IASD M2 at Paris Dauphine

Deep Reinforcement Learning

16: Offline Reinforcement Learning

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Acknowledgement

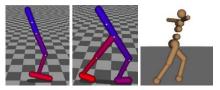
These materials are based on the seminal course of Sergey Levine CS285



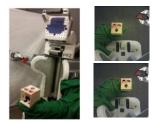
The generalization gap



Mnih et al. '13

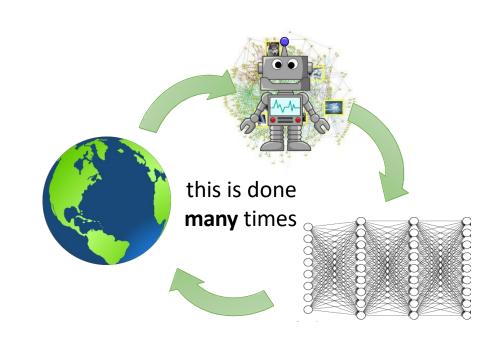


Schulman et al. '14 & '15



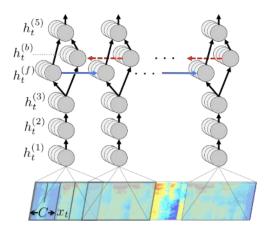
Levine*, Finn*, et al. '16





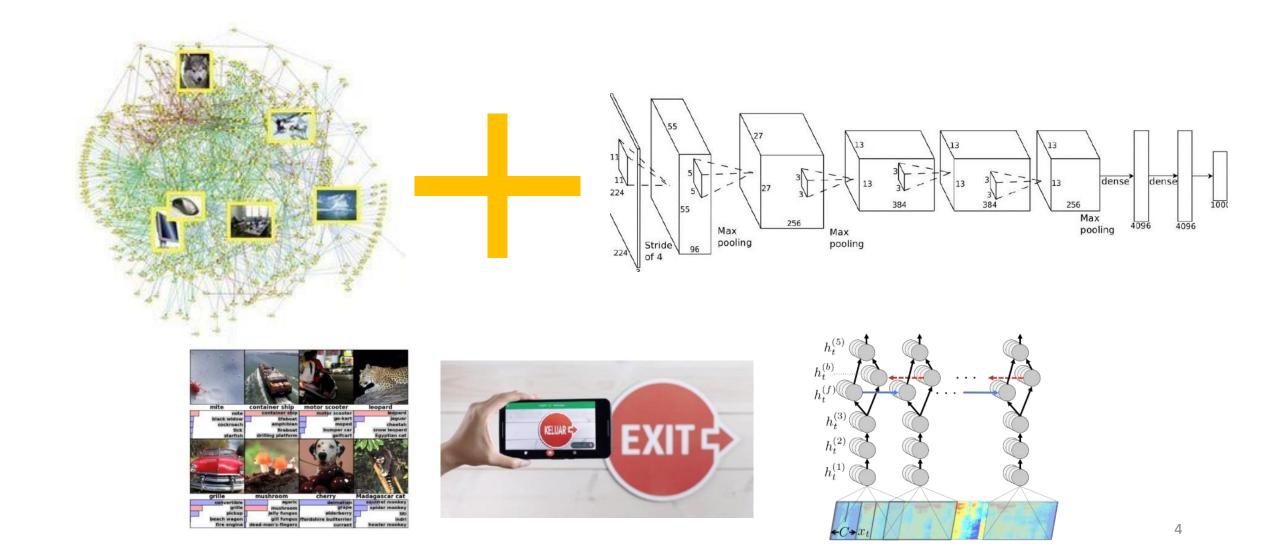




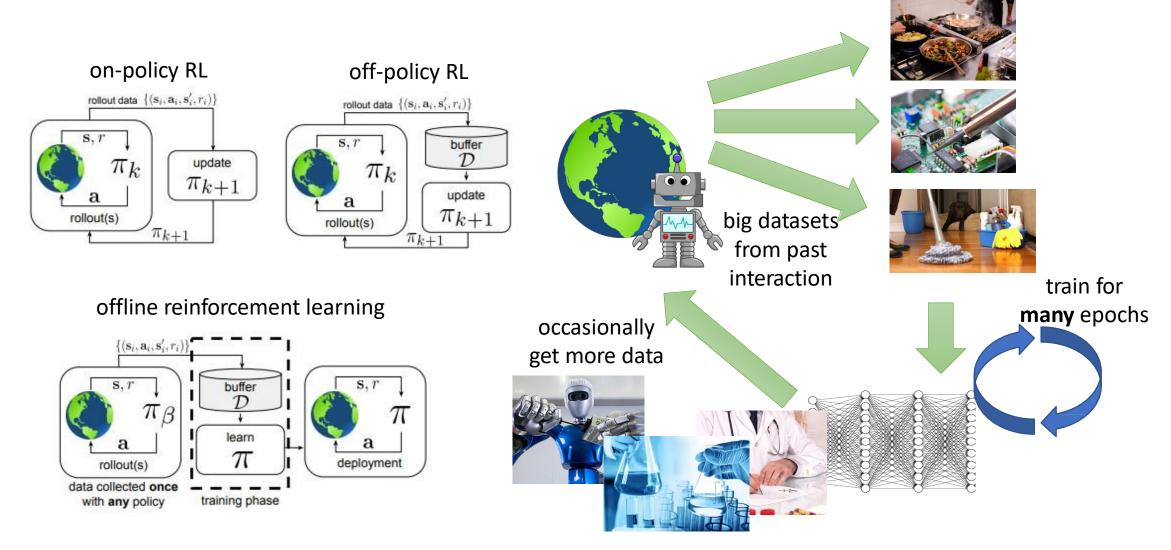




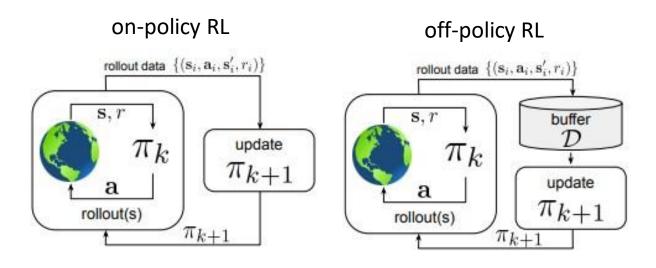
What makes modern machine learning work?



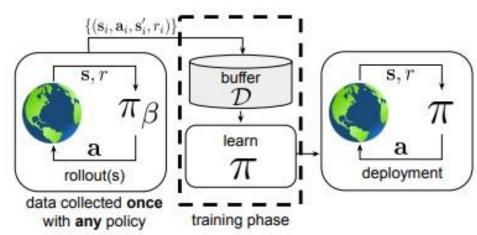
Can we develop data-driven RL methods?



What does offline RL mean?



offline reinforcement learning



Formally:

$$\mathcal{D} = \{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}_i', r_i)\}$$
 $\mathbf{s} \sim d^{\pi_{\beta}}(\mathbf{s})$ generally **not** known $\mathbf{a} \sim \pi_{\beta}(\mathbf{a}|\mathbf{s})$ $\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$ $r \leftarrow r(\mathbf{s}, \mathbf{a})$

RL objective:
