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Lancini, Emiliano ; Pisanu, Francesco

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A HORIZON TOUR OF BOX-TOTAL DUAL INTEGRALITY

Emiliano Lancini, Francesco Pisanu

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A Horizon Tour of Box-Total Dual Integrality

Emiliano Lancini,* Francesco Pisanu †

Abstract

A linear system is *totally dual integral (TDI)* if, for every linear program with integer cost vector defined on it, the dual problem admits an integer optimum whenever it is feasible. A linear system is *box-totally dual integral* (box-TDI) if it remains TDI under the addition of arbitrary rational bounds on its variables. First introduced by Edmonds and Giles in the late 1970s, box-TDIness is a central property in combinatorial optimization, with deep connections to polyhedral integrality, min–max duality, and integer programming. This article provides a self-contained survey of both classical and recent results concerning box-TDI systems and polyhedra. We also discuss complexity aspects and examples from combinatorial optimization where box-TDIness arises naturally. Particular attention is paid to unifying different lines of development in the literature and clarifying the structural properties that underlie the theory. Throughout the paper, we highlight open questions and conjectures, offering a perspective on ongoing and future directions of research.

1 Introduction

Many combinatorial optimization problems can be formulated as Integer Linear Programs (ILP), which are NP-hard [70] in general. However, these problems can be solved efficiently when their continuous relaxations yield integer optimal solutions [62]. According to Schrijver [91, Preface], polyhedral combinatorics was first introduced by Edmonds, and has led to significant algorithmic results in combinatorial optimization, revealing several min-max relations. In this context, several properties that certify polyhedral integrality have been introduced in the literature. Among those, we focus on box-total dual integrality, a strengthening of total dual integrality that we define below.

A linear system $Ax \leq b$, with $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$, is *totally dual integral (TDI)* if the minimum in the linear programming duality equation

$$\max\{w^\top x\colon Ax \leq b, x \geq \mathbf{0}\} = \min\{b^\top y\colon A^\top y \geq w, y \geq \mathbf{0}\} \tag{1}$$

has an integer optimal solution for all integer vectors w for which the optimum is finite. A stronger property is box-total dual integrality. A system $Ax \leq b$ is *box-totally dual integral* (*box-TDI*) if the system $Ax \leq b, \ell \leq x \leq u$ is TDI for all rational vectors ℓ and u, where some of the components of ℓ and u might be unbounded. The prefix “box” comes

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from what we call *box-constraints*, that is the constraints of the form $\ell \leq x \leq u$ with ℓ and u rational vectors with possibly infinite components; these constraints represent bounds on the variables and, when $\ell \geq 0$, are also known as *capacity constraints* in the context of network flows.

TDIness and box-TDIness were introduced by Edmonds and Giles [45] in the late 70's to generalize submodular functions. Reasons why these properties have been intensively studied include min-max relations in linear programming duality, strong integrality properties [55], and matroid theory [15, 45]. Specifically, TDIness characterizes polyhedral integrality [55], and box-TDIness guarantees integrality to be preserved under the addition of arbitrary integer box-constraints [21]. Further studies enriched TDIness theory: for instance, TDI systems are characterized by Hilbert bases (see Section 3.4) and are connected with linear Diophantine equations (see Section 3.4). Duality in linear programming also has applications in game theory. Notably, von Neumann's Minimax Theorem about zero-sum games [98] can be restated in terms of strong duality. Recent results related to box-TDIness in game theory are due to Del Pia et al. [35] and Kleer and Schäfer [71] for congestion games.

The scope of this article is twofold: on one hand, we want to provide an accessible yet rigorous introduction to box-TDIness, to clarify its geometric properties, and to discuss its usefulness and limitations in the context of combinatorial optimization on the other hand, we aim to review classical and recent results on box-TDIness, some recently described box-TDI polyhedra, and several open questions. We refer to the interested reader also to other comprehensive sources of results on (box-)TDIness, such as Chapter 22 of Schrijver's 1986 book [90] in the context of integer linear programming, and Chen et al. [17] from 2013, which focuses on combinatorial results. Whenever possible, we shall leverage geometrical aspects of box-TDIness in order to unify the known results. By doing so, we aim to integrate the framework established in the seminal work of Cook [27] with that of Chervet et al. [21]. Many examples here summarized come from classical results mainly due to Edmonds [44], Cunningham [32], and several more recent works such as Ding et al. [38,39], Chervet et al. [20, 21].

Outline In Section 2, we introduce some fundamental concepts and definitions used throughout the paper, including those of unimodularity and box-integrality. Section 3 provides an overview of TDIness, box-TDIness, and their algebraic foundations. In Section 4, we explore key structural properties of box-TDI systems, including their matricial characterizations. Section 5 is dedicated to the polyhedral properties of box-TDIness. Section 6 presents key complexity results related to (box-)TDI recognition and applications in optimization. In Section 7, we analyze several classical and recent case studies where box-TDIness plays a crucial role, including polymatroids, network flows, matchings, and stable sets. Finally, in Section 8 we briefly discuss the connections of TDIness and box-TDIness with dyadicness and several polyhedral properties stronger than polyhedral integrality.

Throughout the article, we highlight several open questions of general interest. These questions pertain to graph theory, polyhedral verification, and specialized complexity results for box-TDIness instances.

2 Preliminaries

Since Dantzig introduced the simplex method, many combinatorial problems have been treated as relaxations of some ILP. As a consequence, polyhedral integrality has been intensively studied, as well as its connection to matricial descriptions. Here, we recall some basic definitions and foundational results.

2.1 Generalities on Polyhedra

In this section, we review some basic definitions that will be used extensively throughout this work.

Polyhedra A *polyhedron* is the set of points satisfying a system of linear inequalities, that is $\{x\colon Ax \leq b\}$. A polyhedron P is *bounded* if for every point x of P there is no vector u such that $x + tu$ belongs to P for every positive scalar t. A bounded polyhedron is called a *polytope*. An inequality $a^\top x \leq \beta$ is *valid* for the polyhedron P if it is satisfied by every point in P. The *dimension* of P equals the dimension of its affine hull and is denoted by $\dim(P)$.

A face of a polyhedron P is a polyhedron of the form $F = P \cap \{x\colon a^\top x = \beta\}$ where $a^\top x \leq \beta$ is a valid inequality for P. If $P = \{x\colon Ax \leq b\}$, an inequality $a_i^\top x \leq b_i$ of the system $Ax \leq b$ is *tight* for F if $F \subseteq \{x\colon a_i^\top x = b_i\}$, and we denote with $A_F x \leq b_F$ the inequalities from $Ax \leq b$ that are tight for F. Faces whose dimension is equal to $\dim(P) - 1$ are called *facets*. A *vertex* is a face of dimension 0. A polyhedron having a vertex is called *pointed*.

A matrix M is *face-defining* for a face F of P if it has full row rank and the affine space generated by F can be written as $\{x\colon Mx = d\}$ for some vector d of appropriate size. If $P \subseteq \mathbb{R}^n$ and F is a face of P, then, we have that $\dim(F) = n - \operatorname{rank}(M)$, where M is a face-defining matrix of F.

The *k-dilation* of a polyhedron $P = \{x\colon Ax \leq b\}$ is the polyhedron $kP = \{x\colon Ax \leq kb\}$, where $k \in \mathbb{Q}_{>0}$. For a vector t the *t-translation* of P is $t + P = \{t + x\colon x \in P\}$.

(Box-)Integer Polyhedra A polyhedron is *integer* if all its faces contain an integer point. Specifically, a polytope is integer if and only if it is the convex hull of integer points. A polyhedron is *box-integer* if its intersection with the box $\{x\colon \ell \leq x \leq u\}$ is integer for all integer vectors ℓ and u. Similar to the case of box-TDIness, a *box* is indeed the polyhedron defined by box-constraints, and ℓ and u may have some unbounded components.

Figure 1 shows a box-integer polytope, Figure 2 shows an integer polytope that is not box-integer.

A polyhedron is *principally box-integer* if all integer k-dilations are also box-integer. For example, one can see that in $\mathbb{R}^2$, these polyhedra are exactly those whose interior angles are multiples of $\frac{\pi}{4}$. A polyhedron is *fully box-integer* if it is integer and principally box-integer (as the example in Figure 1).

Cones A *(polyhedral) cone* is the set of points satisfying a linear system of the form $Ax \leq \mathbf{0}$. A cone C can also be described as the set of non-negative combinations of a finite set of vectors R. That is, $C = \{x\colon x = \sum \alpha_i R^i, \alpha_i \geq 0\}$, and we say that C is *generated* by R, while the elements of R are called *rays*. We denote by $\operatorname{cone}(R)$ the cone generated

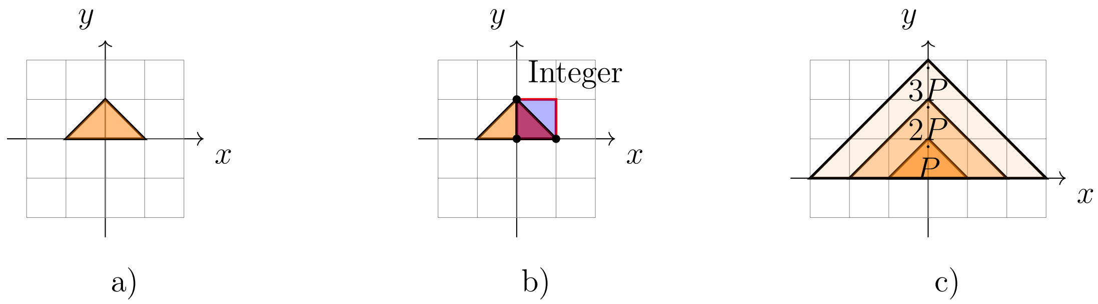

Figure 1: a) An example of an integer polytope in $\mathbb{R}^2$; b) A simple example showing that the intersection with an integer box preserves integrality; c) An example showing that the polytope P is fully box-integer.

Figure 2: a) An example of an integer polytope in $\mathbb{R}^2$; b) Shows that this polytope is not box-integer.

by R. Sometimes we will restrict ourselves to the integer non-negative combinations of a set of finite vectors R, called the *integer cone of* R and denoted by int.cone(R).

A *conic polyhedron* is a polyhedron that is a cone up to translation, that is $C = \{t+x\colon Ax \leq \mathbf{0}\}$, for some $t \in \mathbb{R}^n$. Of course, C is a conic polyhedron if $C = \{x\colon Ax \leq At\}$, for some $t \in \mathbb{R}^n$. For a given face F of a polyhedron $P = \{x\colon Ax \leq b\}$, the *tangent cone* in F is the conic polyhedron $C_F = \{x\colon A_F x \leq b_F\}$. Whenever F is a minimal inclusion-wise face of P, C_F is called *minimal tangent cone* of P. Every polyhedron is the intersection of its minimal tangent cones.

The *polar* of a cone $C = \{x\colon Ax \leq \mathbf{0}\}$ is the cone $C^* = \{x'\colon z^\top x' \leq \mathbf{0}, \text{ for all } z \in C\}$. One can see that $C^{**} = C$. Practically, C^* is the cone generated by the columns of $A^\top$. If C_F is the tangent cone of a face F of a polyhedron P, and $t \in F$, then, $(-t + C_F)^*$ is the set of vectors w such that $\max\{w^\top x\colon x \in P\}$ is achieved by all points of F. We refer to it as the *polar cone* of F (note that the polar cone of a face is also referred to as the normal cone).

2.2 Unimodularity and Generalizations

The integrality of a polyhedron can sometimes be deduced from the matrix describing it. We start by recalling two classical definitions in combinatorial optimization and a few well-known related results.

Unimodular Matrices A $m \times n$ integer matrix is *unimodular* if it has full row rank and all determinants of its square submatrices of rank m are ± 1. The following classical result of Dantzig and Veinnot gives a sufficient condition for the integrality of a polyhedron.

Theorem 2.1 (Dantzig and Veinnot [34]). *Let A be a full row rank matrix. Then, the polyhedron $\{x\colon Ax = b, x \geq \mathbf{0}\}$ is integer for every integer vector b if and only if A is unimodular.*

Totally Unimodular Matrices A matrix is *totally unimodular (TU)* if the determinant of every square submatrix is in $\{-1, 0, 1\}$. For totally unimodular matrices the following result holds.

Theorem 2.2 (Hoffman and Krustal [69]). *The polyhedron $\{x\colon Ax \leq b\}$ is box-integer for every integer vector b if and only if A is TU.*

TU matrices have been largely studied. Schrijver [90] gives the following historical line on the results about the recognition of TU matrices: Auslander and Trent [2, 3], Gould [58], Tutte [96], and Bixby and Cunningham [9]. A milestone result is given by Seymour [93], who showed a decomposition technique for regular matroids, yielding to a polynomial-time algorithm to recognize whether a given matrix is TU, later implemented by Truemper [95].

Different characterizations are known for TU matrices. We refer to [90, Theorem 19.3] for several of them. In this paper, we use the following one, due to Ghouila-Houri (sometimes referred to as the equitable bicoloring of a matrix [24, Section 4.2]).

Theorem 2.3 (Ghouila-Houri [52]). *A matrix is TU if and only if each subset of its rows can be partitioned into two parts such that the sum of the rows in one part minus the sum of the row in the other part is a $0, \pm 1$-vector.*

We provide some definitions generalizing unimodularity and total unimodularity which turn out to be useful due to recent advances (see Section 4.1).

Equimodular Matrices A $m \times n$ rational matrix is *equimodular* if it has full row rank and all its square submatrices of rank m have the same determinant in absolute value. The following theorem links equimodularity, entry-wise integrality, and total unimodularity.

Theorem 2.4 (Heller [65]). *For a full row rank $m \times n$ matrix A, the following statements are equivalent.*

- *A is equimodular;*
- *For each non-singular $m \times m$ submatrix B of A, $B^{-1}A$ is integer;*
- *For each non-singular $m \times m$ submatrix B of A, $B^{-1}A$ is a $0, \pm 1$-matrix;*
- *For each non-singular $m \times m$ submatrix B of A, $B^{-1}A$ is TU;*
- *There exists a non-singular $m \times m$ submatrix B of A such that $B^{-1}A$ is TU.*

A combinatorial interpretation of Theorem 2.4 might be the following: if A is equimodular, all columns of A are $0, \pm 1$ combinations of any maximal set of linearly independent columns.

