornaz: A linear programming formulation for the maximum complete multipartite subgraph problem. *Mathematical Programming, Series B* 105(2-3): 329-344 (2006)
- Denis Cornaz, Jean Fonlupt: Chromatic characterization of biclique covers. *Discrete Mathematics* 306(5): 495-507 (2006)