$$\max_{\pi} \sum_{t=0}^{T} E_{\mathbf{s}_{t} \sim d^{\pi}(\mathbf{s}), \mathbf{a}_{t} \sim \pi(\mathbf{a}|\mathbf{s})} [\gamma^{t} r(\mathbf{s}_{t}, \mathbf{a}_{t})]$$

Types of offline RL problems

off-policy evaluation (OPE):

given
$$\mathcal{D}$$
, estimate $J(\pi) = E_{\pi} \left[\sum_{t=1}^{T} r(\mathbf{s}_{t}, \mathbf{a}_{t}) \right]$

$$\mathcal{D} = \{(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}_i', r_i)\}$$

$$\mathbf{s} \sim d^{\pi_{\beta}}(\mathbf{s})$$

$$\mathbf{a} \sim \pi_{\beta}(\mathbf{a}|\mathbf{s})$$

$$\mathbf{s}' \sim p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$$

$$r \leftarrow r(\mathbf{s}, \mathbf{a})$$

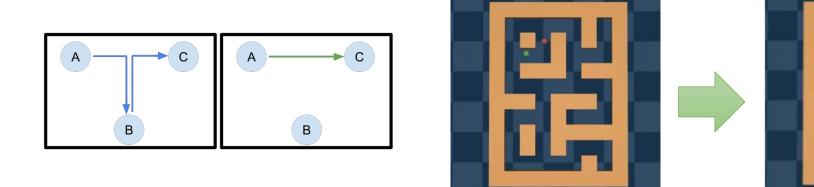
offline reinforcement learning: (a.k.a. batch RL, sometimes fully off-policy RL)

given \mathcal{D} , learn the best possible policy π_{θ}

not necessarily obvious what this means

How is this even possible?

- 1. Find the "good stuff" in a dataset full of good and bad behaviors
- 2. Generalization: good behavior in one place may suggest good behavior in another place
- 3. "Stitching": parts of good behaviors can be recombined

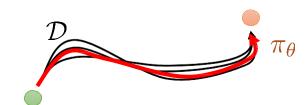


What do we expect offline RL methods to do?