Totally Equimodular Matrices A matrix is *totally equimodular (TE)* if every subset of linearly independent rows is equimodular. By definition, it follows that for each row, all non-zero entries are equal, up to the sign. Thus, one may assume TE matrices to be $0, \pm 1$ entry-wise, as we can always normalize any row of a TE matrix by dividing it by the corresponding non-zero value. Naturally, every TU matrix is a TE matrix for which all minors are bounded by 1 in absolute value. A simple example of TE matrix that is not TU is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

It is easy to verify that this is the smallest non-square ± 1-matrix with this property.

TE matrices recently rose as a subject of interest due to their connection with box-TDIness. In particular, Chervet et al. [22] proved three different characterizations of TE matrices, which we report below.

Theorem 2.5 (Chervet et al. [22])**.** *A square non-singular matrix A is TE if and only if $(A^\top)^{-1}$ is TE.*

Theorem 2.5 generalizes an already known fact for TU matrices. Indeed, it is straightforward to see that if M is a TU matrix, then $M^\top$ and M^{-1} (whenever M is non-singular) are TU.

Theorem 2.6 (Chervet et al. [22])**.** *A matrix A is TE if and only if, after any sequence of Gaussian pivots and removals of a row and a column, the resulting matrix is such that in each row, all the nonzero entries have the same absolute value.*

Theorem 2.6 extends a well-known characterization of TU matrices: indeed, a matrix is TU if and only if any sequence of Gaussian pivots and removals of a row and a column yields a $0, \pm 1$-matrix.

Finally, in the same work, Chervet et al. also proved that a full row rank matrix is TE if and only if its rows can be partitioned into a TU matrix and a collection of specific TE blocks—each not TU—such that the removal of any row from these blocks yields a TU matrix. One can see that all minors of a TE matrix must be powers of 2 via Gaussian elimination. Thanks to their decomposition theorem, Chervet et al. strengthened this result and obtained the following.

Theorem 2.7 (Chervet et al. [22])**.** *The absolute value of the determinant of a TE $0, \pm 1$-matrix of size n is upper bounded by 2^n.*

Note that the bound given in Theorem 2.7 is tight for some matrices, for instance, for the one given below:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}.$$

One can prove via brute-force computation that this is the smallest TE matrix with determinant equal to 2^n, where n is the number of rows. Currently, the only known TE $0, \pm 1$-matrices having this property have size 4 and 6. This leaves open the following question, first proposed by Chervet et al. [22].

Open Question 2.8. *Which are the TE $0, \pm 1$-matrices of size n whose determinant absolute value is exactly 2^n?*

3 Fundamentals

This section aims to examine the general properties of (box-)TDI systems and box-TDI polyhedra as represented in Figure 3. A particular emphasis is devoted to their fundamental algebraic properties together with their connection with polyhedral integrality. We review basic linear algebraic operations and their impact on (box-)TDIness.

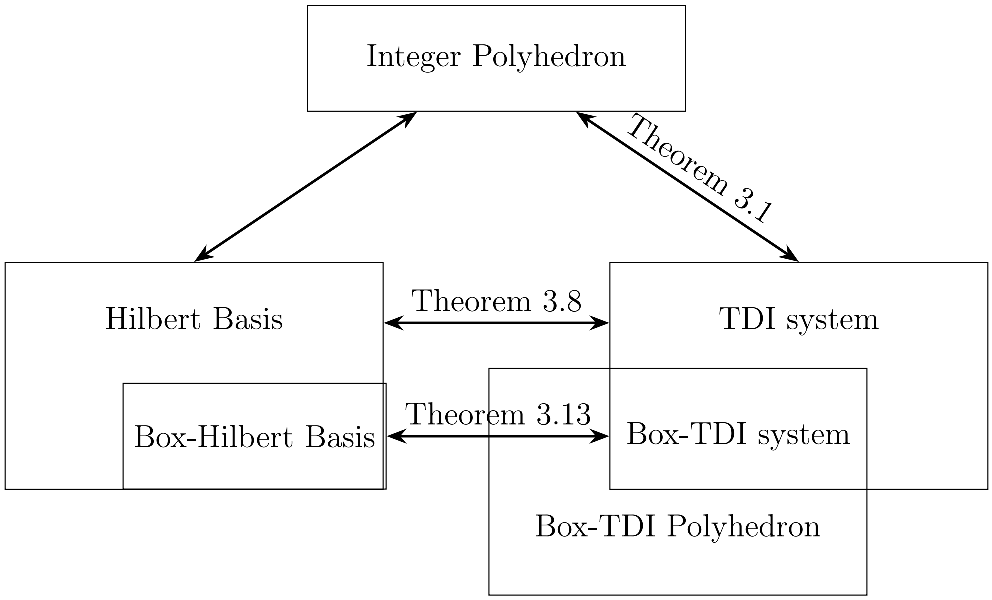

Figure 3: A schematic representation of the results discussed in this section. Each arrow indicates an equivalence between the different objects.

3.1 The Meaning of TDIness

The meaning of the definition of TDI systems, as introduced by Edmonds and Giles, may not be evident at first sight. Their formulation may appear abstract, yet it encodes deep structural properties that have significant implications in combinatorial optimization. In this section, we analyze the rationale behind this definition and explore its main consequences.

Finitness of the Optimum The Duality Theorem [63, Section 0.1] implies that if the primal problem is unbounded, then the dual has no feasible solution. Thus, the optimum of the maximization problem of Equation (1) has to be finite for defining an optimal solution of the dual problem.

Integrality of the Cost Function We want to show the necessity of requiring that the cost vector w has to be integral in Equation (1). Indeed, no system has a dual integer solution for all rational cost vectors. To see this, consider the problem $\max\{w^\top x \colon Ax \le b\}$ whose dual has an integer optimum for a given rational w. Then, there exists a k such that $kA^\top y = kw$ is a system with integer coefficients. If we add $\varepsilon \in (0,1)$ to the right-hand side of the equation, we obtain that an integer combination of y equals a non-integer value, that is, at least one component of y is not integer. Consequently, the dual problem of $\max\{(w + \frac{1}{k}\varepsilon)^\top x \colon Ax \le b\}$ has no integer solution.

Min-max Relations TDIness is strongly related to min-max relations. Many combinatorial min-max relationships stem from the fact that certain linear programs have integer optima (A classical example is the Max-flow Min-cut Theorem [49], that we discuss in Section 7.2). When this happens, we can use the Duality Theorem to deduce a min-max relationship. Indeed, we can always set up the following chain of inequalities:

$$\begin{aligned}\max\{w^\top x : Ax \le b, x \in \mathbb{Z}^n_{\ge 0}\} \le^{(a)} \max\{w^\top x : Ax \le b, x \ge \mathbf{0}\} =^{(b)} \\ = \min\{yb : yA \ge w, y \ge \mathbf{0}\} \le^{(c)} \min\{yb : yA \ge w, y \in \mathbb{Z}^m_{\ge 0}\}.\end{aligned} \tag{2}$$

Inequalities (2) give a bound on the value of the optimal solutions to a combinatorial optimization problem, that is the optimal value of another optimization problem. Equality (b) is a consequence of the Duality Theorem. When the system $Ax \le b$ describes an integer polyhedron, inequality (a) is an equality. Moreover, inequality (c) is an equality whenever $Ax \le b$ is TDI. To conclude, if b is integer and $Ax \le b$ is TDI, all the elements of (2) are equal for all integer w.

Polyhedral Integrality The considerations above prove that a polyhedron is integer whenever it is described by a TDI system with integer right-hand side. Edmonds and Giles also proved the converse.

Theorem 3.1 (Edmonds and Giles [45])**.** *A polyhedron P is integer if and only if there is a TDI system $Ax \le b$ such that $P = \{x\colon Ax \le b\}$ and b is integer.*

It is important to stress that, although every polyhedron (or, equivalently, every linear program) can be described (formulated) using a TDI system, it is not the TDIness alone that guarantees integrality. Instead, integrality is certified only when the corresponding right-hand side is integral. Since recognizing whether a given polyhedron is integer is a well-known co-NP-complete problem [83], TDIness has emerged as a property extensively studied to establish polyhedral integrality. A notable example is the work of Weismantel [99], who provided a linear description of a special case of knapsack polytopes via TDIness.

3.2 On the Existence of TDI systems

So far, we saw why TDI systems are relevant, but how do we deal with them? Is it always possible to find a TDI system describing a polyhedron? And if the answer is affirmative, how can we do that? The following result addresses the fundamental question about the existence of a TDI system.

Theorem 3.2 (Giles and Pulleyblank [55])**.** *Every rational polyhedron can be described by a TDI system $Ax \le b$, with A integer.*

Theorem 3.2 guarantees the existence of a TDI system. However, there are essentially two practical ways to derive a TDI system from a non-TDI one. One approach consists in dividing the existing constraints by some integer, while the other involves adding redundant constraints. Neither method modifies the feasible region, and therefore, they are not to be seen as algorithms for computing the convex hull of a point set.

Division Given a system $Ax \leq b$, we can always obtain a TDI system describing the same set of points simply by dividing the inequalities by an appropriately chosen integer.

Theorem 3.3 (Giles and Pulleyblank [55]). *For each rational system $Ax \leq b$ there exists a natural number k such that $\frac{1}{k}Ax \leq \frac{1}{k}b$ is TDI.*

A valid value for k is the least common multiple among the minors of the constraint matrix. Unfortunately, finding this value is at least as hard as knowing the maximal minor, which is an NP-hard task even in the case of the incidence matrix of a graph [61]. In particular, the case of edge-vertex incidence matrices of graphs is relevant as they are always TE [20], and therefore related to a box-TDIness as we will see later in Section 4.1. A purely theoretical value that one can use to ensure the TDIness of a system is $f(A)!$, where f is any function giving an upper bound of the minors of the constraint matrix A. However, this value is impractical, as it explodes even for $0, \pm 1$-matrices as in the well-known case of Hadamard matrices [64].

Addition of Constraints The second technique that we could use to obtain a TDI system is to add a set of redundant constraints to a non-TDI system.

Theorem 3.4 (Giles and Pulleyblank [55]). *Let $Ax \leq b$ be a non-TDI system, then there exists a TDI system $A'x \leq b'$ obtained by adding redundant constraints to $Ax \leq b$.*

Depending on the objective, the two techniques have different applications. If we are looking for a min-max relation between non-integer objects, if we are looking for the $\frac{1}{k}$-integrality of a polyhedron, or if we want a bound on the gap between two combinatorial values, dividing a system by an integer number can lead to good results. On the other hand, for purely combinatorial applications, like proving the integrality of a polyhedron, or describing a min-max relation between combinatorial objects, we usually look for an integer TDI system. In this case, a valid approach to reach TDIness is to add affine combinations of the existing constraints.

From what we saw in this section, it easily follows that a polyhedron can be generally described by several TDI systems. Schrijver [88] studied the case of *minimal TDI systems*, that is TDI systems such that the removal of any constraint disrupts TDIness. Minimal TDI systems whose constraints matrix is integer are called *Schrijver systems*. In particular, he proved that, for full-dimensional polyhedra, the following result holds.

Theorem 3.5 (Schrijver [88]). *A full-dimensional polyhedron admits a unique Schrijver system describing it.*

3.3 Box-Totally Dual Integral Polyhedra

Unlike TDIness, not every polyhedron admits a box-TDI description. For several years, box-TDIness was regarded merely as a stricter version of TDIness, until Cook [26] established the following.

Theorem 3.6 (Cook [26]). *If a polyhedron can be described by a box-TDI system, then every TDI system describing it is also box-TDI.*

Theorem 3.6 highlights the first fundamental difference between TDIness and box-TDIness: while TDIness is, by all means, a property of systems, box-TDIness is essentially a polyhedral property. This motivates the following definition: a polyhedron is *box-TDI* if it can be described by a box-TDI system. Therefore, Theorem 3.2 cannot be restated by replacing TDIness with box-TDIness. Furthermore, Theorems 3.2 and 3.6 justify the study of box-TDI polyhedra instead of box-TDI systems, as Theorem 3.6 implies that box-TDIness only depends on the polyhedron itself. Remark that as for TDIness, box-TDIness does not imply polyhedral integrality.

Examples of box-TDI polyhedra are all those described by TU and TE matrices (see Section 4 for more details). While the example below presents a non-box-TDI polyhedron.

Example 3.7 (A non-box-TDI polyhedron). *Let us consider the polyhedron* $P = \{x : x_1 - 2x_2 = 0\}$. *Equation* $x_1 - 2x_2 = 0$ *describes* P *and forms a TDI system. However, the polyhedron* $P' = P \cap \{x : 0 \leq x_1 \leq 1\}$ *is not integer since it has* $(1, \frac{1}{2})$ *as a vertex. By Theorem 3.1, the system describing* P' *is not TDI, thus,* P *is not box-TDI by Theorem 3.6.*

Min-max Relations Compared to TDI systems, fewer efforts have been made to explore min-max relations derived from box-TDI systems. A case in which this has been done is the well-known Max-flow Min-cut Theorem (see Section 7.2), where changing the upper bound on the capacities over the arcs does not affect the interpretation of the dual problem.

In general, when we add box-constraints to the primal problem, the new problem arising is a capacitated version of the original problem, on the contrary, the dual problem substantially differs from the original, and we often lack an explicit interpretation. Cornaz et al. [29] proposed an interpretation of the dual of the box-TDI system describing the multicut cone for series-parallel graphs namely the trader multiflow problem.

