Mathematical Modelling – Integer Linear Programming

Exercice

• A logistic firm has to load $m$ trucks $\{1, \ldots, m\}$ in order to serve daily a set of costumers. In the trucks $n$ pallets $\{1, \ldots, n\}$ must be loaded, the $j$-th of them is characterized by a weight $w_j$. The $i$-th truck has a weight capacity of $b_i$. The problem asks for determining the minimum number of trucks to pack all the pallets.

• Write a ILP model to minimize the total number of trucks necessary to serve all the clients. Identify the decisions that must be taken and the corresponding decision variables that have to be used. Identify and comment the objective function of the problem and the constraints.

• Now consider the follow case in which there are 4 pallets ($n = 4$) and 4 available trucks ($m = 4$), the capacity of each truck is $B = 4$. The weights of the pallets and the capacities of the tracks are:

<table>
<thead>
<tr>
<th>$w_j$</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
<th>$j = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b_i$</th>
<th>$i = 1$</th>
<th>$i = 2$</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Write the ILP model for this specific case.

• Try to find a feasible solution of the problem which satisfies the needs the clients without exceeding the capacity of the used trucks. What is the total cost of the proposed solution? Try to justify your choices.
Decision Variables:

\[ x_{ij} = \begin{cases} 
1 & \text{if pallet } j \text{ is packed in truck } i \\
0 & \text{otherwise} 
\end{cases} \quad i = 1, \ldots, m, j = 1, \ldots, n \]

\[ y_i = \begin{cases} 
1 & \text{if truck } i \text{ is used} \\
0 & \text{otherwise} 
\end{cases} \quad i = 1, \ldots, m, \]

ILP Model:

\[ Z(\text{ILP}) = \min \sum_{i=1}^{m} y_i \quad (1) \]

\[ \sum_{i=1}^{m} x_{ij} = 1 \quad j = 1, \ldots, n \quad (2) \]

\[ \sum_{j=1}^{n} w_j x_{ij} \leq b_i y_i \quad i = 1, \ldots, m \quad (3) \]

\[ y_i \in \{0, 1\} \quad i = 1, \ldots, m \quad (4) \]

\[ x_{ij} \in \{0, 1\} \quad i = 1, \ldots, m, j = 1, \ldots, n \quad (5) \]

- the specific model is with 4 pallet and 4 trucks is:

\[ Z(\text{ILP}) = \min y_1 + y_2 + y_3 + y_4 \quad (6) \]

\[ x_{11} + x_{21} + x_{31} + x_{41} = 1 \quad (7) \]

\[ x_{12} + x_{22} + x_{32} + x_{42} = 1 \quad (8) \]

\[ x_{13} + x_{23} + x_{33} + x_{43} = 1 \quad (9) \]

\[ x_{14} + x_{24} + x_{34} + x_{44} = 1 \quad (10) \]

\[ 1x_{11} + 2x_{12} + 3x_{13} + 4x_{14} \leq 4y_1 \quad (11) \]

\[ 1x_{21} + 2x_{22} + 3x_{23} + 4x_{24} \leq 4y_2 \quad (12) \]

\[ 1x_{31} + 2x_{32} + 3x_{33} + 4x_{34} \leq 4y_3 \quad (13) \]

\[ 1x_{41} + 2x_{42} + 3x_{43} + 4x_{44} \leq 4y_4 \quad (14) \]

\[ y_1, y_2, y_3, y_4 \in \{0, 1\} \quad (15) \]

\[ x_{11}, x_{12}, x_{13}, x_{14} \in \{0, 1\} \quad (16) \]

\[ x_{21}, x_{22}, x_{23}, x_{24} \in \{0, 1\} \quad (17) \]

\[ x_{31}, x_{32}, x_{33}, x_{34} \in \{0, 1\} \quad (18) \]

\[ x_{41}, x_{42}, x_{43}, x_{44} \in \{0, 1\} \quad (19) \]