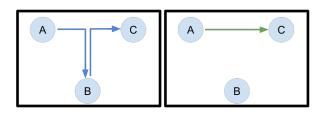
Bad intuition: it's like imitation learning

Though it can be shown to be **provably** better than imitation learning even with optimal data, under some structural assumptions!

See: Kumar, Hong, Singh, Levine. Should I Run Offline Reinforcement Learning or Behavioral Cloning?



Better intuition: get order from chaos



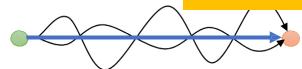
"Macro-scale" stitching



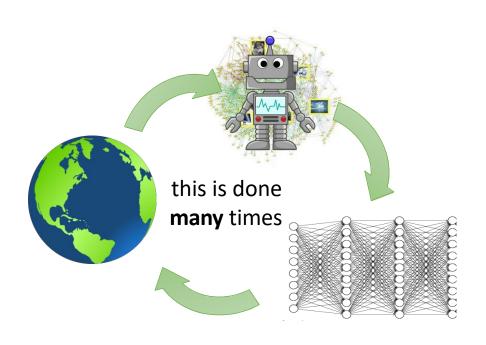
But this is just the clearest example!

"Micro-scale" stitching:

If we have algorithms that properly perform dynamic programming, we can take this idea much further and get near-optimal policies from highly suboptimal data



Why should we care?







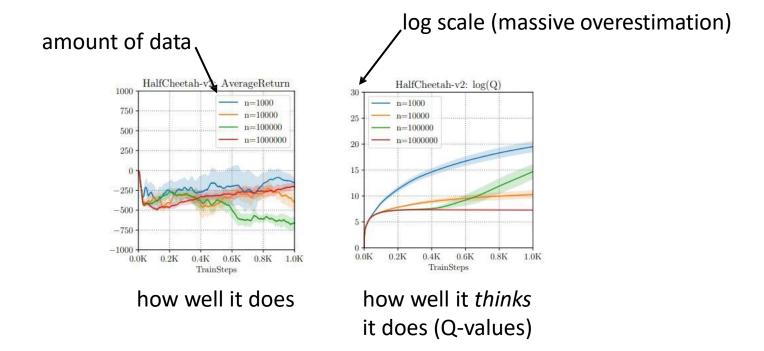




A vivid example

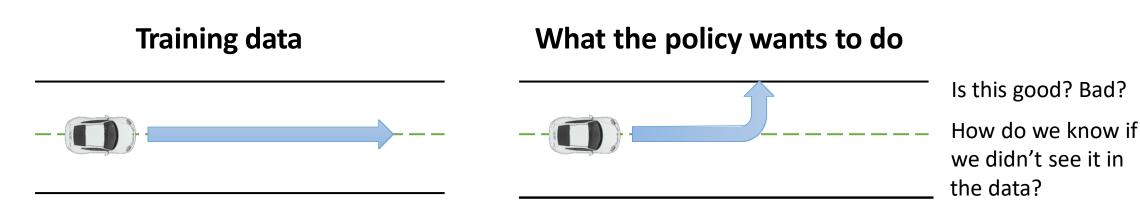


Why is offline RL hard?



Why is offline RL hard?

Fundamental problem: counterfactual queries



Online RL algorithms don't have to handle this, because they can simply **try** this action and see what happens

Offline RL methods must somehow account for these unseen ("out-of-distribution") actions, ideally in a safe way ...while still making use of generalization to come up with behaviors that are better than the best thing seen in the data!

Distribution shift in a nutshell

Example empirical risk minimization (ERM) problem:

$$\theta \leftarrow \arg\min_{\theta} E_{\mathbf{x} \sim p(\mathbf{x}), y \sim p(y|\mathbf{x})} \left[(f_{\theta}(\mathbf{x}) - y)^2 \right]$$

usually we are not worried – neural nets generalize well!

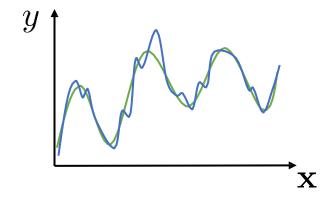
what if we pick $\mathbf{x}^* \leftarrow \arg \max_{\mathbf{x}} f_{\theta}(\mathbf{x})$?

given some \mathbf{x}^* , is $f_{\theta}(\mathbf{x}^*)$ correct?

$$E_{\mathbf{x} \sim p(\mathbf{x}), y \sim p(y|\mathbf{x})} \left[(f_{\theta}(\mathbf{x}) - y)^2 \right]$$
 is low

$$E_{\mathbf{x} \sim p(\mathbf{x}), y \sim p(y|\mathbf{x})} \left[(f_{\theta}(\mathbf{x}) - y)^2 \right]$$
 is not, for general $\bar{p}(\mathbf{x}) \neq p(\mathbf{x})$

what if $\mathbf{x}^* \sim p(\mathbf{x})$? not necessarily...