An analogous economic interpretation can be derived as follows. Consider the linear problem:

$$\max\{w^\top x \colon Ax \leq b, x \geq \mathbf{0}\}, \tag{3}$$

where the linear system $Ax \leq b$ is box-TDI. When we introduce additional box-constraints, the linear programming duality transforms to:

$$\begin{aligned} \max\ & \{w^\top x \colon Ax \leq b, x \geq \mathbf{0}, \ell \leq x \leq u\} = \\ = \min\ & \{b^\top y + u^\top z_u - \ell^\top z_\ell \colon A^\top y + z_u - z_\ell \geq w, y \geq \mathbf{0}, z_u \geq \mathbf{0}, z_\ell \geq \mathbf{0}\}. \end{aligned} \tag{4}$$

By box-TDIness definition, the minimization problem of the linear programming duality equation (4) admits an integer optimum for all rational ℓ and u, with $\mathbf{0} \leq \ell \leq u$, and all integer w such that the maximization problem admits a finite optimum. To understand the nature of the minimization in (4), we compare it to the dual of Problem (3) $\min\{b^\top y \colon A^\top y \geq w, y \geq \mathbf{0}\}$. The variables z_u allow an otherwise non-feasible solution y to become feasible but introduce an additional cost to reach feasibility. The opposite happens for variables z_ℓ: when $z_\ell > 0$, since $\ell \geq 0$, we are improving the value of the objective function, at the cost of tightening some constraints. In this context, the components of the vectors z_u and z_ℓ are the amount of resources that we exchange with a market: when there is a deficit of the resource i, we can compensate by purchasing it at a price u_i. Similarly, if we have a surplus of the resource i, we can sell it at a value ℓ_i. In conclusion, this gives another perspective to the interest in studying box-TDI systems beyond their combinatorial structure, and motivates further investigation into classes of polyhedra for which such interpretations can be derived.

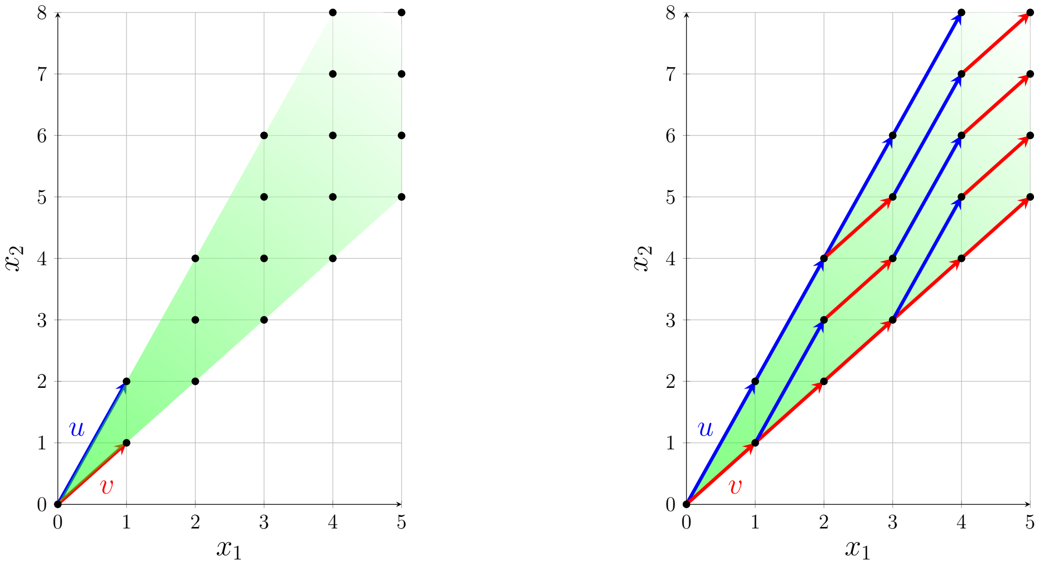

Figure 4: A visual representation: u and v form a Hilbert basis as every integer point in their cone is given by some non-negative integer combination of them.

3.4 Algebraic Properties of (box-)TDI Systems

Several algebraic properties have been established for (box-)TDI systems. We present those that characterize them.

Hilbert Bases TDI systems can be characterized in terms of Hilbert bases. A set $\{v^1, \ldots, v^k\}$ of vectors is a *Hilbert basis* if each integer vector in its conic hull can be expressed as a non-negative integer combination of $v^1, \ldots, v^k$. Figure 4 provides a representation of a Hilbert basis. If, for each face of the polyhedron, the set of row vectors corresponding to the tight constraints forms a Hilbert basis, then any vertex of the dual polyhedron that is an integer combination of the dual coefficients is itself integral.

The following central result was proved by Giles and Pulleyblank [55].

Theorem 3.8 (Giles and Pulleyblank [55])**.** *A system $Ax \leq b$ is TDI if and only if for every face F of $P = \{x : Ax \leq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

The statement of Theorem 3.8 is still valid for TDI systems if we consider only minimal faces, as stated by the following.

Theorem 3.9. *A system $Ax \leq b$ is TDI if and only if for every minimal face F of $P = \{x : Ax \leq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

Theorem 3.9 gives a characterization of Hilbert bases in terms of TDIness, and, conversely, provides a nice result for proving TDIness of systems describing polyhedral cones.

Theorem 3.10. *A system $Ax \leq 0$ is TDI if and only if the rows of A form a Hilbert basis.*

Hilbert bases give an intuition on how Theorem 3.3 and Theorem 3.4 can be used to build a TDI system. We can now show how the two techniques work, thanks to the following example, taken from [57].

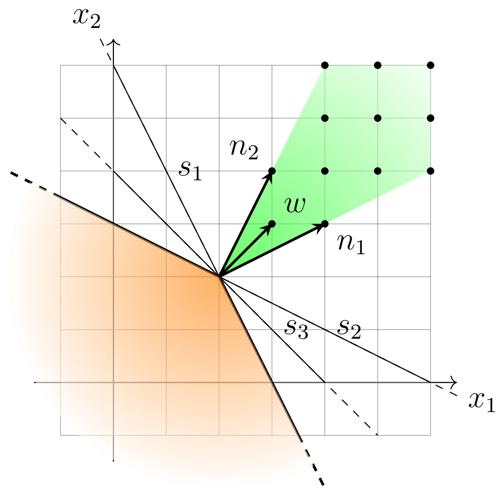

Figure 5: In orange the integer conic polyhedron defined in Example 3.11 and in green the conic polyhedron generated by n_1 and n_2. The straight lines s_1, s_2 and s_3 are defined by $x_1 + 2x_2 = 6$, $2x_1 + x_2 = 6$, and $x_1 + x_2 = 4$ respectively. While $w = [1, 1]^\top$ is the cost vector, and n_1 and n_2 are the normal vectors to s_1 and s_2 pointed at $(2, 2)$.

Example 3.11 (Building a TDI system)**.** *Consider the linear program*

$$\max\left\{x_1 + x_2 \colon \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x \leq \begin{bmatrix} 6 \\ 6 \end{bmatrix}\right\}, \tag{5}$$

whose corresponding polytope is depicted in Figure 5. By considering the dual problem, one may see that

$$\textit{argmin}\left\{6y_1 + 6y_2 \colon \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} y \geq \mathbf{0}\right\} = \left(\frac{1}{3}, \frac{1}{3}\right),$$

hence, the system of Problem (5) *is not TDI. Indeed, the tight constraints at this vertex solution of the maximization problem have normal vectors* $[1, 2]^\top$ *and* $[2, 1]^\top$*, and it is not possible to express the cost vector* $[1, 1]^\top$ *as an integer combination of both of them.*

Now, consider the following problems equivalent to Problem (5)*:*

$$\max\left\{x_1 + x_2 \colon \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\} \textit{ and } \max\left\{x_1 + x_2 \colon \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 6 \\ 6 \\ 4 \end{bmatrix}\right\}.$$

The integer optima of their corresponding dual problems

$$\min\left\{2y_1 + 2y_2 \colon \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, y \geq \mathbf{0}\right\}$$

and

$$\min\left\{6y_1 + 6y_2 + 4y_3 \colon \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, y \geq \mathbf{0}\right\}$$

are the integer points $(1, 1)$ *and* $(0, 0, 1)$*, respectively.*

More generally, let $H_1 = \left\{\frac{1}{3}[2, 1]^\top, \frac{1}{3}[1, 2]^\top\right\}$ *and* $H_2 = \left\{[2, 1]^\top, [1, 2]^\top, [1, 1]^\top\right\}$*, then one can see that* $\textit{cone}\left([2, 1]^\top, [1, 2]^\top\right) \cap \mathbb{Z}^2 = \textit{int.cone}\,(H_1) = \textit{int.cone}\,(H_2)$*. Therefore,* H_1 *and* H_2 *are two Hilbert bases, whose elements correspond to active constraints in* $(2, 2)$*, the only minimal face of the polyhedron. By Theorem 3.9, we deduce that the corresponding systems to* H_1 *and* H_2 *are TDI.*

We conclude the discussion about Hilbert bases by addressing a frequent source of confusion regarding the definition of Hilbert bases. Some authors require a Hilbert basis to be inclusion-wise minimal [18, 92], consistently with the classical notion of basis in a linear space. Other authors define Hilbert basis as a set of vectors R such that int.cone(R) coincides with the intersection of cone(R) with the lattice generated by R [56, 73]. When dealing with TDI systems, the definition we adopt—the one originally introduced by Giles and Pulleyblank [55]—and the latter are not equivalent. Indeed, one can see that, when the elements of R are integer vectors, the intersection of cone(R) with the lattice generated by R is strictly contained in cone$(R) \cap \mathbb{Z}^n$ in general.

Integer Solutions and Local Unimodularity A linear *Diophantine* equation is a linear equation with integer coefficients for which we consider only integer solutions. The existence of an integer optimum of a dual problem is equivalent to the existence of a solution to an associated system of Diophantine equations. The following theorem clarifies this statement.

Theorem 3.12 (Corollary 4.1.c, [90])**.** *Let A be an integer $m \times n$ matrix having full row rank. Then the following are equivalent:*

- *the greatest common divisor of the non-zero minors of A of order m is* 1*;*
- *the system $Ax = b$ has an integer solution x, for each integer vector b;*
- *for each vector y, if $A^\top y$ is integer, then y is integer;*
- *the rows of A are linearly independent and form a Hilbert basis.*

Therefore, one can see that any system with integer constraint matrix respecting the first point of Theorem 3.12 is TDI. However, the condition is too restrictive for the general case: we need only the rows of A corresponding to the inequalities that are tight for each vertex to form a Hilbert basis. This motivates the following definition.

The linear system $Ax \leq b$ describing a polyhedron in $\mathbb{Q}^m$ is *locally unimodular in a vertex p* if the greatest common divisor of the $m \times m$ subdeterminants of A_p is 1, where A_p are the rows of A corresponding to the inequalities that are tight at p. As remarked by Gerards and Sebő [51], if the system $Ax \leq b$ is TDI, then $Ax \leq b$ is locally unimodular in every vertex. Moreover, they strengthen this result as follows. A system $Ax \leq b$ is *locally strongly unimodular* in a vertex p of $P = \{x \colon Ax \leq b\} \subseteq \mathbb{Q}^m$ if A_p has a $m \times m$ submatrix with determinant ± 1. Then, a TDI system describing a full-dimensional pointed polyhedron is locally strongly unimodular at each vertex.

Box-Hilbert Bases In [27], Cook introduced an analogous definition of Hilbert bases for box-TDI systems, to which the first known characterizations of box-TDI systems were associated. A set of integer vectors is a *box-Hilbert basis* if it is a Hilbert basis that is closed under the operation of adding any number of vectors $\pm e_i$, where e_i's are the vectors of the canonical basis of $\mathbb{Q}^n$. Note that this notion came directly from the definitions of box-TDI systems and Hilbert bases. Indeed, this requirement corresponds precisely to preserving TDIness under the addition of a box $x_i \leq u_i$ (or $x_i \geq \ell_i$), for some rational u_i (respectively ℓ_i). In contrast to Hilbert bases, box-Hilbert bases have not been studied as extensively, as evidenced by the limited literature on the subject.

As one may expect, not all Hilbert bases are box-Hilbert bases. For example, $\{[1,2]^\top, [1,3]^\top\}$ is a Hilbert basis but $\{[1,2]^\top, [1,3]^\top, -e_1\}$ is not. A simple example of a box-

Hilbert basis is $\{[\frac{1}{2},\frac{1}{2}]^\top,[\frac{1}{2},-\frac{1}{2}]^\top\}$. The following result is the box-TDIness counterpart of Theorem 3.8.

Theorem 3.13 (Cook [27])**.** *A system $Ax \le b$ is box-TDI if and only if for every face F of $P = \{x : Ax \le b\}$, the rows of A associated with tight constraints for F form a box-Hilbert basis.*

Note that results on minimal faces analogous to Theorems 3.9 and 3.10 do not hold for box-TDI polyhedra (see Section 5 for more details).

3.5 Linear Algebra Operations and (box-)TDIness

As we saw in Example 3.11, TDIness is not necessarily preserved across different systems defining the same polyhedron. Indeed, TDIness and box-TDIness exhibit subtle behaviors under linear algebra operations that are typically considered "safe", as they preserve the feasible region defined by the system. In this section, we describe several operations that either preserve or disrupt TDIness and box-TDIness. Naturally, if an operation disrupts TDIness, it also disrupts system box-TDIness; however, we will explicitly indicate whenever the two properties behave differently. For all unreferenced results, we refer to [90, Section 22.5]. Furthermore, the interested reader is referred to Cook's work [26] for further results.

The multiplication and the division by integers affect TDI systems differently. In fact, multiplying (some of) the constraints defining a TDI system by an integer often disrupts TDIness—even if it does not modify the properties of the corresponding polyhedron. This is because multiplying a constraint by an integer k stretches the corresponding normal vector h, which belongs to a Hilbert basis H associated with the polar cone of a certain face. As a result, it can happen that some points of $\text{int.cone}(H) = \text{cone}(H) \cap \mathbb{Z}^m$ do not belong to $\text{int.cone}(H \setminus \{h\} \cup \{kh\})$. On the contrary, dividing a constraint by k shrinks the corresponding normal vector h. So if $h' = \frac{1}{k}h$, we have that, for every $u \in \text{cone}(H) \cap \mathbb{Z}^m$, $u = \alpha h + \sum_{h_i \in H\setminus\{h\}} \alpha_i h_i = k\alpha h' + \sum_{h_i \in H\setminus\{h\}} \alpha_i h_i$, where all α_i and α are non-negative integers, therefore $H \setminus \{h\} \cup \{h'\}$ is a Hilbert basis. The following elementary example should convince the reader of this fact.

Example 3.14 (Integer Multiplication)**.** *Consider the system $x_1 \le 1, x_2 \le 1$. It is TDI because its constraint matrix is TU. However, if we multiply one of the constraints by 2 and consider the linear programming duality equation*

$$\max\{x_1 + x_2 \colon 2x_1 \le 2, x_2 \le 1\} = \min\{2y_1 + y_2 : 2y_1 = 1, y_2 = 1, y \ge \mathbf{0}\},$$

we see that the unique solution for the minimization problem is $\left(\frac{1}{2}, 1\right)$, which is not integer.

Observation 3.15 (Integer Division)**.** *Let $Ax \le b$ be a (box-)TDI system, and $k \in \mathbb{Z}_{>0}$. The system obtained by dividing by k both sides of any number of its constraints is (box-)TDI.*

Observation 3.16 (Multiplication of the Right-Hand Side)**.** *Let $Ax \le b$ and $\alpha \in \mathbb{Q}_{>0}$. Then $Ax \le b$ is (box-)TDI if and only if $Ax \le \alpha b$ is (box-)TDI.*

In contrast to the previous operations, the one presented in Observation 3.16 modifies the original feasible region, as it corresponds to a dilation of the polyhedron. Nonetheless,

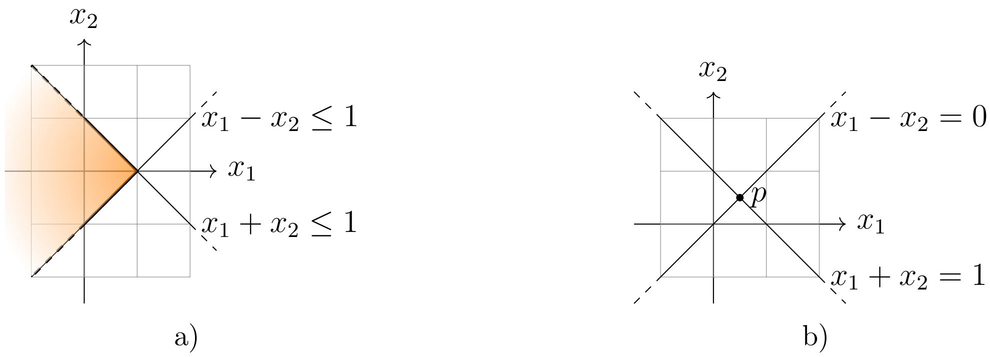

Figure 6: a) The conic polyhedron defined in Example 3.18. b) The projection of $P \cap \{(x,t)\colon t_1 = 0, t_2 = 1\}$ onto x_1 and x_2.

TDIness and box-TDIness are preserved by Theorems 3.8 and 3.13, since the polar cones of the faces of a polyhedron remain invariant under dilation.

TDIness is preserved under the addition of slack variables. However, this behavior does not hold for box-TDI systems and polyhedra. The latter disruption depends on the fact that adding columns of the identity matrix to an equimodular matrix does not preserve equimodularity, hence disrupting box-TDIness by Theorem 4.1.

Observation 3.17 (Adding and Removing Slack Variables)**.** *The system $Ax \leq b$, $ax \leq p$, where a is an integer vector, is TDI if and only if the system $Ax \leq b$, $ax + t = p$, $t \geq 0$, where t is a new variable, is TDI.*

Example 3.18 (Adding Slack Variables Disrupts Box-TDIness)**.** *Consider the following system*

$$\begin{cases} x_1 + x_2 \leq 1, \\ x_1 - x_2 \leq 1, \\ x_1 \leq 1, \end{cases} \tag{6}$$

whose feasible region is represented in Figure 6. This system is box-TDI by Theorem 3.13, since $\{[1,1]^\top, [1,0]^\top, [1,-1]^\top\}$, $\{[1,1]^\top\}$, and $\{[1,-1]^\top\}$ are all box-Hilbert bases.

We now add the slack variables t to System (6). The corresponding polyhedron is $P = \{(x,t)\colon x_1 + x_2 + t_1 = 1, x_1 - x_2 + t_2 = 1, x_1 + t_3 = 1, t \geq \mathbf{0}\}$. If we add the box-constraints $t_1 = 0$ and $t_2 = 1$, the only point of $P \cap \{(x,t)\colon t_1 = 0, t_2 = 1\}$ is $p = (\frac{1}{2}, \frac{1}{2}, 0, 1, \frac{1}{2})$ (see Figure 6). Thus, the system describing $P \cap \{(x,t)\colon t_1 = 0, t_2 = 1\}$ is not TDI by Theorem 3.1, since p is not integer, and hence, the system describing P is not box-TDI.

Moreover, adding the slack variables to System 6 preserves TDIness by Observation 3.17. However, the resulting system is not box-TDI as proved above. Therefore, P is not box-TDI by Theorem 3.6.

Other operations slightly changing the system that preserve (box-)TDIness are the following.

Observation 3.19 (Column Duplication)**.** *Let $Ax \leq b$ be a system, and let α be a column of A. Then the system $Ax + \alpha y \leq b$, where $y \in \mathbb{R}$ is a new variable, is TDI if and only if $Ax \leq b$ is.*

Chervet et al. [21] proved that all box-integer polyhedra can be represented by box-integer polyhedra with only vertices with non-negative coordinates. This result can be easily extended to box-TDI polyhedra.

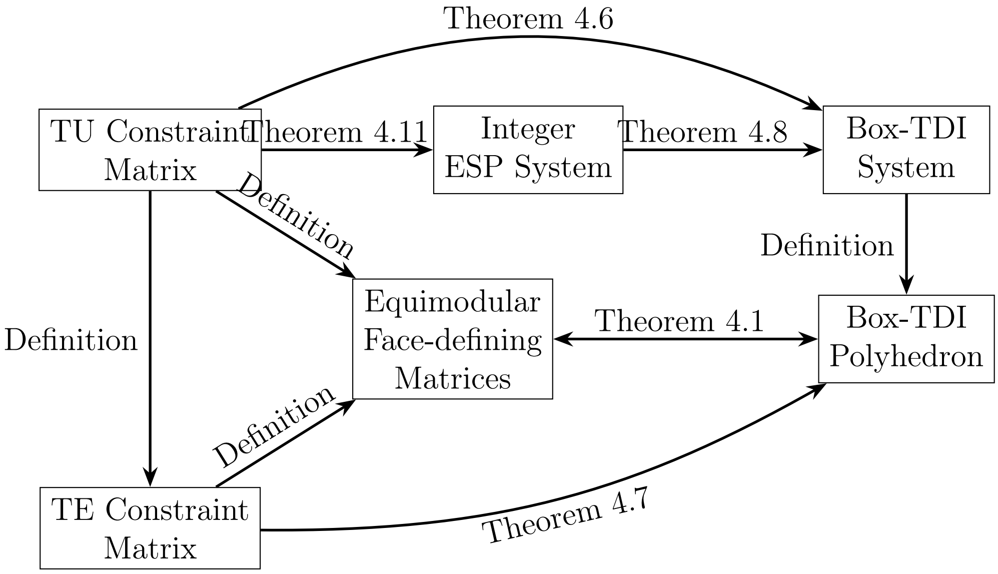

Figure 7: A schematic representation of the results connecting box-TDIness and matrices. Each arrow indicates an implication between the different objects.

Observation 3.20 (Splitting Variables)**.** *Let $P = \{x : Ax \leq b\}$ be a box-TDI polyhedron. The polyhedron obtained by replacing any unbounded variable x_i with $x_i^+ - x_i^-$, with $x_i^+, x_i^- \geq 0$ is box-TDI.*

On a different direction, Cook [26, 27] proved that the Fourier-Motzkin elimination procedure preserves (box-)TDIness for systems that have $0, \pm 1$ coefficients.

Theorem 3.21 (Fourier-Motzkin Elimination)**.** *Let $Ax \leq b$ be a (box-)TDI system. If each coefficient of the variable x_i is in $\{-1, 0, +1\}$, then the system obtained by eliminating x_i via Fourier-Motzkin is also (box-)TDI.*

We conclude this section with the following.

Observation 3.22. *Let $Ax \leq b$ be a (box-)TDI system. Then, the system of $A(x - t) \leq b$ is (box-)TDI for any rational vector t. In particular, rational translations preserve box-TDIness of polyhedra.*

Proof. Let F be a face of the polyhedron described by the (box-)TDI system $Ax \leq b$, and F' be the face corresponding to F of the polyhedron described by $A(x - t) \leq b$. Then, the tight rows of A to F are the same as those of F'. So, they form a (box-)Hilbert basis, and hence the system is (box-)TDI by Theorem 3.8 (Theorem 3.13). □

4 Systems and Matrices

Establishing that a system is TDI is one way to certify the integrality of a given polyhedron. Many proofs of integrality and (box-)TDIness rely on the total unimodularity of the defining matrix. This is particularly true for box-TDI systems, largely because no matricial characterization of box-TDI polyhedra was available until recently. This section focuses on matricial properties of box-TDIness (see Figure 7).

4.1 Matricial Characterization

We open this section by introducing a characterization of box-TDI polyhedra in terms of matrices. This result, due to Chervet et al. [21], is considered both an operative tool for box-TDIness proofs and a generalization of previously known results on the topic.

Theorem 4.1 (Chervet et al. [21])**.** *For a polyhedron P, the following statements are equivalent:*

- *P is box-TDI;*
- *every face-defining matrix of P is equimodular;*
- *every face of P can be described by an equimodular matrix;*
- *every face of P can be described by a TU matrix;*
- *the linear space of every face of P is generated by the columns of a TU matrix.*

Note that the fourth statement of Theorem 4.1 does not imply that every face-defining matrix of a box-TDI polyhedron is TU. In fact, the fourth statement is a direct consequence of Theorem 2.4, which involves a linear transformation applied to the columns of a face-defining matrix. For example, consider the vertex $(1, 0)$ of the cone described in Example 3.18. A face-defining matrix for this vertex is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. By Theorem 2.4, follows that $I_2 x = A^{-1}\mathbf{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where I_2 is the identity matrix of size 2, describes $(1, 0)$. However, the corresponding polyhedral cone $\{x : I_2 x \leq \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$ does not contain the original set of points, and therefore does not allow a valid formulation for the original problem.

It remains unclear whether box-TDI polyhedra always allow a description for which the face-defining matrix of each of their faces is TU. We will return to this point later in this section, in light of a classical result by Schrijver.

Theorem 4.1 leads to another drastic differentiation between TDIness and box-TDIness. Specifically, Theorem 4.1 states that a polyhedron is P box-TDI if and only if every face of P can be described by a minimal system whose corresponding matrix is equimodular, in contrast with one of the main techniques used to obtain TDIness, which is adding redundant constraints. Moreover, Theorem 4.1 provides a matrix-based proof that every face of a box-TDI polyhedron is also box-TDI, as previously observed by Edmonds and Giles.

Observation 4.2 (Edmonds and Giles [45])**.** *Every face of a box-TDI polyhedron is box-TDI.*

Another straightforward consequence of Theorem 4.1, a full-dimensional box-TDI polyhedron can be described by a $0, \pm 1$-matrix, since each facet is described by a unique inequality, up to scalar multiplication. Nonetheless, even this fact was already known.

Theorem 4.3 (Edmonds and Giles [45])**.** *Any box-TDI polyhedron can be described by a $0, \pm 1$-matrix.*

Surprisingly, box-TDI systems can have coefficients that are not in $\{-1, 0, +1\}$, as we show in the following example originally appearing in [90, Section 22.5]. A key remark in this context is that, while all descriptions of a box-TDI polyhedron adhere to Theorem 4.1, this characterization provides no information about their TDIness.

Example 4.4 (A Minimal Box-TDI System with non-Unitary Coefficients). *Consider the following matrices:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \quad \text{and } A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

We want to show that the system $Ax \leq \mathbf{0}$ is box-TDI, while $A'x \leq \mathbf{0}$ is not. First, note that the last row of A corresponds to a redundant inequality, that is, the two systems above describe the same cone. Since A' is a TE matrix, by definition of TE matrix and Theorem 4.1, this cone is box-TDI. Secondarily, the vector $[2, 1, 1, 1]^\top$ belongs to the cone generated by the rows of A', and it cannot be obtained by a non-negative integer combination of them. That is, the rows of A' do not form a Hilbert basis. However, since the least common multiple of the minors of A' is 2, the system $\frac{1}{2}A'x \leq \mathbf{0}$ is TDI as seen in Section 3.2. Since $[2, 1, 1, 1]^\top$ is the only integer vector that one can obtain by non-negative combination with coefficients at most $\frac{1}{2}$ of the rows of A', this is sufficient to prove that the rows of A form a Hilbert basis.

In [90, Section 22.5], Schrijver also notes that no TU matrix can describe the cone given in Example 4.4, implying (as was implicit until now) that not every box-TDI polyhedron can be described by a TU matrix.

4.2 Sufficient Conditions for Box-TDIness

Different sufficient conditions for box-TDIness have been given over the years. For a long time, most of the systems were proved to be box-TDI by showing that the corresponding constraint matrix was TU. In the following section, we present some fundamental results connecting TU matrices with box-TDI systems. We then provide a more recent result on box-TDI polyhedra and TE matrices. Lastly, we present a relatively peculiar approach first proposed by Ding and Zang [40], the ESP property.

4.2.1 Total Unimodularity

The characterization provided in Theorem 3.13 is not the only tool we can use to prove the box-TDIness of a linear system. In fact, the following result of Schrijver provides suitable sufficient conditions for proving it.

Theorem 4.5 (Schrijver [91], Theorem 5.35). *Let $Ax \leq b$ be a linear system. Suppose that for any vector c, $\max\{c^\top x \colon Ax \leq b\}$ has (if finite) an optimal dual solution $y \geq \mathbf{0}$ such that the rows of A corresponding to positive components of y form a TU submatrix of A. Then, $Ax \leq b$ is box-TDI.*

The hypotheses of Theorem 4.5 are too restrictive to characterize box-TDIness, as one can see in Example 4.4. From a historical point of view, Theorem 4.5, is a consequence Theorem 3.13 and the following classical result of Hoffman and Kruskal.

Theorem 4.6 (Hoffman and Kruskal [69]). *An integer matrix A is TU if and only if the system $Ax \leq b$ is box-TDI for every rational b.*

The TDIness guaranteed by Theorem 4.6 stems from Theorem 3.8. Indeed, the volume spanned by any subset of rows of a TU matrix is either 0 or 1. Thus, all integer points in the cone generated by these rows are achieved by the integer sum of them.

As TU matrices are well-characterized and recognizable in polynomial time, Theorem 4.6 is one of the principal instruments used in the literature for proving (box-)TDIness of systems.