Where do we suffer from distribution shift?

$$Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \max_{\mathbf{a}'} Q(\mathbf{s}', \mathbf{a}')$$

$$Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + E_{\mathbf{a}' \sim \pi_{\text{new}}}[Q(\mathbf{s}', \mathbf{a}')]$$

$$y(\mathbf{s}, \mathbf{a})$$

expect good accuracy when $\pi_{\beta}(\mathbf{a}|\mathbf{s}) = \pi_{\text{new}}(\mathbf{a}|\mathbf{s})$

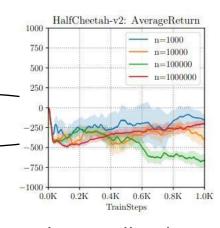
even worse: $\pi_{\text{new}} = \arg \max_{\pi} E_{\mathbf{a} \sim \pi(\mathbf{a}|\mathbf{s})}[Q(\mathbf{s}, \mathbf{a})]$

(what if we pick $\mathbf{x}^* \leftarrow \arg \max_{\mathbf{x}} f_{\theta}(\mathbf{x})$?)

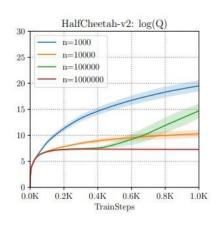
what is the objective?

$$\min_{Q} E_{(\mathbf{s},\mathbf{a}) \sim \pi_{\beta}(\mathbf{s},\mathbf{a})} \left[(Q(\mathbf{s},\mathbf{a}) - y(\mathbf{s},\mathbf{a}))^{2} \right]$$
 target value behavior policy

how often does that happen?

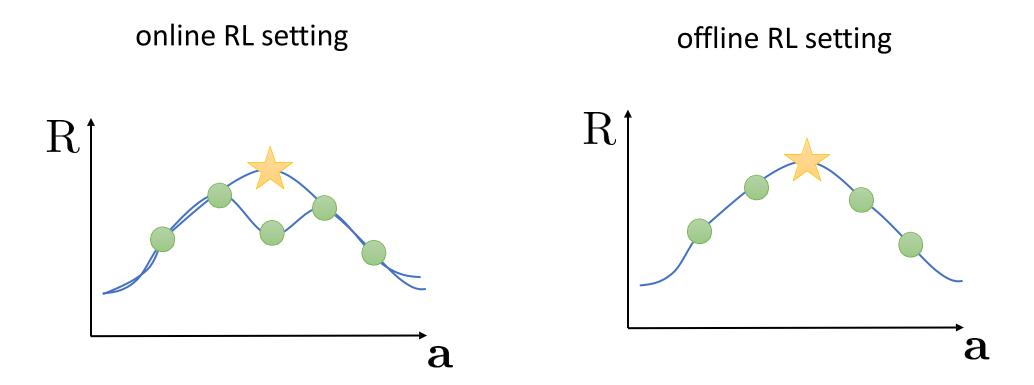


how well it does



how well it *thinks* it does (Q-values)

Issues with generalization are not corrected



Existing challenges with sampling error and function approximation error in standard RL become **much more severe** in offline RL

Batch RL via Importance Sampling

Offline RL with policy gradients

RL objective:
$$\max_{\pi} \sum_{t=0}^{I} E_{\mathbf{s}_{t} \sim d^{\pi}(\mathbf{s}), \mathbf{a}_{t} \sim \pi(\mathbf{a}|\mathbf{s})} [\gamma^{t} r(\mathbf{s}_{t}, \mathbf{a}_{t})]$$

$$\nabla_{\theta} J(\theta) = E_{\tau \sim \pi_{\theta}(\tau)} \left[\sum_{t=0}^{T} \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t} | \mathbf{s}_{t}) \hat{Q}(\mathbf{s}_{t}, \mathbf{a}_{t}) \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t,i}|\mathbf{s}_{t,i}) \hat{Q}(\mathbf{s}_{t,i},\mathbf{a}_{t,i})$$
requires sampling from π_{θ} ! what if we only have samples from π_{β} ?

importance sampling:

$$\nabla_{\theta} J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{\pi_{\theta}(\tau_{i})}{\pi_{\beta}(\tau_{i})} \sum_{t=0}^{T} \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t,i}|\mathbf{s}_{t,i}) \hat{Q}(\mathbf{s}_{t,i},\mathbf{a}_{t,i})$$
importance weight