4.2.2 Total Equimodularity

Despite the fundamental theoretical tool provided by Theorem 4.1, testing the equimodularity of every face-defining matrix may be impractical. Indeed, building on this result, Chervet et al. [20] proved that recognizing whether a given polyhedron is box-TDI is co-NP-complete. This further motivates the study of TE matrices. By Theorem 4.1, every polyhedron whose constraint matrix is TE is box-TDI. It turns out that this characterizes TE matrices.

Theorem 4.7 (Chervet et al. [21])**.** *A matrix A of $\mathbb{Q}^{m\times n}$ is TE if and only if the polyhedron $\{x\colon Ax \le b\}$ is box-TDI for all $b \in \mathbb{Q}^m$.*

Theorem 4.7 shows a parallelism to Theorem 4.6, indeed, TE matrices are to box-TDI polyhedra what TU matrices are to box-TDI systems.

4.2.3 Equitable Subpartitionability

In [40], Ding and Zang characterize the graphs with the min–max relation on packing and covering cycles by defining a property for $0,1$-matrices. Later on, this property has been extended for the case of box-Mengerian hypergraphs in [16]. To suit our purposes, we present the matricial version of this property as given in [39], which generalizes the Ghouila-Houri criterion (Theorem 2.3).

Consider a rational linear system $Ax \le b$, $x \ge \mathbf{0}$, with A of size $n \times m$. Let $[n] = \{1,\dots,n\}$ and $[m] = \{1,\dots,m\}$. For any family Λ (with possibly repeated elements) of elements of $[n]$ and for any element c of $[m]$, define $b(\Lambda) := \sum_{r\in\Lambda} b_r$ and $d_\Lambda(c) := \sum_{r\in\Lambda} A_{rc}$. An *equitable subpartition* of Λ is a couple of families Λ_1 and Λ_2 of elements of $[n]$ such that:

(E1) $b(\Lambda_1) + b(\Lambda_2) \le b(\Lambda)$;

(E2) $d_{\Lambda_1}(c) + d_{\Lambda_2}(c) \ge d_\Lambda(c)$ for all $c \in [m]$;

(E3) $\min\{d_{\Lambda_1}(c), d_{\Lambda_2}(c)\} \ge \lfloor d_\Lambda(c)/2 \rfloor$ for all $c \in [m]$.

The system $Ax \le b$, $x \ge 0$ is *equitably subpartitionable (ESP)* if every family Λ of elements of $[n]$ admits an equitable subpartition.

Theorem 4.8 (Ding et al. [39])**.** *Every ESP system $Ax \le b$, $x \ge \mathbf{0}$, with A integer, is box-TDI.*

In the same paper, the authors state that Theorem 4.8 can be extended by testing the ESP property of every subset of inequalities tight for a face, in analogy with Theorem 4.5.

Interestingly, even though the nature of box-TDI systems relies on optimization problems, the ESP property provides a purely combinatorial tool for proving box-TDIness.

Example 4.9 (Non-Integer ESP Systems are not Box-TDI)**.** *Consider the following system:*

$$\begin{cases} x_1 - \frac{1}{2}x_2 \leq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

By Theorem 4.1, this system describes a cone that is not box-TDI, therefore, it is not a box-TDI system. A simple computation shows that it is ESP with $\Lambda_1 = \{1\}$ *and* $\Lambda_2 = \emptyset$*.*

The ESP property has resulted in characterizing the box-TDIness of some classes of polyhedra [39]. On the other hand, there is no known counterexample showing that the ESP property does not characterize box-TDIness in general. Thus, we leave the following open question concerning ESP systems.

Open Question 4.10. *Can every box-TDI polyhedron be described by an ESP system?*

We answer this question for a few specific cases. Namely, we prove that the answer is affirmative for systems associated with a TU matrix and for box-TDI affine spaces.

Theorem 4.11. *Let* A *be a totally unimodular matrix, then the system* $Ax \leq b$*,* $x \geq \mathbf{0}$ *is ESP.*

Proof. We first observe that properties (E1) and (E2) hold whenever we take Λ_1 and Λ_2 that partition the elements of Λ. Moreover, taking Λ as a family with some elements taken multiple times is no different than considering a system with some rows repeated. As repeating rows of a TU matrix preserves the total unimodularity, we prove that for every set Λ, there exists a partition of Λ respecting property (E3).

Since A is TU, by Theorem 2.3, for any set of rows Λ there exist Λ_+ and Λ_- that partition Λ such that:

$$\sigma(c) = d_{\Lambda_+}(c) - d_{\Lambda_-}(c) \in \{-1, 0, 1\}, \quad \text{for all } c \in [m].$$

As Λ_+ and Λ_- partition Λ properties (E1) and (E2) hold.

Property (E3) holds for the columns c such that $\sigma(c) = 0$: for these columns $d_{\Lambda_+}(c) = d_{\Lambda_-}(c) = d_\Lambda(c)/2$. On the other hand, when $\sigma(c) = 1$, we have that $d_{\Lambda_-}(c) = d_{\Lambda_+}(c) - 1$, that implies $d_{\Lambda_-}(c) = (d_\Lambda(c) - 1)/2$. The case where $\sigma(c) = -1$ is the same if we swap the roles of Λ_+ and Λ_-. □

Corollary 4.12. *Every box-TDI polyhedron* $P = \{x\colon Ax = b, x \geq \mathbf{0}\}$ *admits an ESP system describing it.*

Proof. Suppose that P is non-empty and that A has full row rank. By Theorem 4.1, A is an equimodular matrix. Hence, by Theorem 2.4, there exists a TU matrix A' such that $P = \{x\colon A'x = b', x \geq \mathbf{0}\}$. Thus, the system $A'x = b', x \geq \mathbf{0}$ is ESP by Theorem 4.11 and the statement follows. □

5 Polyhedra and Efficiency

In this section, we explore the geometric aspects of box-TDI polyhedra. In particular, we explore the role played by tangent and polar cones in box-TDI characterizations. Moreover, we discuss the diameter of box-TDI polyhedra, as it is known to be relevant in optimization.

As a consequence of Observation 3.16, one can see that the dilation of a box-TDI polyhedron is still box-TDI. Thus, whenever the dilation of a box-TDI polyhedron is integer, integrality is preserved under intersection with integer boxes thanks to Theorems 3.1 and 3.6. Chervet et al. [21] provided a purely geometrical characterization of box-TDI polyhedra in these terms.

Theorem 5.1 (Chervet et al. [21])**.** *A polyhedron is box-TDI if and only if it is principally box-integer.*

Corollary 5.2 (Chervet et al. [21])**.** *An integer polyhedron is box-TDI if and only if it is fully box-integer.*

5.1 Cones

Let $C \subseteq \mathbb{Q}^n$ be a cone. Then, C has the *box-property* if for any vector r in C and any partition $\{U, L\}$ of $\{1, \ldots, n\}$, there exists an integer vector r' in C such that: $r'_i = r_i$ for any i for which r_i is integer, $\lfloor r_i \rfloor \leq r'_i$ for any $i \in L$, and $r'_i \leq \lceil r_i \rceil$ for any $i \in U$. Schrijver [90, Section 22.4] used an equivalent formulation: a cone C has the box-property if, for any vector r in C, C contains a vertex of the integer box $\{x : \lfloor r \rfloor \leq x \leq \lceil r \rceil\}$.

Theorem 5.3 (Schrijver [90], Theorem 22.9)**.** *A polyhedron P is box-TDI if and only if the polar cone of each face of P has the box-property.*

This result can be equivalently restated in terms of box-Hilbert bases. Indeed, any polyhedron P can be described by a TDI system by Theorem 3.2. Hence, for each face F of P, there exists a Hilbert basis generating the polar cone of F. Thus, P is box-TDI if and only if every such Hilbert basis is indeed a box-Hilbert basis. This gives the following.

Theorem 5.4 (Cook [27])**.** *Let H be a Hilbert basis and let $C = cone(H)$. Then H is a box-Hilbert basis if and only if C has the box-property.*

Theorems 5.3 and 5.4 imply that verifying box-TDIness is a co-NP problem. In fact, one can check in polynomial time if a given cone has the box-property [27]. Several other results involving polyhedral operations like projections and dominants stem from this result.

Since cones are invariant under dilations, the following stronger characterization holds.

Corollary 5.5 (Chervet et al. [21])**.** *A cone is box-TDI if and only if it is box-integer.*

Moreover, the generators of box-integer cones are $0, \pm 1$-vectors.

Observation 5.6 (Schrijver [90] Remark 22.2)**.** *Every box-TDI cone can be generated by $0, \pm 1$-vectors.*

It is important to remark that the converse does not hold, as shown by the following example due to Murota and Tamura [79]:

Example 5.7 (A non-box-TDI $0, \pm 1$-cone)**.** *Consider the following cone:*

$$C = cone \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

One can see that intersecting C with $\{x \colon x_1 = x_2 = x_3 = 1\}$ gives the point $(1, 1, 1, \frac{3}{2})$ which is not integer. Therefore, C is not box-integer.

The intersection of all minimal tangent cones of a polyhedron gives the polyhedron itself. Thus, Theorem 5.1 gives the following.

Theorem 5.8 (Chervet et al. [21]). *A polyhedron is box-TDI if and only if every minimal tangent cone is box-TDI.*

We now can show that we cannot replace TDI with box-TDI in Theorem 3.9. In particular, it is not sufficient to check if the rows associated with tight constraints for every minimal face form a box-Hilbert basis to prove box-TDIness of a polyhedron. For example, consider the cone C described by $2x_1 + x_2 \leq 0$ and $x_1 + x_2 \leq 0$, and let D be the cone in Figure 4. One can see that C and D are not box-TDI since they are not box-integer, by Theorem 5.5. The unique minimal face of C is $(0, 0)$, whose tight constraints are $2x_1 + x_2 = 0$ and $x_1 + x_2 = 0$. The vectors of these two rows are respectively u and v generating D, and $\{u, v\}$ is a box-Hilbert basis.

5.2 Polyhedral Operations and box-TDIness

This segment is dedicated to polyhedral operations and their interactions with box-TDIness. We start with a trivial observation that follows from the definitions.

Observation 5.9 (Box Intersection). *Box-TDIness is preserved under intersection with rational boxes. Box-integrality is preserved under intersection with integer boxes.*

Minkowski Sum The *Minkowski sum* of two polyhedra P and Q is the polyhedron $P + Q := \{x : x = y + z, y \in P, z \in Q\}$. The *dominant* of a polyhedron P, denoted by $\text{dom}(P)$, is the polyhedron $\{y : y \geq x, x \in P\}$. By definition, $\text{dom}(P)$ is the Minkowski sum of P with the cone $\{x : x \geq \mathbf{0}\}$. Similarly, the *submissive* of a polyhedron, denoted $\text{sub}(P)$, is the set of points obtained by the Minkowski sum of P with $\{x : x \leq \mathbf{0}\}$ (see Figure 8).

In general, the Minkowski sum of two box-TDI polyhedra is not box-TDI, as shown by the cone in Example 5.7, that is the Minkowski sum of the box-TDI cones generated by each vector. Indeed, the Minkowski sum can disrupt box-TDIness by generating a new face that is not box-TDI. At the same time, the Minkowski sum of two non-box-TDI polyhedra can be a box-TDI polyhedron. For example, $\mathbb{Q}^2_{\geq 0} = \{x : x_1 - 2x_2 \geq 0, x \geq \mathbf{0}\} + \{x : x_1 - 2x_2 \leq 0, x \geq \mathbf{0}\}$ is box-TDI, while the two addends of the Minkowski sum are not.

The Minkowski sum of a box-TDI polyhedron and an orthant gives a box-TDI polyhedron. In particular, we have the following result by Cook [27].

Theorem 5.10 (Cook [27]). *The dominant and the submissive of a box-TDI polyhedron are box-TDI polyhedra.*

This result leads to the following.

Observation 5.11 (Boxes of Dominants). *Let P be a polytope and ℓ and u such that $P \subseteq \{x : \ell \leq x \leq u\}$. Then, dom($P$) (resp. sub($P$)) is box-TDI if and only if $\{x : \ell \leq x \leq u\} \cap$ dom(P) (resp. $\{x : \ell \leq x \leq u\} \cap$ sub(P)) is box-TDI.*

Proof. Since the non-empty intersection of two boxes is a box, if $\text{dom}(P)$ is box-TDI, then, of course, also $\{x : \ell \leq x \leq u\} \cap \text{dom}(P)$ is. Conversely, if $\{x : \ell \leq x \leq u\} \cap \text{dom}(P)$ is box-TDI, then $\text{dom}(\{x : \ell \leq x \leq u\} \cap \text{dom}(P)) = \text{dom}(P)$. By Theorem 5.10, $\text{dom}(P)$ is box-TDI. □

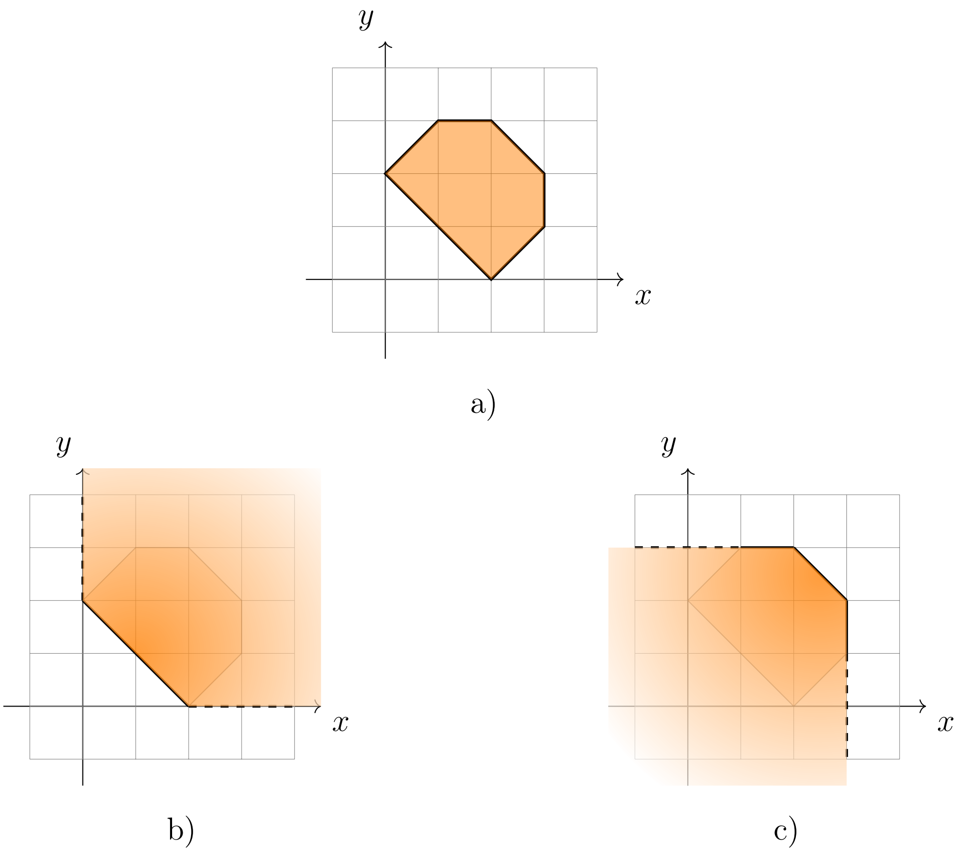

Figure 8: a) An example of a (box-TDI) polytope in $\mathbb{R}^2$; b) The dominant of this polypote; c) The submissive of this polytope

Projections Theorem 3.21 implicitly leverages the following result.

Theorem 5.12 (Cook [27]). *The projection onto a subset of variables of a box-TDI polyhedron is box-TDI.*

Characteristic Cones For a polyhedron P, the *characteristic cone* (often called *recession cone*) is the set char.cone$(P) = \{y \colon x + y \in P,\ \text{for all } x \in P\}$. Equivalently, if $P = \{x \colon Ax \leq b\}$, then char.cone$(P) = \{y \colon Ay \leq \mathbf{0}\}$. In [79], Murota and Tamura give the following interesting results on characteristic cones of box-integer polyhedra.

Theorem 5.13 (Murota and Takamura [79]). *The characteristic cone of a box-integer polyhedron is box-integer, and can be generated by* $0, \pm 1$*-vectors.*

Theorem 5.14 (Murota and Takamura [79]). *Every box-integer polyhedron can be represented as the Minkowski sum of a box-integer polytope and a box-integer cone.*

Grappe [59] independently extended Theorem 5.13 to box-TDI polyhedra.

Theorem 5.15 (Grappe [59], Consequence 3.20). *The characteristic cone of a box-TDI polyhedron is box-TDI.*

As a direct consequence of Observation 5.6 and Theorem 5.15, we have the following.

Corollary 5.16. *The characteristic cone of a box-TDI polyhedron is generated by* $0, \pm 1$*-vectors*

As a consequence, we can give an equivalent of Theorem 5.14 for box-TDI polyhedra.

Corollary 5.17. *Every box-TDI polyhedron can be represented as the Minkowski sum of a box-TDI polytope and a box-TDI cone.*

Proof. Let $P \subseteq \mathbb{R}^n$ be a box-TDI polyhedron. Let k be an integer such that all vertices of P are contained in the hypercube $[-k, k]^n$. Then, by definition of char.cone(P) and Minkowski sum, $P = \text{char.cone}(P) + (P \cap [-k, k]^n)$. The polytope $P \cap [-k, k]^n$ is box-TDI by Observation 5.9, and char.cone(P) is box-TDI by Theorem 5.15, therefore P is the Minkowski sum of a box-TDI polytope and a box-TDI cone. □

Polar Cones For cones, box-TDIness is preserved under cone polarity.

Theorem 5.18 (Chervet et al. [21]). *A cone C is box-TDI if and only if C^* is.*

By Theorems 5.3, 5.8, and 5.18 we have the following.

Corollary 5.19. *For a polyhedron P the following statements are equivalent:*

- *P is box-TDI;*
- *the polar cone of each face of P is box-TDI;*
- *the polar cone of each face of P has the box-property.*

5.3 Efficiency and Diameters

The performance of the simplex algorithm is lower bounded by the polyhedral diameter, that is the maximum, over all pairs of vertices, of the shortest edge-path between them. Nöbel and Steiner [80] recently showed that computing this diameter is NP-hard even when the polytope arises from a TU matrix.

We present several results concerning diameters and introduce different natural questions linking box-TDI polyhedra and diameters, in particular for the case of polyhedra described by TE matrices. Indeed, several known upper bounds on diameters depend on the subdeterminants of the matrix used to describe a polyhedron.

Let $P = \{x \colon Ax \leq b\} \subseteq \mathbb{Q}^m_{\geq 0}$, where A is a $m \times n$ TE $0, \pm 1$-matrix, and Δ is the maximum absolute value of the minors of A. Note that $\Delta \leq 2^s$, where $s = \min\{n, m\}$, by Theorem 2.7. Bonifas et al. [10] results imply that the diameter of P is bounded by $O(2^{2s} n^4 \log 2^s n)$, that is an exponential bound on s. In particular, they prove that the diameter of polyhedra described by TU matrices is bounded by $O(n^4 \log n)$.

Suppose that P is full-dimensional and define $b_0 = \max\{b_i\}$. In [36], Deza and Pournin proved that the diameter of an n-dimensional polytope with vertices of integer coordinates ranging between 0 and k is bounded by $kn - \lceil \frac{2}{3} n \rceil$. Since translations and dilations preserve the diameter of a polytope, this result can be adapted to find a bound for any polytope. The vertices of the dilation ΔP are component-wise bounded by Δb_0. Thus, $n \Delta b_0 - \lceil \frac{2}{3} n \rceil$ is an upper bound for the diameter of ΔP, and consequently for P. Hence, Deza and Pournin's result provides another exponential bound on the diameter of P with respect to s.

Following the TU case, it is legitimate to ask whether the diameter of any box-TDI polyhedron defined by a TE matrix is polynomially bounded. We therefore leave the following question open.

Open Question 5.20. *Is the diameter of a polyhedron defined by TE matrix polynomially bounded?*

A notable case is given by the edge relaxation of the stable set polytope (see Section 7.5 for more details). Chervet et al. [21], proved that the edge relaxation of the stable set polytope is box-TDI by showing that the edge-vertex incidence matrix of a graph is always TE. Michini and Sassano [77] show that the diameter of the edge relaxation of the stable set polytope of a graph is at most the number of its vertices. Subsequently, they show that the well-known Hirsch conjecture[1] holds for this family of polytopes.

A last interesting result in this topic is the one of Topkis [94], stating that the diameter of polymatroids—that are a class of box-TDI polyhedra [45]—of dimension n is bounded by $\min\{2n, \frac{n}{2}(n-1) + 1\}$.

6 Complexity results

The recognition of TDI systems is a topic that interests many academics. Let $Ax \leq b$ be a rational system with A integer, how complex is to determine whether it is TDI or box-TDI?

Generally, the problems "Does the linear system $Ax \leq b$ describe an integer polyhedron?", "Is the linear system $Ax \leq b$ TDI?", and "Is the linear system $Ax \leq b$ box-TDI?"

[1]We recall that the Hirsch conjecture [100, Section 3.3] has been disproved by Santos [87]. However, the validity of the Hirsch conjecture is still an object of study for several classes of polytopes.

belong to co-NP [90, Section 22.9]. Cook [27] showed that the problem “Does the linear system $Ax \leq b$ describe a box-TDI polyhedron?" is in co-NP too. Papadimitriou and Yannakakis [83] proved that deciding whether a given system describes an integer polyhedron is a co-NP-complete problem. The recognition of TDI and box-TDI systems has been proved to be co-NP-complete by Ding et al. [38]. Similarly, Chervet et al. [20] proved that deciding whether a given linear system describes a box-TDI polyhedron is co-NP-complete. In particular, both works focus on the dominants of a class of polytopes contained in the hypercube (see Section 7.4 for more details). Thus, by Observation 5.11, their complexity results on box-TDIness can also be extended to polytopes. Pap [82] proved that the decision problem is co-NP-complete for TDIness even under the assumption that the system has only binary coefficients and that the defined polyhedron is a cone.

If we assume the rank of A as fixed, Cook et al. [28] proved that we can decide whether $Ax \leq b$ is TDI in polynomial time; their work extended previous results of Chandrasekaran and Shirali [13]. Starting from a characterization of Hilbert bases of Sebő [92], Dueck et al. [42] proved that can be decided in polynomial time whether the linear system $Ax \leq b$ is TDI, if we assume that the codimension of the described polyhedron is fixed.

By Theorem 3.10, we deduce equivalent results for Hilbert bases. Complementary results on Hilbert bases can be found in [66–68]. Chervet al. [21] showed that the following problems belong to co-NP.

Open Question 6.1. *Given a matrix A, what is the complexity of deciding if the system $Ax \leq \mathbf{0}$ is box-TDI?*

Open Question 6.2. *Given a matrix A, what is the complexity of deciding if the cone $C = \{x \colon Ax \leq \mathbf{0}\}$ is box-TDI?*

Recognizing equimodularity can be done in polynomial time by Theorem 2.4. However, it is not clear if TE matrices can be recognized in polynomial time. Currently, Chervet et al. [21] proved that the recognition of TE matrix is co-NP, leaving open the following question.

Open Question 6.3. *Given a matrix A, what is the complexity of deciding if A is TE?*

The recent decomposition theorem on full row rank TE matrices, due to Chervet et al. [22], is not sufficient to answer the previous question. The authors also state that it is challenging to understand how to extend their decomposition theorem to non-full row rank matrices.

Chervet et al. [20] proved that integer programming over box-TDI polyhedra is NP-hard as a consequence of the fact that the edge-vertex incidence matrix of a graph is always TE. Specifically, they proved that the edge-relaxation of the stable set polytope is box-TDI for every graph; therefore, finding an integer optimal solution remains NP-hard even in the case there exists a box-TDI relaxation whose integral vertices encode the feasible solutions of a combinatorial problem. Finally, Nöbel and Steiner [80] proved that it is NP-hard to find the diameter of the perfect matching polytope of certain bipartite graphs. This implies that finding the diameter of a polyhedron described by a TU matrix is NP-hard.

In Table 1, we summarize all known complexity results related to the main objects introduced so far.

Problem	Class	Reference
Does the system $Ax \le b$ describe an integer polyhedron?	co-NP-complete	[83]
Is $Ax \le b$ TDI?	co-NP-complete	[38]
Is the system $Ax \le \mathbf{0}$ TDI?	co-NP-complete	[82]
Is the system $Ax \le b$ box-TDI?	co-NP-complete	[38]
Is the system $Ax \le b$ describing a polytope box-TDI?	co-NP-complete	Observation 5.11
Is $\{x\colon Ax \le b\}$ a box-TDI polyhedron?	co-NP-complete	[20]
Is $\{x\colon Ax \le b\}$ a box-TDI polytope?	co-NP-complete	Observation 5.11
Is $\{x : Ax \le \mathbf{0}\}$ a box-TDI cone?	co-NP	[21]
Is A an equimodular matrix?	P	[21]
Is A a TU matrix?	P	[93]
Is A a TE matrix?	co-NP	[21]
Integer programming on box-TDI polyhedra.	NP-hard	[20]
Finding the diameter of $P = \{x\colon Ax \le b\}$, with A TU.	NP-hard	[80]

Table 1: Hardness of some fundamental problems related to (box-)TDIness.

Here we give two open questions that are related to each other. Indeed, on one hand, if the answer to the first question is “there exists a polynomial-time algorithm whose output is an integer TDI system describing a given polyhedron” so it would be for the second. On the other hand, if the answer to the last one is “co-NP-complete", then the answer to the first would be “NP-hard".

Open Question 6.4. *Given a box-TDI polyhedron, what is the complexity of finding an integer TDI system describing it?*

Open Question 6.5. *Given a box-TDI polyhedron P, what is the complexity of deciding whether P is integer?*

7 On the Box-TDIness of Classical Packing and Covering Problems

This section examines various classes of box-TDI systems and polyhedra, highlighting both classic and recent results. Many examples were already presented in Schrijver’s work [89]. We present these results as well as subsequent developments and generalizations from the literature. The notations and common vocabulary used here follow [91].

The vast majority of classical results on box-TDIness are related to TU matrices. These matrices were often used to prove the integrality of the polyhedron described by a linear system. Therefore, the list of box-TDI polyhedra associated with these matrices cannot be exhaustive as the first known studies of TU matrices date back to Poincaré’s work on $0, \pm 1$-matrices [85]. Some notable examples of TU matrices include the incidence matrix of bipartite graphs [69], incidence matrix of directed graphs [33,69], network matrices [97], adjacency matrices of bipartite graphs with no Eulerian tour of length congruent 2 mod 4 [81], and edge coloring matrices of trees [75].

7.1 Polymatroids

The content of this section is largely derived from Chapters 44-46 of Schrijver’s book [91].

Let X be a finite set, a set function f defined on the parts of X is *submodular* if for all $S, T \subseteq X$ we have:

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

Similarly, f is *supermodular* if for all $S, T \subseteq X$ we have:

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T).$$

A function is *modular* if it is both supermodular and submodular.

Let $X \neq \emptyset$ be a finite set and let f be a submodular set function on X. The polyhedron $EP_f := \{x \in \mathbb{Q}^{|X|} : x(U) \leq f(U), \forall U \subseteq X\}$ is called the *extended polymatroid* associated with f. The polytope $EP_f \cap \{x \colon x \geq \mathbf{0}\}$ is called the *polymatroid* associated with f.

Let g be a supermodular function, the polyhedron $EQ_g := \{x \in \mathbb{Q}^{|X|} : x(U) \geq g(U), \forall U \subseteq X\}$ is called the *extended contrapolymatroid* associated with g. The polyhedron $EQ_g \cap \{x \colon x \geq \mathbf{0}\}$ is called the *contrapolymatroid* associated with g.

Polymatroids are particularly important for their interaction with matroids:

Observation 7.1. *Let M be a matroid on the ground set X, with rank function r. The polyhedron $\{x \in \mathbb{Q}^{|X|} : x \geq \mathbf{0}, x(U) \leq r(U), \forall U \subseteq X\}$ is a polymatroid.*

We denote by $\mathcal{M}(X)$ the family of all polymatroids, contrapolymatroids, and their extended counterparts on the set X, and we refer to the systems given above as their *canonical descriptions*. The following results stem from Theorem 4.5.