Offline RL with policy gradients

$$\nabla_{\theta} J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{\pi_{\theta}(\tau_{i})}{\pi_{\beta}(\tau_{i})} \sum_{t=0}^{T} \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t,i}|\mathbf{s}_{t,i}) \hat{Q}(\mathbf{s}_{t,i},\mathbf{a}_{t,i}) - E_{\pi_{\theta}} \left[\sum_{t'=t}^{T} \gamma^{t'-t} r_{t'} \right] \approx \sum_{t'=t}^{T} \gamma^{t'-t} r_{t',i}$$

$$\frac{\pi_{\theta}(\tau)}{\pi_{\beta}(\tau)} = \frac{p(\mathbf{s}_1) \prod_t p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) \pi_{\theta}(\mathbf{a}_t|\mathbf{s}_t)}{p(\mathbf{s}_1) \prod_t p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) \pi_{\beta}(\mathbf{a}_t|\mathbf{s}_t)}$$

this is exponential in T weights likely to be degenerate as T becomes large

can we fix this?

$$\nabla_{\theta} J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \left(\prod_{t'=0}^{t-1} \frac{\pi_{\theta}(\mathbf{a}_{t',i}|\mathbf{s}_{t',i})}{\pi_{\beta}(\mathbf{a}_{t',i}|\mathbf{s}_{t',i})} \right) \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t,i}|\mathbf{s}_{t,i}) \left(\prod_{t'=t}^{T} \frac{\pi_{\theta}(\mathbf{a}_{t',i}|\mathbf{s}_{t',i})}{\pi_{\beta}(\mathbf{a}_{t',i}|\mathbf{s}_{t',i})} \right) \hat{Q}(\mathbf{s}_{t,i},\mathbf{a}_{t,i}) \right)$$

accounts for difference in probability of landing in $\mathbf{s}_{t,i}$ we have $\mathbf{s}_t \sim d^{\pi_{\beta}}(\mathbf{s}_t)$, but want $\mathbf{s}_t \sim d^{\pi_{\theta}}(\mathbf{s}_t)$

why is this a reasonable approximation?

accounts for having the incorrect $\hat{Q}(\mathbf{s}_{t,i}, \mathbf{a}_{t,i})$

Estimating the returns

$$\nabla_{\theta} J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_{\theta} \gamma^{t} \log \pi_{\theta}(\mathbf{a}_{t,i} | \mathbf{s}_{t,i}) \left(\prod_{t'=t}^{T} \frac{\pi_{\theta}(\mathbf{a}_{t',i} | \mathbf{s}_{t',i})}{\pi_{\beta}(\mathbf{a}_{t',i} | \mathbf{s}_{t',i})} \right) \hat{Q}(\mathbf{s}_{t,i}, \mathbf{a}_{t,i})$$

$$E_{\pi_{\theta}} \left[\sum_{t'=t}^{T} \gamma^{t'-t} r_{t'} \right] \approx \sum_{t'=t}^{T} \gamma^{t'-t} r_{t',i}$$

$$\sum_{t'=t}^{T} \left(\prod_{t''=t}^{T} \frac{\pi_{\theta}(\mathbf{a}_{t'',i}|\mathbf{s}_{t'',i})}{\pi_{\beta}(\mathbf{a}_{t'',i}|\mathbf{s}_{t'',i})} \right) \gamma^{t'-t} r_{t',i}$$

but this is *still* exponential!

To avoid exponentially exploding importance weights, we **must** use value function estimation!

imagine we knew $Q^{\pi_{\theta}}(\mathbf{s}, \mathbf{a})$

We'll see how to do this shortly, but first let's conclude our discussion of importance sampling

The doubly robust estimator

$$V^{\pi_{\theta}}(\mathbf{s}_{t}) \approx \sum_{t'=t}^{T} \left(\prod_{t''=t}^{t'} \frac{\pi_{\theta}(\mathbf{a}_{t'',i}|\mathbf{s}_{t'',i})}{\pi_{\beta}(\mathbf{a}_{t'',i}|\mathbf{s}_{t'',i})} \right) \gamma^{t'-t} r_{t',i}$$

this is exponential!