Theorem 7.2 (Edmonds and Giles [45])**.** *Let $P \in \mathcal{M}(X)$, with $X \neq \emptyset$ a finite set. Then, the canonical description of P is box-TDI.*

Interestingly, the intersection of any two of these objects preserves box-TDIness.

Theorem 7.3 (Schrijver [91], Chapter 46)**.** *Let P and Q be two polyhedra in $\mathcal{M}(X)$, with $X \neq \emptyset$ finite set. Then, the system obtained by concatenating the canonical descriptions of P and Q is box-TDI.*

7.2 Flows and Edge-Connectivity

Several results presented in this section can be found in [89].

7.2.1 Flows

A classical result concerning both connectivity and flows is the following theorem of Menger.

Theorem 7.4 (Menger [76])**.** *Let G be a graph, and let s, t be two vertices. Then, the maximum number of pairwise edge-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ cut.*

Theorem 7.4 was later independently generalized by Elias, Feinstein, and Shannon [48] and by Ford and Fulkerson [49].

Theorem 7.5 (Max-Flow Min-Cut Theorem)**.** *Let s and t be two vertices of a directed graph G, the maximum $s-t$ flow is equal to the minimum capacity of an $s-t$ cut.*

This result is one of the most used examples to show a case of min-max relation among combinatorial objects that can be obtained through the TDIness of a system. Moreover, one can generalize Theorems 7.4 and 7.5 in terms of box-TDIness as follows:

Theorem 7.6 (Schrijver [89]). *Let $D = (V, A)$ be a digraph. Then, the system:*

$$x(\delta^+(v)) - x(\delta^-(v)) = 0 \quad \forall v \in V, \tag{7}$$

is box-TDI.

Proof. By Theorem 2.3, the constraint matrix is TU, as each column contains exactly one $+1$ and one -1. Thus, System 7 is box-TDI by Theorem 4.6. □

One can deduce Theorem 7.5 from Theorem 7.6 by removing the constraints associated to s and t, and fixing to 0 all variables associated to arcs in $\delta^+(s)$ or to $\delta^-(t)$. Thus, it suffices to remark that the capacity constraints on the edges are just box-constraints. Once these constraints are added, the dual of the maximization problem of a linear function over the polyhedron defined by System (7) corresponds to the minimum cut problem.

7.2.2 Edge-Connectivity

One classical result we can achieve by the application of Theorem 7.2 concerns the spanning tree polytope, that is the convex hull of all spanning trees of a graph. More precisely, one can prove that the spanning tree polytope, the forest polytope, and the connector polyhedron are all box-TDI. These results are considered folklore by the community, nevertheless, we provide a proof for the sake of completeness.

Theorem 7.7 (Folklore). *The spanning tree polytope, the forest polytope, and the connector polyhedron are box-TDI for all graphs.*

Proof. Let $G = (V, E)$ be a graph. The forest polytope is a polymatroid, hence, it is box-TDI by Theorem 7.2. The spanning tree polytope is a face of the forest polytope, namely the one obtained by setting to equality the constraint $x(E) \leq |V| - 1$. Therefore, by Theorem 4.5, it is box-TDI. The connector polyhedron is the dominant of the spanning tree polytope, thus it is box-TDI as well thanks to Theorem 5.10. □

In the case of directed graphs, we have that the *r-arborescence polytope*, that is the convex hull of all arborescences rooted in r, is box-TDI, as proved by Schrijver [91, Corollary 52.4]. Note that it is not possible to derive the box-TDIness of the arborescence polytope from this result, as the convex hull of two or more box-TDI polyhedra is not necessarily box-TDI.

Further results with respect to the Menger's Theorem 7.4 highlight a natural connection between series-parallel graphs and box-TDIness of polyhedra associated with flows and edge-connectivity. We summarize subsequent advancement in this topic due to Barbato et al. [4], Chen et al. [14], Chervet et al. [21], and Cornaz et al. [29] in the following theorem.

Theorem 7.8. *Let G be a graph and $k \in \mathbb{Z}$, $k \geq 2$. Then, each of the following polyhedra is box-TDI if and only if G is series-parallel.*

- *the flow cone of G;*
- *the cone of conservative functions of G;*

- *the cycle cone of G;*
- *the cut cone of G;*
- *the cut polytope of G;*
- *the multicut polytope of G;*
- *the k-edge-connected spanning subgraph polyhedron of G.*

A remark about the last point of Theorem 7.8 is the fact that this was the first instance of a polyhedron proved to be box-TDI without using a TDI system describing it. Moreover, in [4] the authors prove that the classical system for this problem, given by Didi Biha and Mahjoub [37] is TDI for series-parallel graphs.

7.3 Matchings

One of the first min-max results between combinatorial objects is the well-known Kőnig's Theorem [72], generalized by Egerváry [47] to the weighted case.

Theorem 7.9 (Kőnig's Theorem)**.** *Let G be a bipartite graph. Then, the size of a maximum matching equals the size of the smallest vertex cover.*

This result can be seen as a particular case of the Max-flow Min-cut Theorem, and it is also one of the most classic examples of duality in combinatorial optimization. In fact, Theorem 7.9 celebrates TDIness as shown in the next section.

For the completeness of this compendium, we introduce the matching polytope along with two special cases: the perfect matchings and the extendable matchings.

7.3.1 The Matching Polytope

Let $G = (V, E)$ be a graph. The *matching polytope* of G is the convex hull of the incidence vectors of all matchings in G. The system

$$\begin{cases} x(E(U)) \leq \frac{|U|-1}{2}, \text{ for each } U \subseteq V \text{ with } |U| \geq 3 \text{ odd}, \\ x(\delta(u)) \leq 1, \text{ for each } u \in V, \\ x \geq \mathbf{0}, \end{cases} \tag{8}$$

describes the matching polytope of G [43] and it is known as *Edmonds system.* When G is bipartite, the inequalities $x(E(U)) \leq (|U| - 1)/2$ are redundant for each $U \subseteq V$ with $|U| \geq 3$ and odd cardinality. Moreover, in this case, the incidence matrix is TU [69] and the matching polytope is box-TDI. This is sufficient to prove Theorem 7.9. Interestingly, Cunningham and Marsh [32] proved that System (8) is TDI for any graph G. Thus, System (8) is box-TDI if and only if the matching polytope is. Ding et al. [39] characterize the graphs for which this polytope is box-TDI, in terms of odd subdivision. A *fully odd subdivision* of a graph is a graph obtained by replacing an edge with a path composed of an odd number of edges.

Theorem 7.10 (Ding et al. [39])**.** *The matching polytope of a graph G is box-TDI if and only if G does not include any fully odd subdivision of G_1, G_2, G_3, and G_4 as a subgraph.*

Theorem 7.10 was proved by showing when System (8) is ESP.

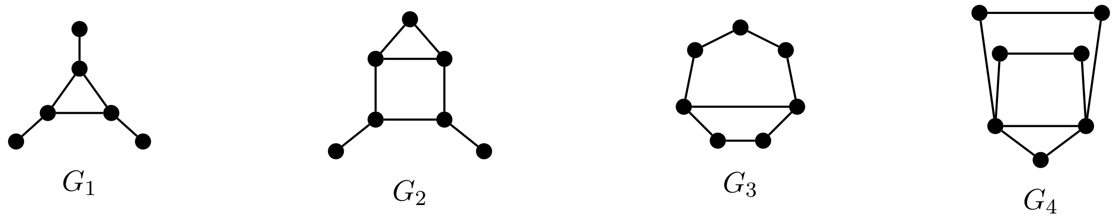

Figure 9: Forbidden subgraphs for the box-TDIness of the matching polytope.

7.3.2 Capacitated b-matching

Given a graph $G = (V, E)$ and two vectors $c \in \mathbb{Z}^E_{\geq 0}$ and $b \in \mathbb{Z}^V_{\geq 0}$, a *$c$-capacitated b-matching* is an integer point of the system $x(\delta(u)) \leq b_u$, $\mathbf{0} \leq x \leq c$. Clearly, c-capacitated b-matchings are a generalization of matchings.

In his Ph.D. thesis, Cook [25] provides a TDI system for the convex hull of the c-capacitated b-matchings, generalizing a previous result due to Pulleyblank [86]. We report the result as stated in Schrijver's book [91].

Theorem 7.11 (Schrijver [91], Theorem 32.3)**.** *Let $G = (V, E)$ be a graph, b and c be a rational vector over the edges of G. Then, the system*

$$\begin{cases} x(E(U)) + x(F) \leq \left\lfloor \frac{b(U)+c(F)}{2} \right\rfloor, \ \text{for each } U \subseteq V \text{ and } F \subseteq \delta(U), \\ x(\delta(u)) \leq b_u, \ \text{for each } u \in V, \\ \mathbf{0} \leq x \leq c, \end{cases} \tag{9}$$

is TDI system and describes the c-capacitated matching polytope of G.

We remark that Theorem 7.11 gives a TDI system for each fixed c. However, since its right-hand side depends on the capacity constraints, this result does not provide information about the box-TDIness of the system. This leaves open the following question.

Open Question 7.12. *When is System* (9) *box-TDI?*

7.3.3 The Perfect Matching Polytope

Let $G = (V, E)$ be a graph. The *perfect matching polytope* of G is the convex hull of the incidence vectors of all perfect matchings in G. The system

$$\begin{cases} x(\delta(U)) \geq 1, \ \text{for each } U \subseteq V \text{ with } |U| \geq 3 \text{ odd}, \\ x(\delta(u)) = 1, \ \text{for each } u \in V, \\ x \geq \mathbf{0}, \end{cases} \tag{10}$$

describes the perfect matching polytope of G as proved in [43]. Moreover, in the same work Edmonds also showed that System (10) is not TDI when G is non-bipartite.

It is straightforward to see that the perfect matching polytope is a face of the matching polytope. Thus, by Theorem 4.1, Theorem 7.10 gives sufficient but not necessary conditions for the box-TDIness. In fact, contrary to what happens for matchings, the forbidden graph G_1 of Figure 9 has only one perfect matching polytope, and hence the corresponding perfect matching polytope is box-TDI. Indeed, integer points are box-TDI polytopes by Theorem 5.2).

Grappe et al. [60], proved that if the perfect matching polytope of a graph G is box-TDI, then it is described by the compact formulation $x(\delta(U)) = 1$, for each *tight cut* $\delta(U)$, and $x \geq \mathbf{0}$, where a cut is tight if $|\delta(U) \cap M| = 1$ for each perfect matching M of G. This result leads to the following geometrical characterization.

Theorem 7.13 (Grappe et al. [60])**.** *The perfect matching polytope of a graph G is box-TDI if and only if its affine hull is box-TDI.*

In the same work, the authors characterize the box-TDIness of the perfect matching polytope in terms of forbidden structure for bicritical graphs and some other special classes of graphs. Where a graph is *bicritical* if the removal of any couple of vertices gives a graph that has a perfect matching.

To complete the characterization of the box-TDIness of the perfect matching polytope in terms of forbidden structures, the case of barrier cuts—a fundamental class of tight cuts—still needs to be settled. In a graph G, a *barrier* is a set of vertices U of G such that $G \setminus U$ has exactly $|U|$ odd connected components. A *barrier cut* of G is a cut such that one of the node sets defining it is an odd component left from removing a barrier. Thus, Grappe et al. [60] also let the following problem open.

Open Question 7.14. *When does the presence of barrier cuts prevent the perfect matching polytope from being box-TDI?*

7.3.4 The Extendable Matching Polytope

A matching is *extendable* if it is included in a perfect matching. The *extendable matching polytope* is the convex hull of the incidence vectors of the extendable matchings. Cunningham and Green-Krotki [31], gave a description of this polyhedron.

Theorem 7.15. *The extendable matching polytope is box-TDI if and only if the perfect matching polytope is box-TDI.*

Proof. The extendable matching polytope is the submissive of the perfect matching polytope intersected with the positive orthant. Therefore, if the perfect matching polytope is box-TDI so is the extendable matching polytope by Theorem 5.10.

Since the perfect matching polytope is a face of the extendable matching polytope, whenever the latter is box-TDI, so is the first, thanks to Theorem 4.1. □

7.4 Edge Covers

The *edge cover polytope* of G, denoted by $EC(G)$, is the convex hull of the edge covers of G. Edmonds [46] gives the following description of $EC(G)$:

$$\begin{cases} x(E(U) \cup \delta(U)) \geq \left\lceil \frac{|U|}{2} \right\rceil, \text{ for each } U \subseteq V, \\ \mathbf{0} \leq x \leq \mathbf{1}. \end{cases} \tag{11}$$

The characterization of the box-TDIness of the edge cover polytope is not known, thus we have the following.

Open Question 7.16. *When is System* (11) *box-TDI?*

A partial answer to Open Question 7.16 has been given by Ding et al. [38], who studied the box-TDIness of the linear system

$$\begin{cases} x(\delta(u)) \geq 1, \text{ for each } u \in V, \\ x \geq \mathbf{0}, \end{cases} \tag{12}$$

describing the *dominant of the edge relaxation of the edge cover polytope*, for a given graph G of vertex-set V. We denote this polyhedron by $DEC(G)$.

A graph is *quasi-bipartite* if the removal of the vertex-set of any odd cycle gives a graph having an isolated vertex.

Theorem 7.17 (Ding et al. [38]). *For a simple graph G the following are equivalent:*

- *G is quasi-bipartite and different from K_4;*
- *System* (12) *is TDI;*
- *System* (12) *is box-TDI.*

Theorem 7.18 (Chervet et al. [20]). *Let G be a simple graph. Then, $DEC(G)$ is box-TDI if and only if G is either a circuit or a quasi-bipartite different from K_4.*

By Theorem 7.17, if $G \neq K_4$ is quasi-bipartite, then $EC(G) = DEC(G) \cap \{x \colon x \leq \mathbf{1}\}$. In this case, $EC(G)$ is box-TDI by Observation 5.11. However, this is not a characterization since there exist graphs outside of this class for which $EC(G)$ is box-TDI. For example, $EC(K_3)$ is box-TDI, since System (11) is box-TDI for this graph.

7.5 Stable Sets

The *stable set polytope* of a graph G is the convex hull of all stable sets of G. The following system describes the *edge relaxation of the stable set polytope*:

$$\begin{cases} x_u + x_v \leq 1, \text{ for each } uv \in E, \\ x \geq \mathbf{0}. \end{cases} \tag{13}$$

Chervet et al. [20] showed that the edge-vertex incidence matrix of a graph is always TE, thus proving the following, thanks to Theorem 4.7.

Theorem 7.19 (Chervet et al. [20]). *The edge relaxation of the stable set polytope is box-TDI.*

Note that the above result does not give information about the integrality of the polytope, since System (13) is not TDI in general. When G is bipartite, the edge-vertex incidence matrix is TU. Thus, System 13 is box-TDI by Theorem 4.6, and describes the stable set polytope by Theorem 3.1.

Karp [70] proved that finding a maximum stable set for a given graph is NP-hard in general. Thus, by Theorem 7.19 finding an integer optimal solution over a non-integer box-TDI polyhedron is NP-hard [20].

A graph is *perfect* if, for every node-induced subgraph, the chromatic number equals the size of the largest clique. Perfect graphs have been introduced by Berge [8]. For a given graph G the *clique-vertex matrix* of G is the matrix whose rows correspond to the characteristic vectors of maximal cliques of G. A *clutter* is a hypergraph such that every hyperedge is inclusion-wise maximal. Let M be the hyperedge-vertex incidence matrix of a clutter. Chvátal [23] proved that the polytope $\{x\colon Mx \le 1, x \ge \mathbf{0}\}$ is integral if and only if M is the clique-vertex matrix of a perfect graph. That is, the stable set polytope of G is described by $\{x\colon Mx \le 1, x \ge \mathbf{0}\}$ if and only if G is perfect. Verifying whether a given matrix is the clique-vertex of a graph can be done in polynomial time [30, Corollary 3.10]. These facts are related to TDIness by the following.

Theorem 7.20 (Lovász [74]). *Let M be the clique-vertex matrix of a graph G. Then, G is perfect if and only if the linear system $x \ge \mathbf{0}, Mx \le \mathbf{1}$ is TDI.*

A graph is *box-perfect* if it is perfect and its stable set polytope is box-TDI. Edmonds and Cameron [12] posed the following open question which turned out to be seminal for some research streams.

Open Question 7.21. *When is a given graph box-perfect?*

Cameron [11] proved that the removal and the duplication of vertices of a box-perfect graph result in a box-perfect graph. Moreover, Cameron proved that the comparability and incomparability graphs are box-perfect. Ding et al. [41] proved that a graph such that for any couple of vertices all induced paths between them have the same parity is box-perfect. Chervet and Grappe [19] characterize the problem in terms of complementary graphs. Precisely, they proved that a graph G and its complement $\overline{G}$ are both box-perfect if and only if $\overline{G}^+$ is box-perfect, where G^+ is obtained by adding a universal vertex to G.

7.6 Integrally Convex Sets

Box-TDI polyhedra play a role in the theory of discrete convex sets and functions. A set of integer points S is *integrally convex* if its convex hull can be expressed as the union of the convex hulls of its integer points within small local regions around every point. More precisely, for every $x \in \mathbb{R}^n$, define its integer neighborhood $IN(x)$ as the set of integer points $\{z \in \mathbb{Z}^n : \lfloor x_i \rfloor \le z_i \le \lceil x_i \rceil\}$. Then, a set S is integrally convex if $\text{conv.hull}(S) = \bigcup_{x \in \mathbb{R}^n} \text{conv.hull}(S \cap IN(x))$, where conv.hull$(X)$ denotes the convex hulls of a set of points X. Integrally convex sets play an important role in discrete convexity.

The interconnection between integrally convex set and polyhedral theory is shown by different results. First, it is easy to prove (see for instance, Murota and Takamura [79]) the following property:

Observation 7.22. *The convex hull of an integrally convex set is a box-integer polyhedron. Conversely, the set of integer points of a box-integer polyhedron is integrally convex.*

Thus, we have a correspondence between integrally convex sets and box-integer polyhedra. As an immediate consequence, one can see that:

Corollary 7.23. *The set of integer points of an integer box-TDI polyhedron is integrally convex.*

These sets are also called *box-TDI sets* in the literature. Not all integrally convex sets arise from box-TDI polyhedra, however, some key classes of integrally convex sets do, as shown in the remainder of this section.

L-Convex Sets A set S is *L-convex* if it satisfies two properties:

- for all $x, y \in S$ both $x \wedge y$ and $x \vee y$ belong to S, where $\wedge$ and $\vee$ represent respectively the component-wise minimum and the component-wise maximum,
- for all $x \in S$, $x \pm \mathbf{1} \in S$.

A set S is *$L^\natural$-convex* if it can be obtained as the intersection of an L-convex set with some coordinated hyperplanes. Finally, a set is *L_2-convex* (respectively *$L_2^\natural$-convex*) if it is given by the intersection of two L-convex (respectively $L^\natural$-convex) sets.

The relation between L-convex sets and box-TDI polyhedra is highlighted by the following results.

Theorem 7.24 (Moriguchi and Murota [78]). *The convex hull of a $L_2^\natural$-convex set is box-TDI.*

Since the class of $L_2^\natural$-convex sets contains strictly $L^\natural$-convex sets, L_2-convex sets, and L-convex sets, we can derive the following.

Corollary 7.25. *The convex hulls of $L^\natural$-convex sets, L_2-convex sets, and L-convex sets are box-TDI.*

M-Convex Sets A set of integer vectors $S \subseteq \mathbb{Z}^n$ is *M-convex* if for every pair of distinct vectors $x, y \in S$, and for every index $i \in [n]$ such that $x_i > y_i$, there exists an index $j \in [n]$ such that:

- $x_j < y_j$ and
- $x - e_i + e_j$ belongs to S.

A set S is *$M^\natural$-convex* if it is the projection of an M-convex set along a coordinate axis. Similarly to what happened for L-convex sets, a set is *M_2-convex* (respectively *$M_2^\natural$-convex*) if it is given by the intersection of two M-convex (respectively $M^\natural$-convex) sets.

These sets are closely related to polymatroids, and this relation can be exploited to prove that the convex hull of any of these sets is box-TDI.

Theorem 7.26 (Moriguchi and Murota [78]). *The convex hull of a $M_2^\natural$-convex set is box-TDI. Moreover, so are the convex hulls of $M^\natural$-convex sets, M_2-convex sets, and M-convex sets.*

8 Beyond the Core Theory

This section is devoted to analyzing the connection between box-TDIness and several other properties present in the literature.

Discrete Convex Functions A function $\phi : \mathbb{Z} \to \mathbb{Z}$ is *discrete convex* if, for all $z \in \mathbb{Z}$ we have that $\phi(z-1) + \phi(z+1) \geq 2\phi(z)$. A function $\Phi : \mathbb{Z}^n \to \mathbb{Z}$ is *separable discrete convex* if $\Phi(z) = \sum_{i=1}^{n} \phi_i(z_i), \quad \forall z \in \mathbb{Z}^n$, where ϕ_i are discrete convex functions and z_i are the components of z.

Frank and Murota [50] give a min-max theorem for minimizing separable integer-valued convex functions over box-TDI polyhedra. Indeed, they show that the minimum of such a function over the integer points of a box-TDI polyhedron can be characterized by a dual optimization problem, where optimality is certified by the existence of a dual vector satisfying a discrete subgradient condition. Notably, their work provides a novel application of box-TDI polyhedra to nonlinear combinatorial optimization.

Dyadicness A vector is *p-adic* if each of its entries is of the form a/p^k for some integers a, k with $k \geq 0$ and p prime. When $p = 2$, we call the vector *dyadic*. A system is *totally dual p-adic* if whenever $\min\{b^\top y \colon A^\top y = w, y \geq \mathbf{0}\}$ for w integer, has an optimal solution, it has a p-adic optimal solution. A vector x is *half-integer* if $2x$ is integer. A system is *totally dual half-integral* if whenever $\min\{b^\top y \colon A^\top y = w, y \geq \mathbf{0}\}$ for w integer, has an optimal solution, it has a half-integer optimal solution. Of course, totally dual half-integral systems are totally dual dyadic. Despite this fact, totally dual half-integral systems are a notable class deserving a mention due to their connection with the matching polytope [32].

Dyadic and half-integral vectors present an intriguing relation with TDI systems as stated in the following.

Theorem 8.1 (Section 22.7, [90])**.** *A system $Ax \leq b$ is TDI if and only if:*

- *for each vector $y \geq \mathbf{0}$ with $A^\top y$ integer, there exists a dyadic vector $y' \geq \mathbf{0}$ with $A^\top y' = A^\top y$;*
- *for each $\{0, \frac{1}{2}\}$-vector y with $A^\top y$ integer, there exists an integer vector $y' \geq \mathbf{0}$ with $A^\top y' = A^\top y$ and $b^\top y' \leq b^\top y$.*

Abdi et al. [1] have shown that TDI systems $Ax \leq b$, with A and b entry-wise integer, describing a pointed polyhedron are totally dual p-adic for all prime numbers p.

Similarly, suppose that $P = \{x \colon Ax \leq b\}$ is a pointed polyhedron, with A and b entry-wise integer. If $Ax \leq b$ is totally dual p-adic and totally dual q-adic, for two distinct prime numbers p and q, then P is an integer polyhedron.

Dyadic Polyhedra A polyhedron is *p-adic* for some prime number p if every non-empty face contains a p-adic point. As integer TDI systems certify polyhedral integrality, total dual p-adic systems certify polyhedral p-adicness [1]. Moreover, TE $0, \pm 1$-matrices define dyadic polyhedra [22, 84]. Thus, every negative complexity result holding for TE matrices immediately transfers to dyadic polyhedra. For instance, Chervet et al. [20] hardness result on TE matrices implies that integer programming over dyadic polyhedra is NP-hard.

Integer Rounding Property A system $Ax \leq b$ has the *integer rounding property* if, for every integer vector c, its dual problem respects

$$\min\{y^\top b \colon y^\top A = c, y \in \mathbb{Z}^n_{\geq 0}\} = \lceil \min\{y^\top b \colon y^\top A = c, y \geq \mathbf{0}\} \rceil,$$

whenever the optimum of the right-hand side of the equality is finite.

This notion is related to TDIness by the following.

Theorem 8.2 (Giles and Orlin [54]). *Let $Ax \leq b$ be a feasible system and x' be a variable that differs from every component of x. Then, $Ax \leq b$ has the integer rounding property if and only if the system $Ax - bx' \leq \mathbf{0}$, $x' \geq 0$ is TDI.*

By Theorems 3.8 and 8.2, the system $Ax \leq b$ has the integer rounding property if and only if the rows of the matrix $\begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$ form a Hilbert basis [54]. If a system has the integer rounding property then one can solve any ILP over it in polynomial time [7].

Integer Decomposition Property A polyhedron P has the *integer decomposition property* if for each integer positive k and every $x' \in kP \cap \mathbb{Z}^n$, there exists a *proper integer decomposition*, that is a set of points $x_1, \ldots, x_k \in P \cap \mathbb{Z}^n$ such that $x' = x_1 + \ldots + x_k$.

In general, box-TDIness and integer decomposition property are mutually independent properties as shown in Section 6.3 of [21] and Section 5 of [60]. Nevertheless, when a polyhedron is described by a TU matrix, the following holds.

Theorem 8.3 (Baum and Trotter [5]). *A polyhedron described by a TU matrix has the integer decomposition property. Moreover, it is possible to find a proper integer decomposition in polynomial time.*

Later the same authors extended this result to polymatroids [6]. In [60], Grappe et al. remark that the same result holds when the perfect matching polytope is box-TDI. These results lead to the following.

Open Question 8.4. *Let P be a box-TDI polyhedron having the integer decomposition property. Is it possible to find in polynomial time a proper integer decomposition every $x \in kP \cap \mathbb{Z}^n$, with $k \in \mathbb{Z}_{>0}$?*

Another open question about the integer decomposition property of box-TDI smooth polytopes is presented in [21, Section 6.3].

Open Question 8.5. *Do smooth fully box-integer polyhedra have the integer decomposition property?*

Integer Carathéodory Property A polyhedron P has the *integer Carathéodory property* if for each integer positive k, every $x' \in kP \cap \mathbb{Z}^n$ is such that $x' = \alpha_1 x_1 + \ldots + \alpha_m x_m$, where $x_1, \ldots, x_m \in P \cap \mathbb{Z}^n$ are affinely independent and $\alpha_i \in \mathbb{Z}_{\geq 0}$ for all i. One can check that $\alpha_1 + \ldots + \alpha_m = k$, thus, the integer Carathéodory property implies the integer decomposition property. In particular, Theorem 8.3 can be extended to the integer Carathéodory property thanks to the results of Baum and Trotter [5].

Several results connect box-TDI polyhedra with the integer Carathéodory property. In particular, a large class of polyhedra having the integer Carathéodory property has been introduced by Gijswijt and Regts [53], which are box-TDI, as remarked in [21]. Moreover, Gijswijt and Regts raise the following.

Open Question 8.6. *Does the r-arborescence polytope have the integer Carathéodory property?*

Recently, Chervet et al. [22] proved that the box-TDI cones generated by a set of linearly independent non-negative vectors have the integer Carathéodory property.

Conclusions

From its origins in the work of Edmonds and Giles [45] to recent characterizations via equimodular and TE matrices [21], box-TDIness offers structural properties of algorithmic and polyhedral interest. Beyond summarizing the main developments, we have highlighted several open questions, ranging from structural characterizations to algorithmic recognition. These questions reflect both the maturity of the field and its openness to new directions of research.

We reviewed box-TDIness by highlighting its deep theoretical foundations, geometric interpretations, and connections to classical combinatorial optimization. A discussed key point is that box-TDIness is not only a strengthening of TDIness, but a fundamentally different property with polyhedral and algebraic characterizations. Discussed examples show that several polyhedra arising in classical combinatorial problems—such as polymatroids, network flows, and matchings—are box-TDI even in non-trivial cases. The connections between polyhedral integrality, Hilbert bases, and equimodularity also indicate several directions that can be further studied.

Although the matricial criteria by Chervet et al. [21] offer a new way for understanding box-TDIness and help unify different strands of the literature, most of the recognition problems related to box-TDIness remain hard to solve. This suggests that, unless $P = NP$, efficient recognition algorithms are likely to exist only in restricted cases.

Finally, we have emphasized the role of box-TDIness in ensuring strong min-max relations, robust duality under variable bounds, and the preservation of integrality in bounded relaxations. This interpretation makes box-TDIness interesting in the theoretical context as well as in the applied one.

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