The doubly robust estimator

$$V^{\pi_{\theta}}(\mathbf{s}_{0}) \approx \sum_{t=0}^{T} \left(\prod_{t'=0}^{t} \frac{\pi_{\theta}(\mathbf{a}_{t'}|\mathbf{s}_{t'})}{\pi_{\beta}(\mathbf{a}_{t'}|\mathbf{s}_{t'})} \right) \gamma^{t} r_{t}$$

$$= \sum_{t=0}^{T} \left(\prod_{t'=0}^{t} \rho_{t'} \right) \gamma^{t} r_{t}$$

$$= \rho_{0} r_{0} + \rho_{0} \gamma \rho_{1} r_{1} + \rho_{0} \gamma \rho_{1} \gamma \rho_{2} r_{2} + \dots$$

$$= \rho_{0} (r_{0} + \gamma(\rho_{1}(r_{1} + \gamma(\rho_{2}(r_{2} + \gamma \dots)))))$$

$$= \bar{V}^{T} \qquad \text{where } \bar{V}^{T+1-t} = \rho_{t}(r_{t} + \gamma \bar{V}^{T-t})$$

$$\bar{V}_{\mathrm{DR}}^{T+1-t} = \hat{V}(\mathbf{s}_{t}) + \rho_{t}(r_{t} + \gamma \bar{V}_{\mathrm{DR}}^{T-t} - \hat{Q}(\mathbf{s}_{t}, \mathbf{a}_{t}))$$

this is exponential!

doubly robust estimation (bandit case)

$$V_{\rm DR}(s) = \hat{V}(s) + \rho(s, a)(r_{s,a} - \hat{Q}(s, a))$$

model or function approximator

model or function approximator

Jiang, N. and Li, L. (2015). Doubly robust off-policy value evaluation for reinforcement learning.

Marginalized importance sampling

Main idea: instead of using $\prod_t \frac{\pi_{\theta}(\mathbf{a}_t|\mathbf{s}_t)}{\pi_{\beta}(\mathbf{a}_t|\mathbf{s}_t)}$, estimate $w(\mathbf{s}, \mathbf{a}) = \frac{d^{\pi_{\theta}}(\mathbf{s}, \mathbf{a})}{d^{\pi_{\beta}}(\mathbf{s}, \mathbf{a})}$ if we can do this, we can estimate $J(\theta) \approx \frac{1}{N} \sum_i w(\mathbf{s}_i, \mathbf{a}_i) r_i$ typically this is done for off-policy evaluation, rather than policy learning

how to determine $w(\mathbf{s}, \mathbf{a})$? typically solve some kind of consistency condition example (Zhang et al., GenDICE):

$$d^{\pi_{\beta}}(\mathbf{s}', \mathbf{a}')w(\mathbf{s}', \mathbf{a}') = (1 - \gamma)p_0(\mathbf{s}')\pi_{\theta}(\mathbf{a}'|\mathbf{s}') + \gamma \sum_{\mathbf{s}, \mathbf{a}} \pi_{\theta}(\mathbf{a}'|\mathbf{s}')p(\mathbf{s}'|\mathbf{s}, \mathbf{a})d^{\pi_{\beta}}(\mathbf{s}, \mathbf{a})w(\mathbf{s}, \mathbf{a})$$
probability of starting in $(\mathbf{s}', \mathbf{a}')$ probability of transitioning into $(\mathbf{s}', \mathbf{a}')$

solving for $w(\mathbf{s}, \mathbf{a})$ typically involves some fixed point problem

Additional readings: importance sampling

Classic work on importance sampled policy gradients and return estimation:

Precup, D. (2000). Eligibility traces for off-policy policy evaluation.

Peshkin, L. and Shelton, C. R. (2002). Learning from scarce experience.

Doubly robust estimators and other improved importance-sampling estimators:

Jiang, N. and Li, L. (2015). Doubly robust off-policy value evaluation for reinforcement learning.

Thomas, P. and Brunskill, E. (2016). Data-efficient off-policy policy evaluation for reinforcement learning.

Analysis and theory:

Thomas, P. S., Theocharous, G., and Ghavamzadeh, M. (2015). High-confidence off-policy evaluation.

Marginalized importance sampling:

Hallak, A. and Mannor, S. (2017). Consistent on-line off-policy evaluation.

Liu, Y., Swaminathan, A., Agarwal, A., and Brunskill, E. (2019). Off-policy policy gradient with state distribution correction.

Additional readings in our offline RL survey: Section 3.1, 3.2, 3.3, 3.4: https://arxiv.org/abs/2005.01643

Batch RL via Linear Fitted Value Functions

Offline value function estimation

How have people thought about it before?

Extend existing ideas for approximate dynamic programming and Q-learning to offline setting

Derive tractable solutions with simple (e.g., linear) function approximators

How are people thinking about it now?

Derive approximate solutions with highly expressive function approximators (e.g., deep nets)

The primary challenge turns out to be **distributional shift**

We'll discuss some older offline/batch RL methods next for completeness



generally not concerned with distributional shift before

(maybe it was not such a big problem with linear models)

Warmup: linear models

 Φ – feature matrix, $|S| \times K$ could also think of as a vector-valued function $\Phi(\mathbf{s})$

Can we do (offline) model-based RL in feature space?

- 1. Estimate the reward
- 2. Estimate the transitions
- 3. Recover the value function
- 4. Improve the policy
- 1. Reward model: $\Phi \mathbf{w}_r \approx r$

transition matrix

2. Transition model: $\Phi \mathbf{P}_{\Phi} \approx \mathbf{P}^{\pi} \Phi$

estimated feature-space

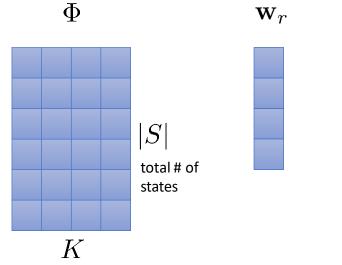
 $K \times K$

real transition matrix (on states) $|S| \times |S|$

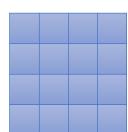
least squares: $\mathbf{w}_r = (\Phi^T \Phi)^{-1} \Phi^T \vec{\mathbf{r}}$

least squares: $\mathbf{P}_{\Phi} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{P}^{\pi} \Phi$

total # of features



 \mathbf{P}_{Φ}^{π}



vector of rewards for all state-action tuples but we'll talk about sample-based setting soon!

all of this is for a fixed policy π

Recovering the value function

- 1. Reward model: $\Phi \mathbf{w}_r \approx r$ least squares: $\mathbf{w}_r = (\Phi^T \Phi)^{-1} \Phi \vec{\mathbf{r}}$
- 2. Transition model: $\Phi \mathbf{P}_{\Phi} \approx \mathbf{P}^{\pi} \Phi$ least squares: $\mathbf{P}_{\Phi} = (\Phi^{T} \Phi)^{-1} \Phi \mathbf{P}^{\pi} \Phi$
- 3. Estimate $V^{\pi} \approx V_{\Phi}^{\pi} = \Phi \mathbf{w}_{V}$ can apply the same equation in feature space: $\mathbf{w}_{V} = (\mathbf{I} \gamma \mathbf{P}_{\Phi})^{-1} \mathbf{w}_{r}$ substitute

but wait – do we even *need* the model?

$$\mathbf{w}_V = (\mathbf{I} - \gamma(\Phi^T\Phi)^{-1}\Phi^T\mathbf{P}^\pi\Phi)^{-1}(\Phi^T\Phi)^{-1}\Phi^T\vec{\mathbf{r}}$$
 after a bit of algebra...

$$\mathbf{w}_V = (\Phi^T \Phi - \gamma \Phi^T \mathbf{P}^{\pi} \Phi)^{-1} \Phi^T \vec{\mathbf{r}}$$

 $\frac{|S|}{\text{total \# of states}}$

 \mathbf{W}_r

 \mathbf{P}_{Φ}^{π}

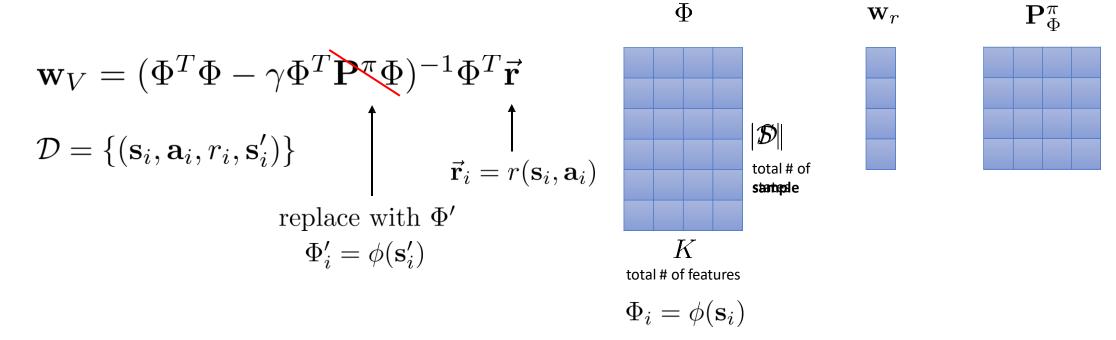
Aside: solving for V^{π} in terms of \mathbf{P}^{π} and \mathbf{r} :

$$V^{\pi} = \mathbf{r} + \gamma \mathbf{P}^{\pi} V^{\pi}$$
$$(\mathbf{I} - \gamma \mathbf{P}^{\pi}) V^{\pi} = \mathbf{r}$$
$$V^{\pi} = (\mathbf{I} - \gamma \mathbf{P}^{\pi})^{-1} \mathbf{r}$$

this is called least-squares temporal difference (LSTD)

Φ

Doing it all with samples



Everything else works **exactly** the same way, only now we have some sampling error

Improving the policy

- 1. Estimate the reward
- 2. Estimate the transitions
- 3. Recover the value function
- 4. Improve the policy

typical policy improvement step:

$$\pi'(\mathbf{s}) \leftarrow \text{Gree V}(\Phi \mathbf{w}_V)$$

$$= \text{estimate } V^{\pi'}$$

That's not going to work for offline RL!

or just do these together with LSTD!

$$\mathbf{w}_{V} = (\Phi^{T}\Phi - \gamma\Phi^{T}\mathbf{P}^{\pi}\Phi)^{-1}\Phi^{T}\vec{\mathbf{r}}$$

$$\mathcal{D} = \{(\mathbf{s}_{i}, \mathbf{a}_{i}, r_{i}, \mathbf{s}'_{i})\} \qquad \uparrow$$

$$\vec{\mathbf{r}}_{i} = r(\mathbf{s}_{i}, \mathbf{a}_{i})$$

$$replace with Φ'

$$\Phi'_{i} = \phi(\mathbf{s}'_{i})$$$$

this requires samples from $\pi!$

Least-squares policy iteration (LSPI)

Main idea: replace LSTD with LSTDQ – LSTD but for Q-functions

$$\mathbf{w}_Q = (\Phi^T \Phi - \gamma \Phi^T \Phi')^{-1} \Phi^T \vec{\mathbf{r}}$$

$$\mathcal{D} = \{ (\mathbf{s}_i, \mathbf{a}_i, r_i, \mathbf{s}_i') \} \qquad \uparrow$$

$$\vec{\mathbf{r}}_i = r(\mathbf{s}_i, \mathbf{a}_i)$$

$$\Phi_i' = \phi(\mathbf{s}_i', \pi(\mathbf{s}_i'))$$
encode the action π would take

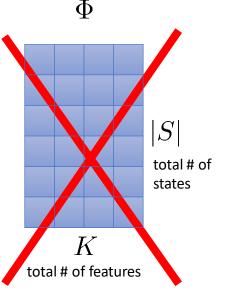
LSPI:

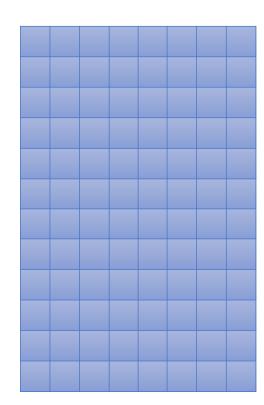
1. compute \mathbf{w}_Q for π_k

not the action in the data

2.
$$\pi_{k+1}(\mathbf{s}) = \arg \max_{\mathbf{a}} \phi(\mathbf{s}, \mathbf{a}) \mathbf{w}_Q$$

3. Set
$$\Phi'_i = \phi(\mathbf{s}'_i, \pi_{k+1}(\mathbf{s}'_i))$$



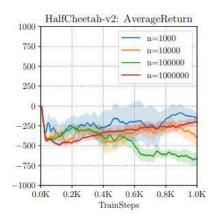


Φ

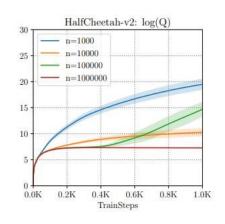
|S||A| total # of states-action tuples

 $K' \\ \text{total \# of features typically} \\ \text{replicated for each action} \\$

What's the issue?



how well it does



how well it thinks it does (Q-values)

In general, all approximate dynamic programming (e.g., fitted value/Q iteration) methods will suffer from action distributional shift, and we **must** fix it!

$$Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + E_{\mathbf{a}' \sim \pi_{\text{new}}}[Q(\mathbf{s}', \mathbf{a}')]$$
$$y(\mathbf{s}, \mathbf{a})$$

$$\min_{Q} E_{(\mathbf{s},\mathbf{a}) \sim \pi_{\beta}(\mathbf{s},\mathbf{a})} \left[(Q(\mathbf{s},\mathbf{a}) - y(\mathbf{s},\mathbf{a}))^2 \right]$$
 target value behavior policy

expect good accuracy when $\pi_{\beta}(\mathbf{a}|\mathbf{s}) = \pi_{\text{new}}(\mathbf{a}|\mathbf{s})$

how often does that happen?

even worse: $\pi_{\text{new}} = \arg \max_{\pi} E_{\mathbf{a} \sim \pi(\mathbf{a}|\mathbf{s})}[Q(\mathbf{s}, \mathbf{a})]$